Maurice Auslander Distinguished Lectures
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Quiver mutations

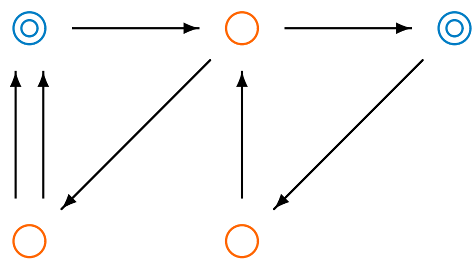
based on joint work with
Andrei Zelevinsky

Tensor diagrams and cluster algebras

based on joint work with
Pavlo Pylyavskyy

Quivers

A *quiver* is a finite oriented graph.



Multiple edges are allowed.

No loops, no oriented cycles of length 2.

Two types of vertices: "frozen" and "mutable."

Ignore edges connecting frozen vertices to each other.

Quiver mutations

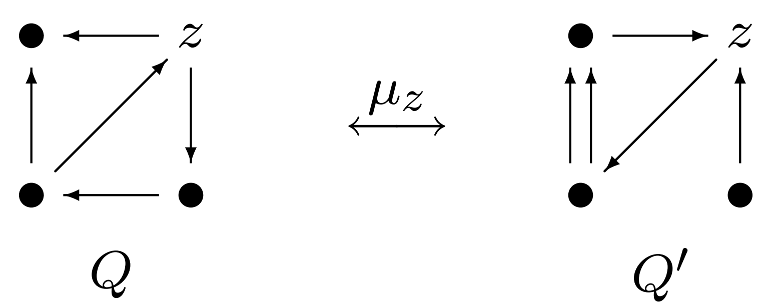
Pick a mutable vertex z.

Quiver mutation $\mu_z : Q \mapsto Q'$ is computed in three steps.

Step 1. For each instance of $x \to z \to y$, introduce an edge $x \to y$.

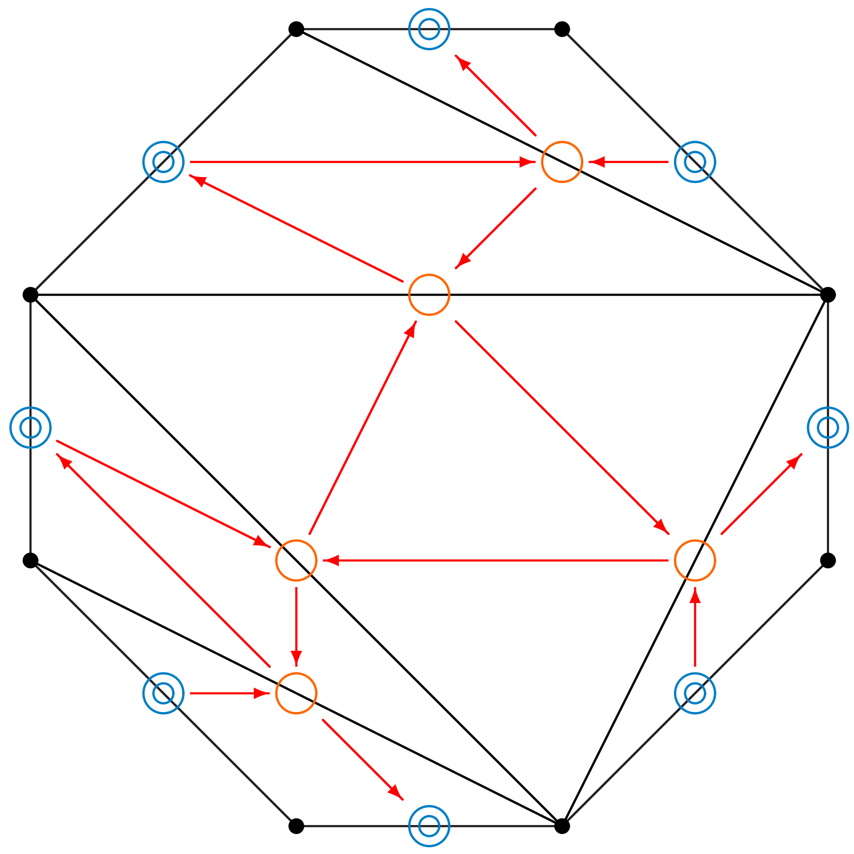
Step 2. Reverse the direction of all edges incident to z.

Step 3. Remove oriented 2-cycles.



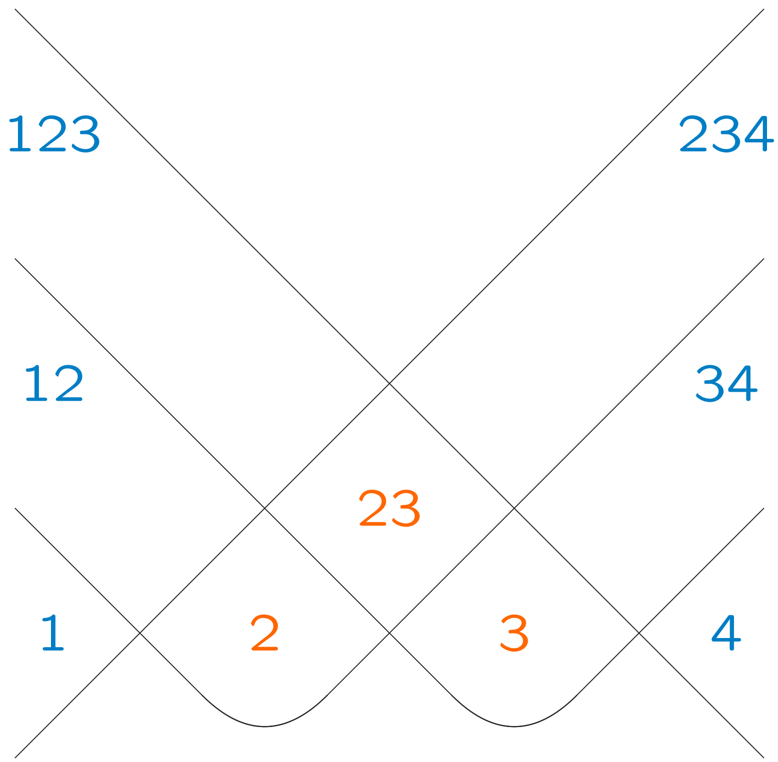
Mutation of Q' at z recovers Q.

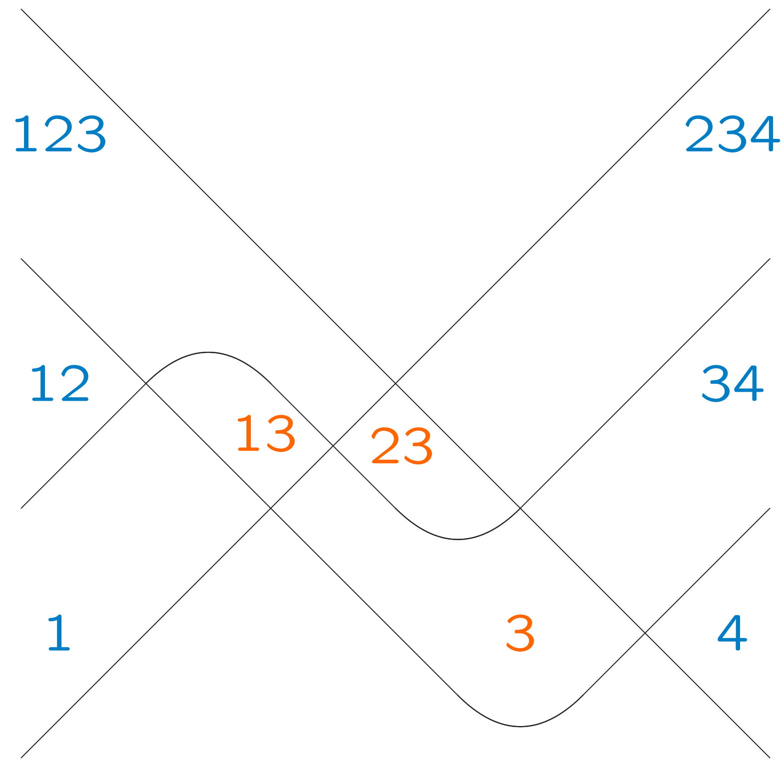
Example: quivers associated with triangulations

