Definability in monoidal additive categories

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1 Definable subcategories of module categories

1.1 Functor categories and free abelian categories

• R is a multisorted ring = a (skeletally) small preadditive category = a ring with many objects
• Mod-R is the category of right R-modules, mod-R the subcategory of finitely presented right R-modules, R-Mod the category of left R-modules.
• Everything is additive and "subcategory" always means full subcategory.
• $(\text{mod-}R, \mathbf{Ab})^{\text{fp}}$ denotes the category of finitely presented abelian-group-valued functors on finitely presented R-modules. A typical object of this category has projective presentation of the form $(B,-) \xrightarrow{(f,-)} (A,-) \to F \to 0$ where $f : A \to B$ is a morphism in mod-R.

The **free abelian category** on R, $i : R \to \text{Ab}(R)$, is defined by the universal property that, for every (additive) functor $f : R \to \mathcal{B}$ with $\mathcal{B}$ abelian, there is an essentially unique *exact* functor $g : \text{Ab}(R) \to \mathcal{B}$ with $f = gj$.

$$\begin{tikzcd} R \arrow[r, "j"] \arrow[dr, "f"'] & \text{Ab}(R) \arrow[d, dotted, "g"] \\ & \mathcal{B} \end{tikzcd}$$

(We could write M in place of f and then could denote g by $M^{\text{eq}+}$.)

The free abelian category on $R^{(\text{op})}$ exists, indeed:

$$\text{Ab}(R^{\text{op}}) \simeq (\text{mod-}R, \mathbf{Ab})^{\text{fp}} \simeq \mathbb{L}_R^{\text{eq}+}$$

where $\mathbb{L}_R^{\text{eq}+}$ is the category of pp-imaginaries for right R-modules - its objects are the pp-pairs and its morphisms are the pp-definable maps between them.

Note that this category acts on Mod-R. If we think of this category concretely as $(\text{mod-}R, \mathbf{Ab})^{\text{fp}}$ and if $F \in \text{mod-}R$, then we denote by $\overrightarrow{F}$ the $\varinjlim$-commuting extension of F to all of Mod-R.

A subcategory $\mathcal{D}$ of Mod-R is **definable** if it is additive and axiomatisable, equivalently if it is closed under direct products, directed colimits, and pure submodules.

Let $\mathcal{S}_{\mathcal{D}} = \{F \in (\text{mod-}R, \mathbf{Ab})^{\text{fp}} : \overrightarrow{F}D = 0 \ \forall D \in \mathcal{D}\}$ be the annihilator of $\mathcal{D}$ in $(\text{mod-}R, \mathbf{Ab})^{\text{fp}}$. Then $\mathcal{S}_{\mathcal{D}}$ is a Serre subcategory of $(\text{mod-}R, \mathbf{Ab})^{\text{fp}}$ and so we have the localisation (an exact functor) $(\text{mod-}R, \mathbf{Ab})^{\text{fp}} \to (\text{mod-}R, \mathbf{Ab})^{\text{fp}}/\mathcal{S}_{\mathcal{D}}$. We set $\text{fun}(\mathcal{D}) = (\text{mod-}R, \mathbf{Ab})^{\text{fp}}/\mathcal{S}_{\mathcal{D}}$. It is a skeletally small abelian category and it acts faithfully on $\mathcal{D}$.

1.2 $\mathbb{ABEX}$ and $\mathbb{DEF}$

A **definable** additive category is one equivalent to a definable subcategory of a module category, equivalently to a definable subcategory of an additive finitely accessible category with products.

Purity is intrinsic in a definable category since a sequence is pure-exact iff some ultrapower of it is split (and ultraproducts are directed colimits of direct products, so definable categories have these).

Also the model theory of a definable category $\mathcal{D}$ is intrinsic since $\text{fun}(\mathcal{D})$ may be obtained as the category $(\mathcal{D}, \mathbf{Ab})^{\prod \to}$ of functors on $\mathcal{D}$ which commute with direct products and directed colimits. In the other direction, $\mathcal{D}$ is the category $\text{Ex}(\text{fun}(\mathcal{D}), \mathbf{Ab})$ of exact functors from $\text{fun}(\mathcal{D})$ to $\mathbf{Ab}$.

Indeed:

Theorem 1.1. *There is an anti-equivalence of 2-categories:*

$$\mathbb{DEF} \simeq^{\text{op}} \mathbb{ABEX}.$$

where $\mathbb{DEF}$ is the 2-category with objects the definable additive categories, morphisms the functors which commute with direct products and directed colimits (= the interpretation functors) and 2-arrows the natural transformations,
and where $\mathbb{ABEX}$ is the 2-category with objects the skeletally small abelian categories, arrows the exact functors and 2-arrows the natural transformations.