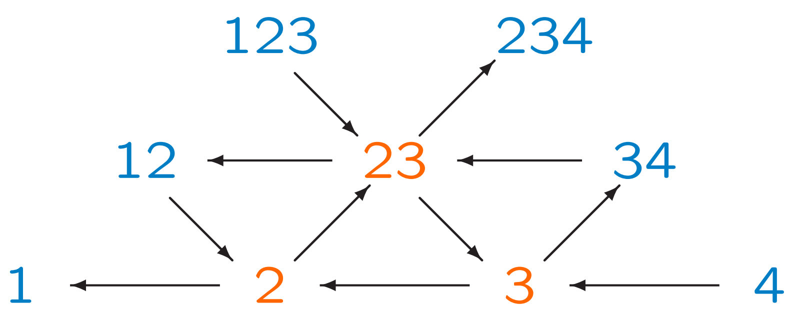


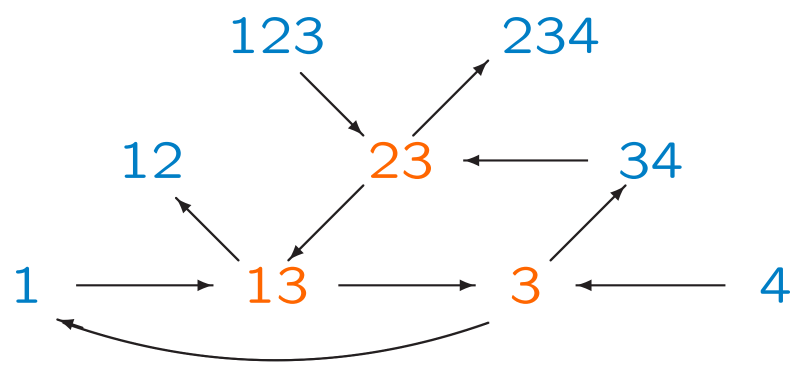
Mutations correspond to *flips*.

Example: braid moves









Other occurences of quiver mutation

- *Seiberg dualities* in string theory

- *urban renewal* transformations of planar graphs

- *tropical Y-systems*

- *A'Campo–Gusein-Zade diagrams* of morsified curve singularities

- *star-triangle* transformations of electric networks

Mutation-acyclic quivers

A quiver is *mutation-acyclic* if it can be transformed by iterated mutations into a quiver whose mutable part is acyclic.

Theorem 1 [A. Buan, R. Marsh, and I. Reiten, 2008]
A full subquiver of a mutation-acyclic quiver is mutation-acyclic.

Classification of quivers of finite mutation type

A quiver has *finite mutation type* if its mutation equivalence class consists of finitely many quivers (up to isomorphism).

Theorem 2 [A. Felikson, P. Tumarkin, and M. Shapiro, 2008]
Apart from 11 exceptions, a quiver has finite mutation type if and only if its mutable part comes from a triangulated surface.

Seeds and clusters

Let $\mathcal{F} \supset \mathbb{C}$ be a field. A *seed* in $\mathcal{F}$ is a pair $(Q, \mathbf{z})$ consisting of

- a quiver Q as above;
- an *extended cluster* $\mathbf{z}$, a tuple of algebraically independent (over $\mathbb{C}$) elements of $\mathcal{F}$ labeled by the vertices of Q.

coefficient variables	$\longleftrightarrow$	frozen vertices
cluster variables	$\longleftrightarrow$	mutable vertices

The subset of $\mathbf{z}$ consisting of cluster variables is called a *cluster*.

Seed mutations

Pick a mutable vertex. Let z be the corresponding cluster variable.

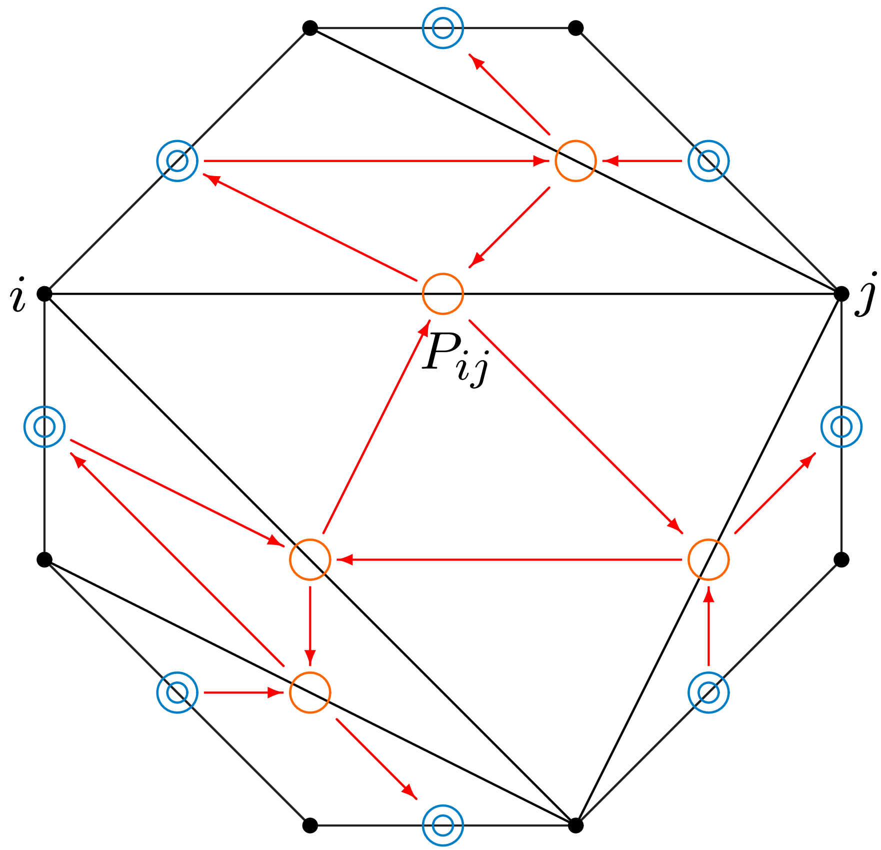
A *seed mutation* μ_z replaces z by the new cluster variable z' defined by the *exchange relation*

$$z\,z' = \prod_{z \leftarrow y} y + \prod_{z \rightarrow y} y\,.$$

The rest of cluster and coefficient variables remain unchanged.

Then mutate the quiver Q at the chosen vertex.

Example: Grassmannian $\mathrm{Gr}_{2,N}$



Ptolemy (or Grassmann–Plücker) relations:

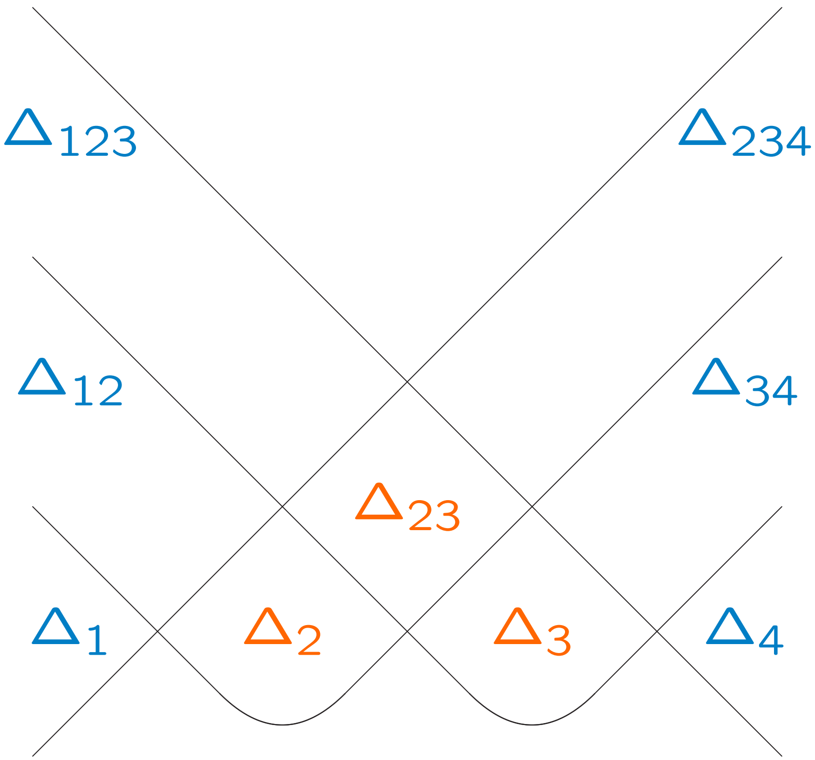
$$P_{ac}\,P_{bd} = P_{bc}\,P_{ad} + P_{ab}\,P_{cd}\,.$$

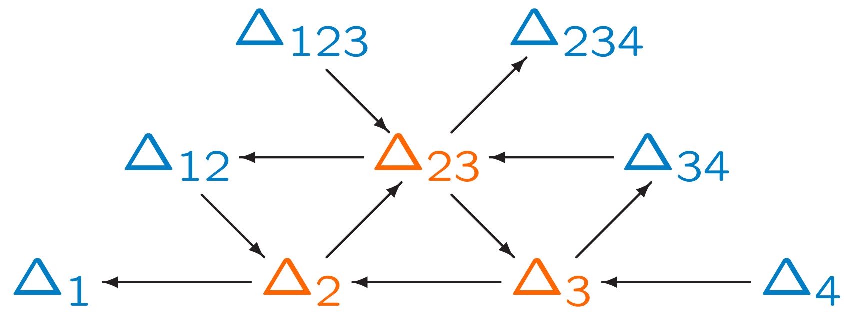
Mutation dynamics on general surfaces

Seed mutations associated with flips on arbitrary triangulated surfaces (oriented, with boundary) describe transformations of the corresponding *lambda lengths*, a.k.a. Penner coordinates on the appropriately defined *decorated Teichmüller space*.

See [SF–D. Thurston, `arXiv:1210.5569`].

Example: chamber minors



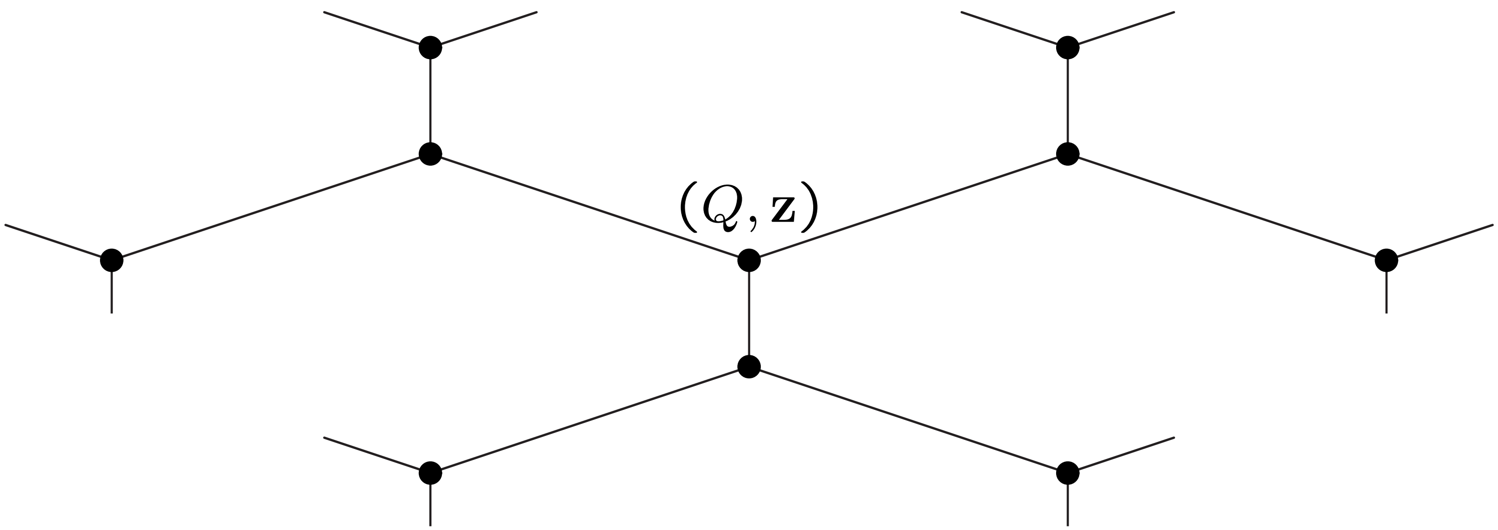


$$\Delta_2 \Delta_{13} = \Delta_{12} \Delta_3 + \Delta_1 \Delta_{23} \, .$$

See [SF, ICM 2010].

Cluster algebra

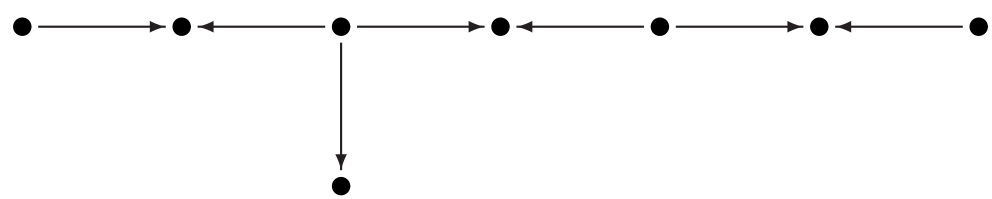
The *cluster algebra* $\mathcal{A}(Q, \mathbf{z})$ is generated inside $\mathcal{F}$ by all elements appearing in the seeds obtained from $(Q, \mathbf{z})$ by iterated mutations.



More precisely, we defined cluster algebras of *geometric type* with *skew-symmetric* exchange matrices.

Finite type classification

The classification of cluster algebras with finitely many seeds is completely parallel to the *Cartan-Killing classification.*



The Laurent phenomenon

Theorem 3 *Every cluster variable in* $\mathcal{A}(Q, \mathbf{z})$ *is a Laurent polynomial in the elements of* $\mathbf{z}$.

No "direct" description of these Laurent polynomials is known.

They are conjectured to have positive coefficients.

The Starfish Lemma

Lemma 4 *Let R be a polynomial ring. Let $(Q, \mathbf{z})$ be a seed in the field of fractions for R. Assume that*

- *all elements of $\mathbf{z}$ belong to R, and are pairwise coprime;*
- *all elements of clusters adjacent to $\mathbf{z}$ belong to R.*

Then $\mathcal{A}(Q, \mathbf{z}) \subset R$.

Problem: Under these assumptions, give "polynomial" formulas for all cluster variables.

Open for any cluster algebra of infinite mutation type.

The Starfish Lemma for rings of invariants

Many important rings have a natural cluster algebra structure. Here we focus on classical rings of invariants.

Lemma 5 *Let G be a group acting on a polynomial ring R by ring isomorphisms. Let $(Q, \mathbf{z})$ be a seed in the field of fractions for the ring of invariants R^G. Assume that*

- *all elements of $\mathbf{z}$ belong to R^G, and are pairwise coprime;*
- *all elements of clusters adjacent to $\mathbf{z}$ belong to R^G.*

Then $\mathcal{A}(Q, \mathbf{z}) \subset R$.

If, in addition, the set of cluster and coefficient variables for $\mathcal{A}(Q, \mathbf{z})$ is known to contain a generating set for R^G, then $R^G = \mathcal{A}(Q, \mathbf{z})$.

Example: base affine space.

Cluster structures in Grassmannians

The homogeneous coordinate ring of the Grassmannian

$$\mathrm{Gr}_{k,N} = \{\text{subspaces of dimension } k \text{ in } \mathbb{C}^N\},$$

with respect to its Plücker embedding, has a standard cluster structure, explicitly described by J. Scott [2003]. It can be obtained as an application of the Starfish Lemma.

Although this cluster algebra has been extensively studied, our understanding of it is still very limited for $k \geq 3$.

Cluster structures in classical rings of invariants

The homogeneous coordinate ring of $\mathrm{Gr}_{k,N}$ is isomorphic to the ring of polynomial SL_k-invariants of configurations of N vectors in a k-dimensional complex vector space.

We anticipate natural cluster algebra structures in arbitrary rings of SL_k-invariants of collections of vectors and linear forms.

We establish this for $k = 3$.

Tensors

Let $V \cong \mathbb{C}^k$. A *tensor* T of *type* (a, b) is a multilinear map

$$T : \underbrace{V^* \times \cdots \times V^*}_{a \text{ copies}} \times \underbrace{V \times \cdots \times V}_{b \text{ copies}} \longrightarrow \mathbb{C}.$$

In coordinate notation, T is an $(a+b)$-dimensional array indexed by tuples of a "row indices" and b "column indices."

Kronecker tensor: the standard pairing $V^* \times V \to \mathbb{C}$.

Fix a volume form on V. This defines:

- the *volume tensor* of type $(0, k)$;
- the *dual volume tensor* of type $(k, 0)$.

Contraction of tensors with respect to a pair of arguments: a vector argument and a covector argument.

SL(V) **invariants**

The action of SL(V) on $(V^*)^a \times V^b$ defines the ring

$$R_{a,b}(V) = \mathbb{C}[(V^*)^a \times V^b]^{\mathsf{SL}(V)}$$

of SL(V)-invariant polynomial functions of a covariant and b contravariant arguments.

First Fundamental Theorem of Invariant Theory

Theorem 6 (H. Weyl, 1930s) *The ring $R_{a,b}(V)$ is generated by the following* SL(V)*-invariant multilinear polynomials (tensors):*

- *the <u>Plücker coordinates</u> (volumes of k-tuples of vectors);*
- *the <u>dual Plücker coordinates</u> (volumes of k-tuples of covectors);*
- *the <u>pairings</u> of vectors with covectors.*

Signatures

We distinguish between incarnations of $R_{a,b}(V)$ that use different orderings of the contravariant and covariant arguments.

A *signature* is a binary word encoding such an ordering:

covector arguments	$\circ$
vector arguments	$\bullet$

$$R_\sigma(V) \overset{\text{def}}{=} \{\mathsf{SL}(V) \text{ invariants of signature } \sigma\}$$

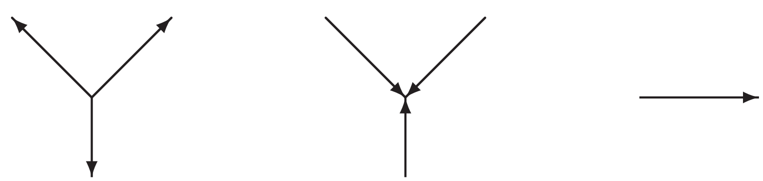
$$R_{\circ\bullet\bullet}(V) \cong R_{\bullet\circ\bullet}(V) \cong R_{\bullet\bullet\circ}(V) \cong R_{1,2}(V)$$

(signatures of type $(1,2)$)

Tensor diagrams

From now on: $k = 3$, $V \cong \mathbb{C}^3$.

Tensor diagrams are built using three types of building blocks which correspond to the three families of Weyl's generators:

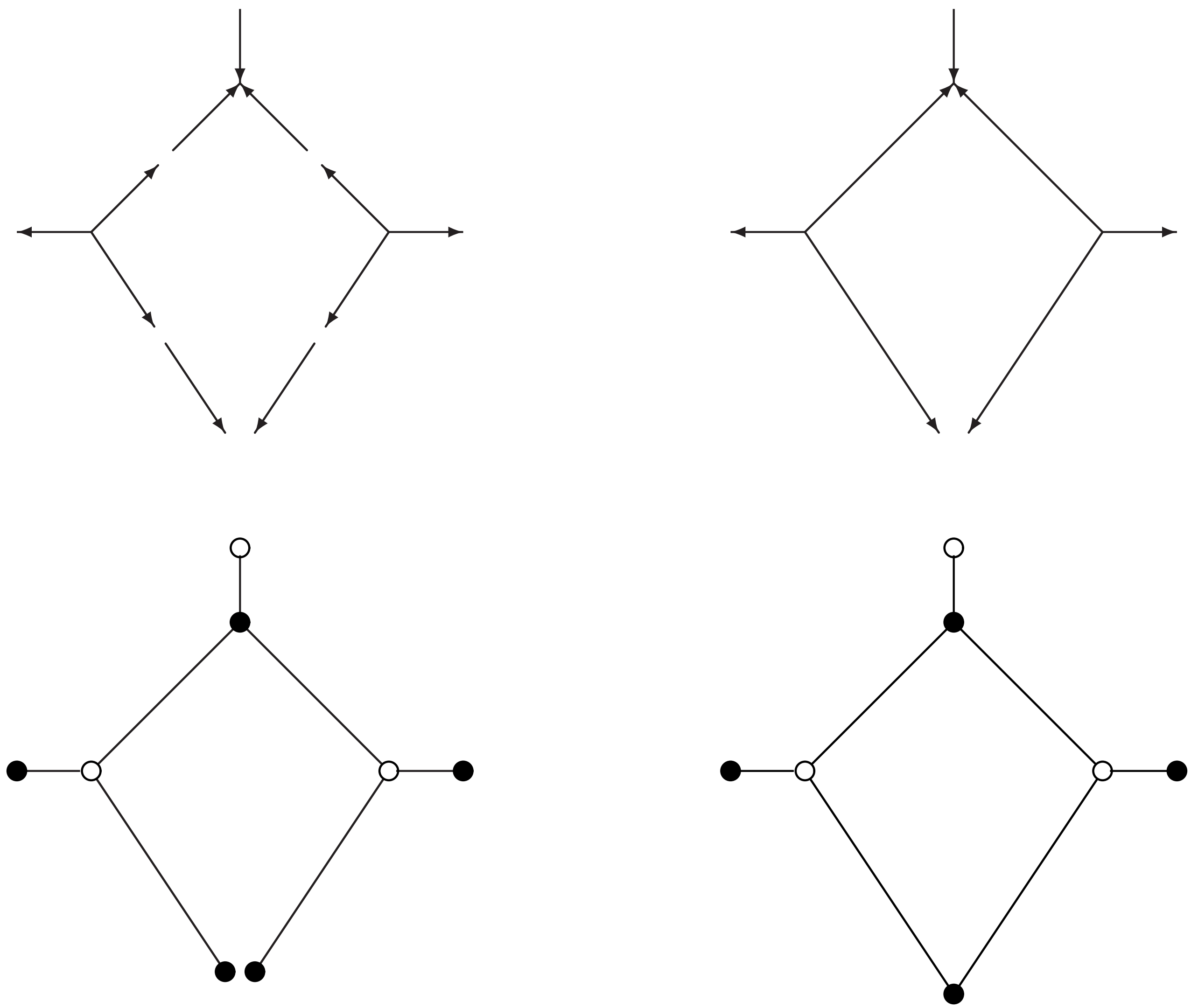


At trivalent vertices, a cyclic ordering must be specified.

Operations on invariants and tensor diagrams

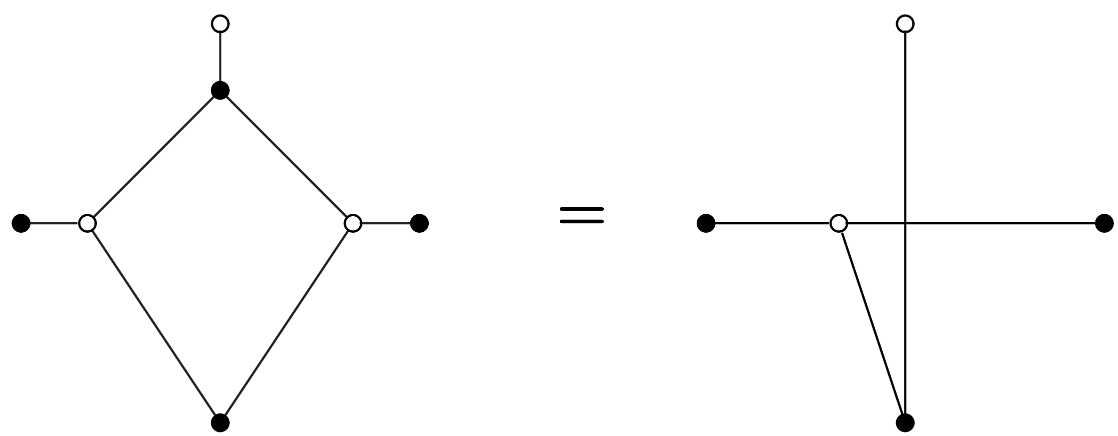
invariants	**tensor diagrams**
addition	formal sum
multiplication	superposition
contraction	plugging in
restitution	clasping of endpoints
polarization	unclasping

Assembling a tensor diagram

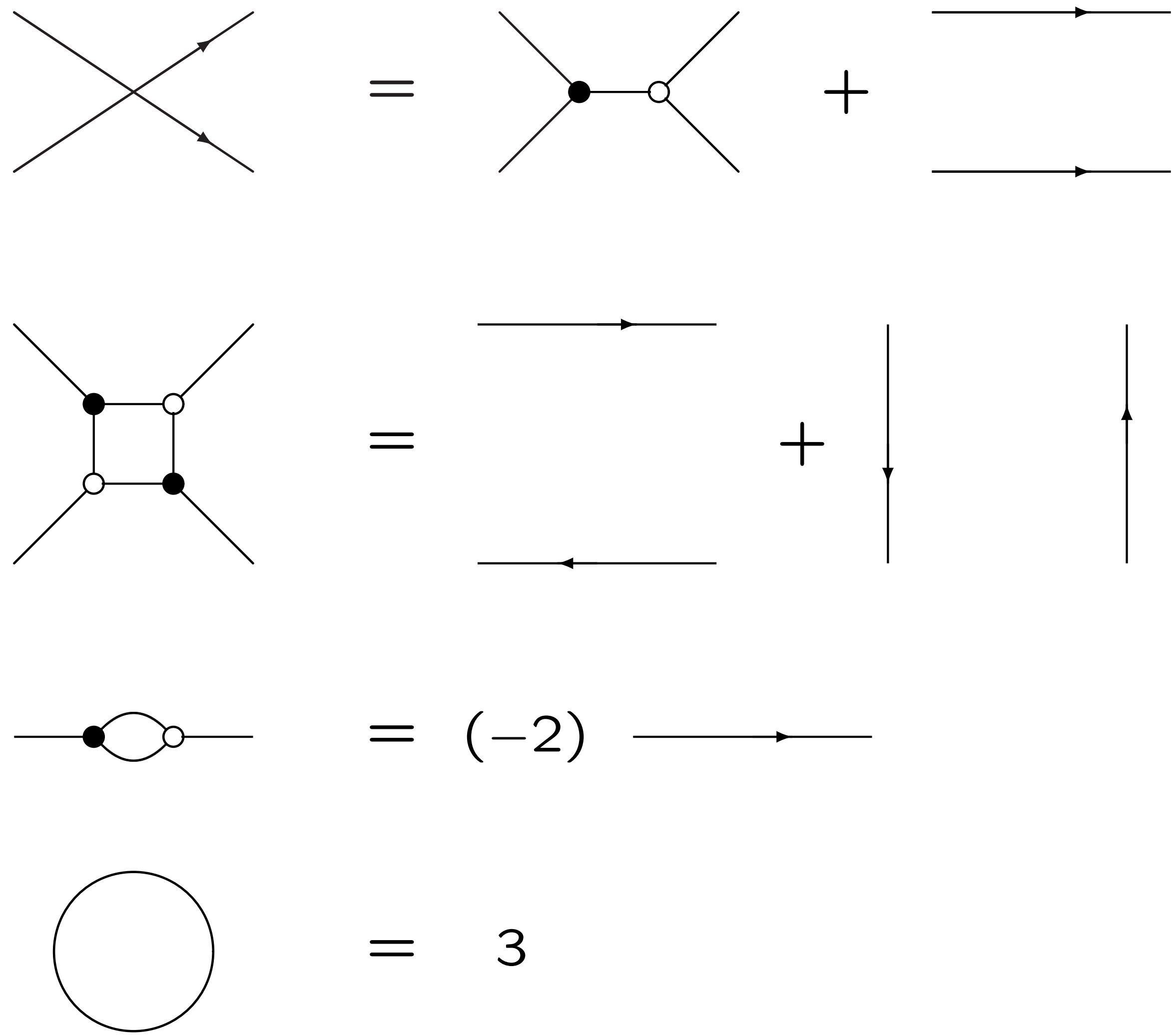


Tensor diagram D of signature $[\bullet \bullet \bullet \circ]$ of type $(1,3)$
representing an invariant $[D]$ of multidegree $(1,2,1,1)$

Different tensor diagrams may define the same invariant



Skein relations



+ two relations involving a vertex on the boundary

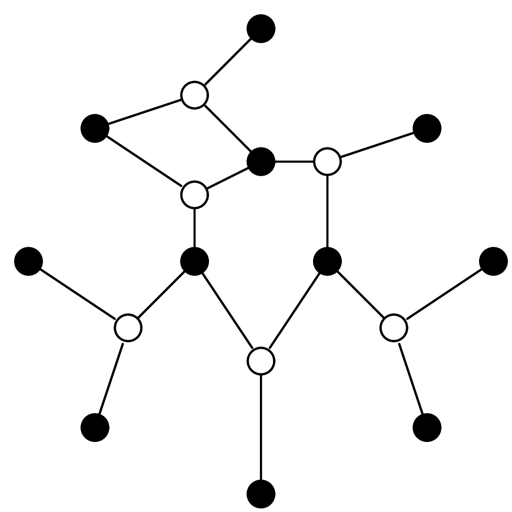
Webs

(after G. Kuperberg [1996])

Planar tensor diagrams are called *webs*.

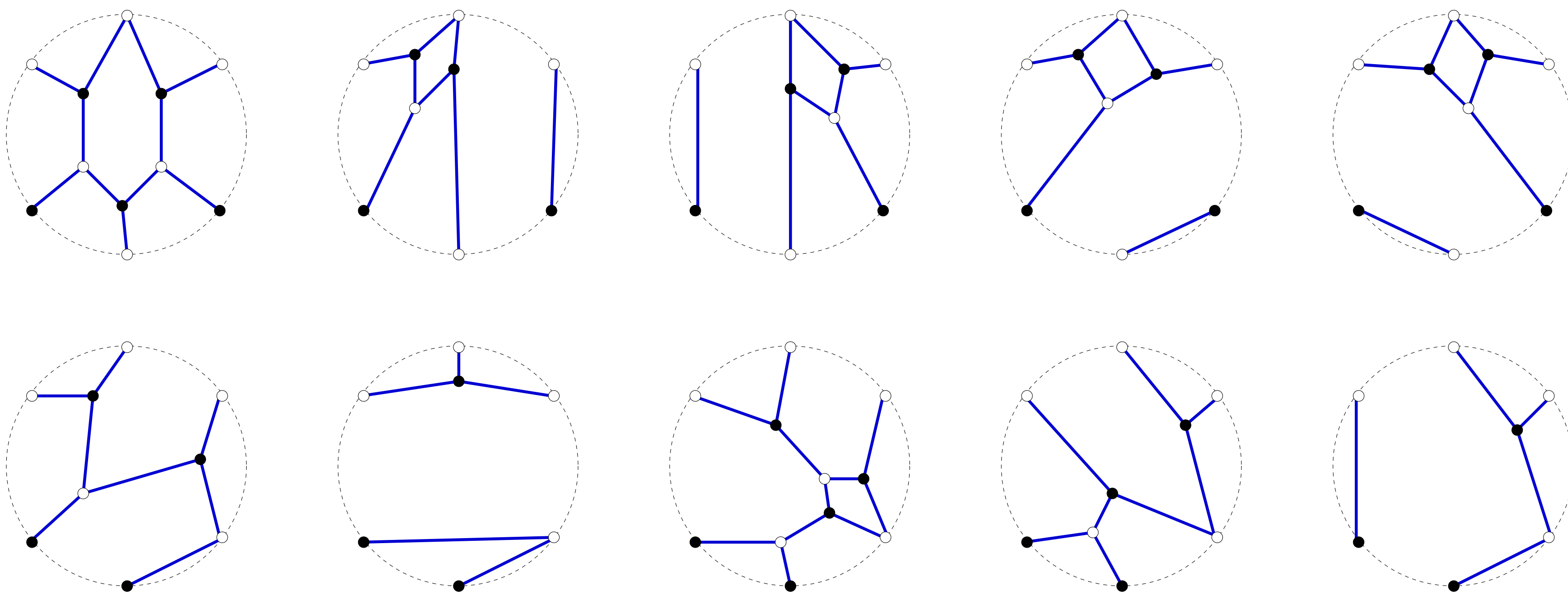
More precisely, a web of signature σ is a planar tensor diagram drawn inside a convex $(a+b)$-gon whose vertices have been colored according to σ. The cyclic ordering at each vertex is clockwise.

An invariant $[D]$ associated with a web D with no multiple edges and no internal 4-cycles is called a *web invariant*.



The web basis

Theorem 7 (G. Kuperberg) *Web invariants of signature σ form a linear basis in the ring of invariants $R_\sigma(V)$.*



Towards a cluster structure in $R_\sigma(V)$

Fix a *non-alternating* signature σ of type (a, b) with $a + b \geq 6$.

Goal: construct a cluster algebra structure in $R_\sigma(V)$.

Idea: describe a family of "special" seeds defining such a structure.

Step 1: Describe cluster variables appearing in these seeds.

Step 2: Explain how they group into clusters.

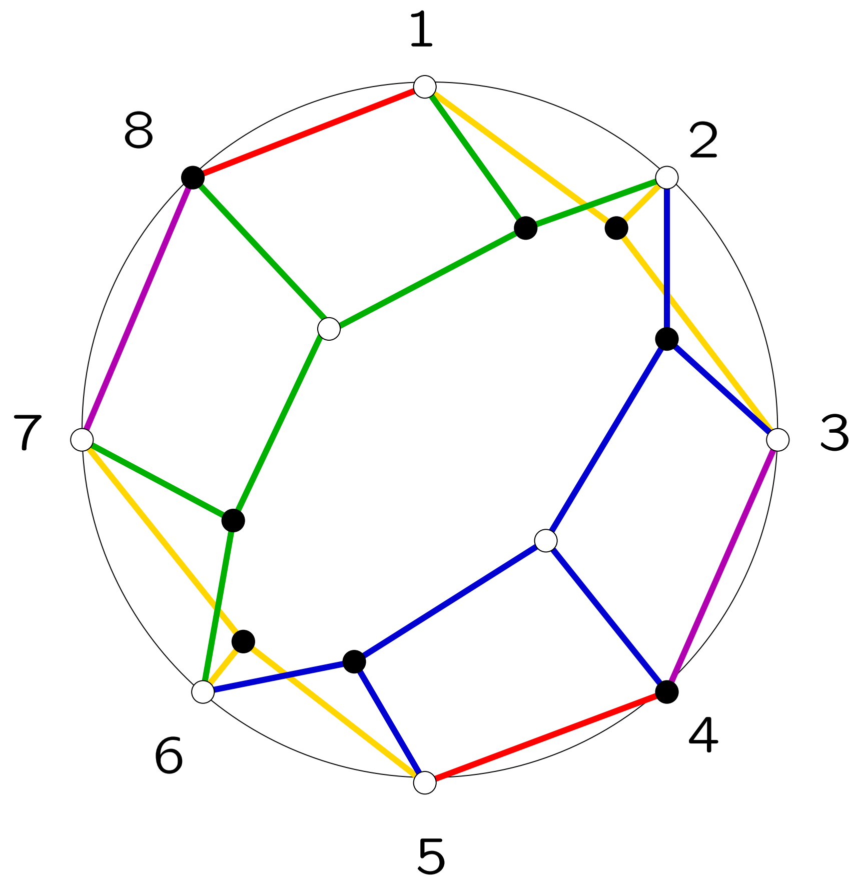
Step 3: Define the associated quivers.

Step 4: Verify the conditions of the Starfish Lemma.

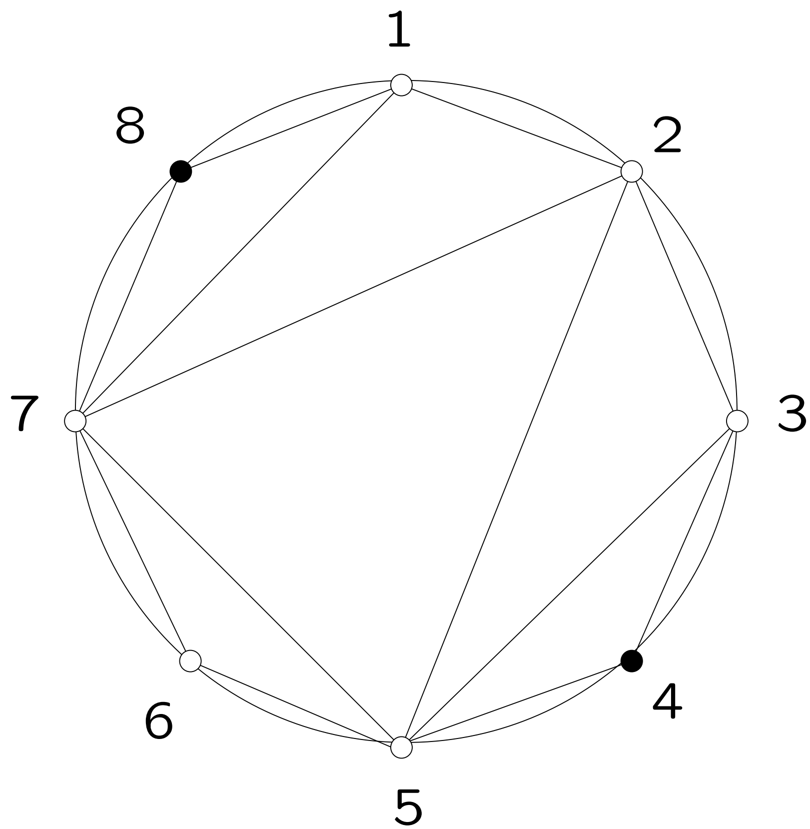
Step 5: Check that all special seeds are mutation equivalent.

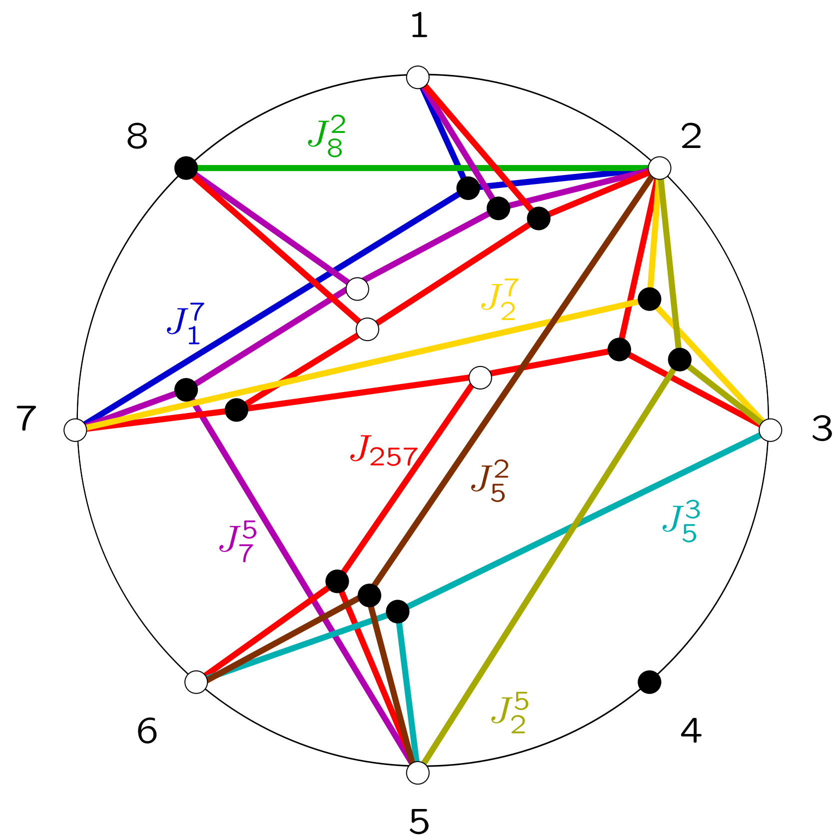
Step 6: Check that all Weyl generators appear.

Coefficient variables



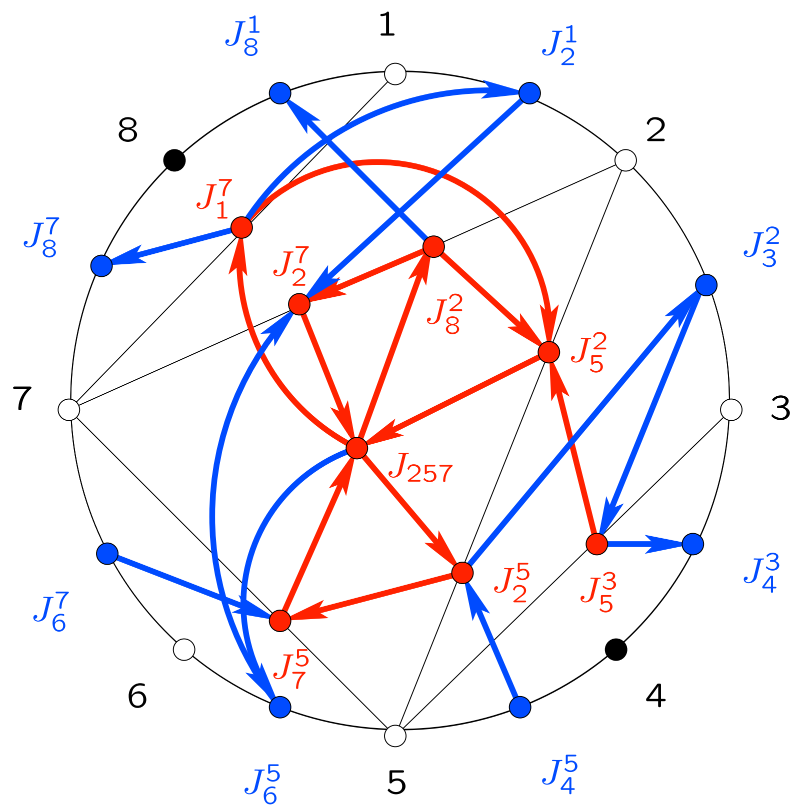
Special seed associated to a triangulation





All cluster and coefficient variables appearing in these special seeds are web invariants.

Quiver associated with a triangulation



Main theorem

Theorem 8 *Our construction defines a cluster structure on the ring of invariants* $R_\sigma(V)$*. This cluster structure does not depend on the choice of a triangulation* T*.*

Each seed in $R_\sigma(V)$ has $2(a+b)-8$ cluster variables and $a+b$ coefficient variaables.

Cluster types of $R_\sigma(V)$

$a+b=6$

a		
$a=0$	● ● ● ● ● ●	D_4
$a=1$	● ● ● ● ● ○	A_4
$a=2$	● ● ● ● ○ ○	A_4
	● ● ● ○ ● ○	A_4
	● ● ○ ● ● ○	$A_2 \sqcup A_2$
$a=3$	● ● ● ○ ○ ○	D_4
	● ● ○ ● ○ ○	$A_3 \sqcup A_1$

$a+b=7$

a		
$a=0$	● ● ● ● ● ● ●	E_6
$a=1$	● ● ● ● ● ● ○	E_6
$a=2$	● ● ● ● ● ○ ○	D_6
	● ● ● ● ○ ● ○	$D_5^{(1)}$
	● ● ● ○ ● ● ○	D_6
$a=3$	● ● ● ● ○ ○ ○	E_6
	● ● ● ○ ○ ● ○	E_6
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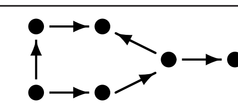
	● ● ○ ● ○ ● ○	

Cluster types of $R_\sigma(V)$, $a+b=8$

$a=0$

● ● ● ● ● ● ● ●	E_8

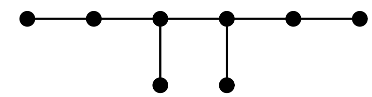
$a=1$

● ● ● ● ● ● ● ○	$E_7^{(1)}$

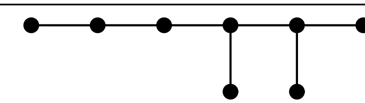
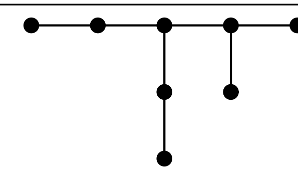
$a=2$

● ● ● ● ● ● ○ ○	E_8
● ● ● ● ● ○ ● ○	T_{433}
● ● ● ● ○ ● ● ○	T_{433}
● ● ● ○ ● ● ● ○	E_8

$a=3$

● ● ● ● ● ○ ○ ○	E_8
● ● ● ● ○ ○ ● ○	T_{433}
● ● ● ○ ○ ● ● ○	E_8
● ● ● ○ ● ○ ● ○	T_{433}
● ● ○ ● ● ○ ● ○	

$a=4$

● ● ● ● ○ ○ ○ ○	$E_7^{(1)}$
● ● ● ○ ○ ○ ● ○	$E_7^{(1)}$
● ● ● ○ ○ ● ○ ○	$E_7^{(1)}$
● ● ○ ○ ● ● ○ ○	D_8
● ● ○ ○ ● ○ ● ○	
● ● ○ ● ○ ○ ● ○	

Functoriality

Let σ and σ' be two non-alternating signatures related in one of the two ways shown below:

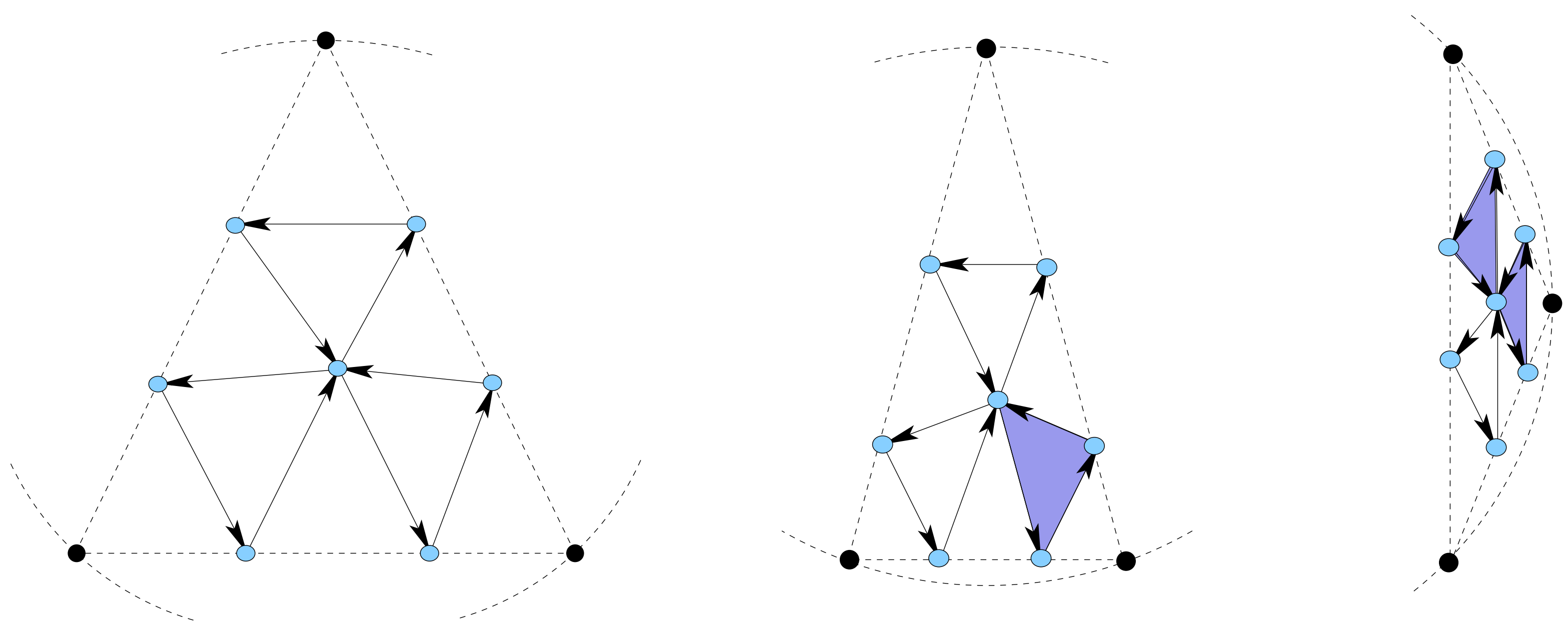
$$\begin{array}{c}\sigma = \cdots \bullet \bullet \circ \bullet {\color{red}\bullet} \circ \bullet \circ \circ \cdots \\ \downarrow \\ \sigma' = \cdots \bullet \bullet \circ \bullet \quad \circ \bullet \circ \circ \cdots \end{array} \qquad \Bigg\Vert \qquad \begin{array}{c}\sigma = \cdots \bullet \bullet \circ \bullet {\color{red}\bullet} \, {\color{red}\bullet} \circ \bullet \circ \circ \cdots \\ \downarrow \\ \sigma' = \cdots \bullet \bullet \circ \bullet \; {\color{red}\circ} \; \circ \bullet \circ \circ \cdots \end{array}$$

Then $R_{\sigma'}(V)$ is naturally identified with a subring of $R_\sigma(V)$:

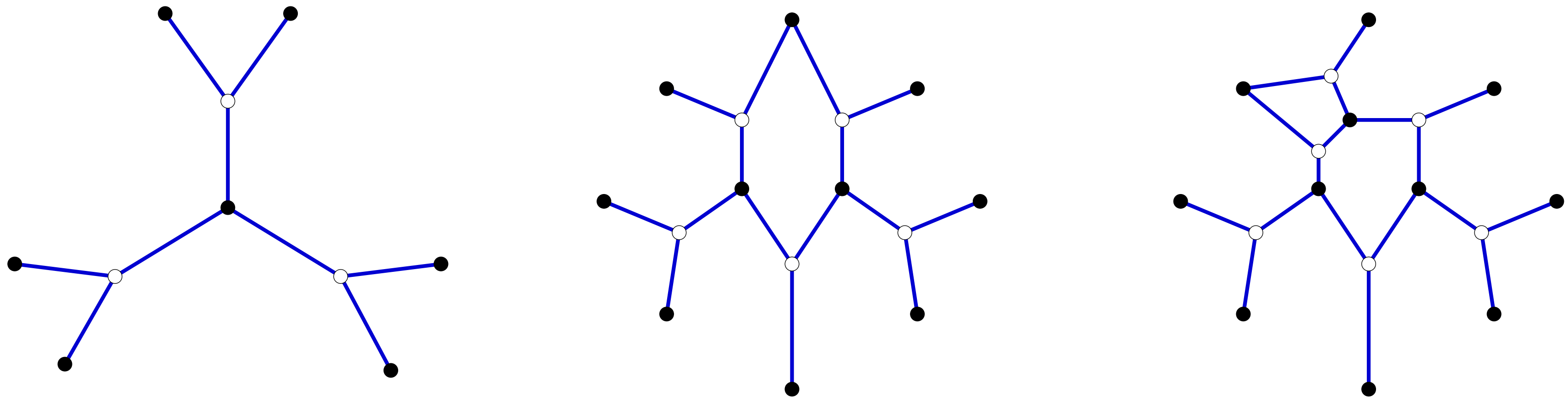
Theorem 9 *$R_{\sigma'}(V)$ is a cluster subalgebra of $R_\sigma(V)$.*

Grassmannians, revisited

Theorem 10 *The canonical isomorphism between* $R_{0,N}(V)$ *and the homogeneous coordinate ring of the Grassmannian* $\mathrm{Gr}_{3,N}$ *identifies the cluster algebra structure described above with the standard cluster structure in the Grassmannian.*



Grassmannians $\mathrm{Gr}_{3,N}$ of finite cluster type



Non-Plücker cluster variables in $R_{0,N}(V)$, *for* $N \in \{6, 7, 8\}$.

Main conjectures

Conjecture 11 *All cluster variables are web invariants.*

Conjecture 12 *Cluster variables lie in the same cluster if and only if their product is a web invariant.*

Conjecture 13 *Given a finite collection of distinct web invariants, if the product of any two of them is a web invariant, then so is the product of all of them.*

Cluster monomials

Given a cluster algebra, a *cluster monomial* is a monomial in the elements of any extended cluster.

Theorem 14 [G. Cerulli Irelli, B. Keller, D. Labardini-Fragoso, and P.-G. Plamondon, 2012]. *For cluster algebras defined by quivers, cluster monomials are linearly independent.*

Tantalizing problem: construct an additive basis containing all cluster monomials. Solutions are only known in special cases:
- acyclic quivers [H. Nakajima *et al.*];
- surface quivers [G. Musiker, R. Schiffler, L. Williams];
- rank 2 quivers [A. Zelevinsky *et al.*].

Conjecture 15 *In the cluster algebra* $R_\sigma(V)$*, Kuperberg's web basis contains all cluster monomials.*

Strong positivity conjecture

Conjecture 16 *Any cluster algebra has a basis that includes all cluster monomials and has nonnegative structure constants.*

Conjecture 16 implies Laurent positivity.

Conjecture 16 suggests the existence of a *monoidal categorification* [B. Leclerc–D. Hernandez, H. Nakajima, Y. Kimura–F. Qin].

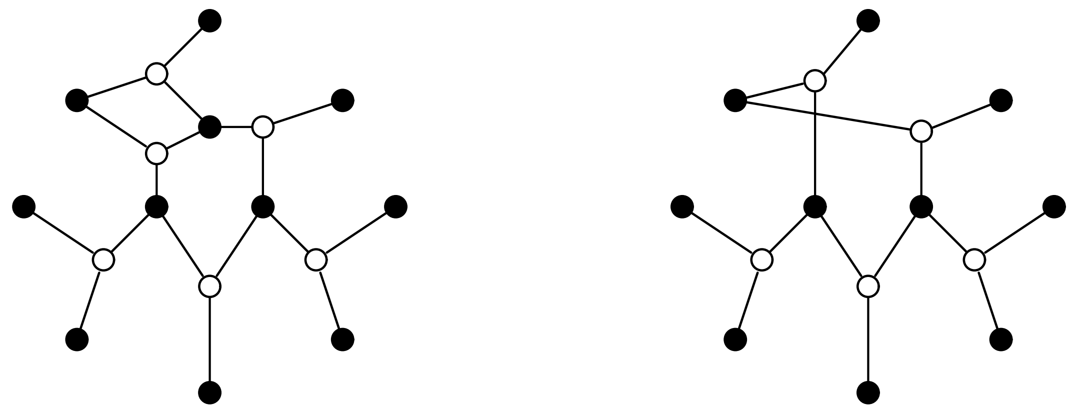
For some choices of σ, some structure constants for the web basis are negative. ☹

M. Khovanov and G. Kuperberg [1999]: the web basis is generally different from G. Lusztig's *dual canonical basis*.

It may however coincide with the *dual semicanonical basis*. ☺

Which web invariants are cluster variables?

Conjecture 17 *A web invariant is a cluster or coefficient variable if and only if it can be given by a* <u>*tree*</u> *tensor diagram.*



Theorem 18 *If a tensor diagram* D *is a planar tree, then* $[D]$ *is a cluster or coefficient variable in* $R_\sigma(V)$*.*

Our *arborization algorithm* conjecturally determines whether a given web invariant can be given by a tree (resp., forest).

Arborization algorithm