On objects the anti-equivalence is given by:

$$\mathcal{D} \mapsto (\mathcal{D}, \mathbf{Ab})^{\prod \to}.$$

$$\mathcal{A} \mapsto \text{Ex}(\mathcal{A}, \mathbf{Ab})$$

References for this

• P. Freyd, Representations in abelian categories, pp. 95-120 *in* Proceedings of the Conference on Categorical Algebra, La Jolla, 1965, Springer-Verlag, 1966. (free abelian categories)
• M. Adelman, Abelian categories over additive ones, J. Pure Appl. Algebra, 3(2) (1973), 103-117. (free abelian categories)
also e.g., • S. Posur, On free abelian categories for theorem proving, J. Pure Appl. Algebra, 226(7) (2022), Paper No. 106994
• I. Herzog, Elementary duality of modules, Trans. Amer. Math. Soc., 340(1) (1993), 37-69. ($\mathbb{L}_R^{\text{eq}+}$)
• H. Krause, Exactly definable categories, J. Algebra, 201(2) (1998), 456-492. (coherent functors)
• M. Prest, Definable additive categories: purity and model theory, Mem. Amer. Math. Soc., 210(987) (2011). (coherent functors)
• M. Prest and R. Rajani, Structure sheaves of definable additive categories, J. Pure Appl. Algebra 214(8) (2010), 1370–1383 2-equivalence. (the 2-category equivalence)

• M. Ziegler, Model theory of modules, Ann. Pure Appl. Logic, 26(2) (1984), 149-213. (the spectrum behind definable subcategories)
also
• M. Makkai, A theorem on Barr-exact categories with an infinitary generalization, Ann. Pure Appl. Logic, 47 (1990), 225-268. (some of this, set in a very general, non-additive, context)
• A. Kuber and J. Rosický, Definable categories, J. Pure Appl. Algebra, 222(5) (2018), 1006-1025. (extension of the equivalence to non-additive locally finitely presentable categories)
• S. Lack and G. Tendas, Enriched regular theories, J. Pure Appl. Algebra 224(6), Paper No. 106268. (further extension to enriched categories)
• G. Garkusha and D. Jones, Derived categories for Grothendieck categories of enriched functors, pp.23-45 *in* Model Theory of Modules, Algebra and Categories, Contemporary Mathematics, Vol. 730, Amer. Math. Soc., 2019. (enriched functor categories)

1.3 Monoidal abelian categories

What if there is a monoidal = tensor product structure on one of the categories? How is this reflected in the other categories?

By a monoidal structure on a category $\mathcal{C}$ we mean (roughly) a bifunctor $\otimes$ which we will assume to be symmetric (so $A \otimes B \simeq B \otimes A$) and we will assume that there is a tensor-unit $\mathbb{1}$ (so $A \otimes \mathbb{1} \simeq A$).

Day convolution: if there is a tensor product on the small preadditive category R then this induces a tensor product on the module category R-mod (define $(A, -)\otimes(B, -)$ to be $(A \otimes B, -)$ and use right exactness of $\otimes$ to extend to finitely presented modules) and that extends to R-Mod (since every module is a direct limit of finitely presented modules).

There are various monoidal analogues to the free abelian construction which start with a skeletally small preadditive category R equipped with a monoidal structure. See, for instance:
• L. Barbieri-Viale, A, Huber, M. Prest, Tensor structure for Nori motives, Pacific J. Math. 306(1) (2020), 1–30.
• L. Barbieri-Viale and B. Kahn, A universal rigid abelian tensor category, arXiv:2111.11217
• B. Kahn, Universal rigid abelian tensor categories and Schur finiteness, arXiv:2203.03572
and references therein.

1.4 Definable subcategories of a monoidal category

Suppose that $\mathcal{C}$ is an additive finitely accessible category with products. Since every object of $\mathcal{C}$ is a directed colimit of a diagram in $\mathcal{C}^{\mathrm{fp}}$, we can regard every object of $\mathcal{C}$ as a right $\mathcal{C}^{\mathrm{fp}}$-module (and $\mathcal{C}$ is a definable subcategory of the category of right $\mathcal{C}^{\mathrm{fp}}$-modules).

Suppose further that there is a closed symmetric monoidal structure $\otimes$ on $\mathcal{C}$ such that $\mathcal{C}^{\mathrm{fp}}$ is closed under $\otimes$. "Closed" means that there is a right adjoint $[-, -]$ **internal hom** for $\otimes$: so for $X, Y, Z \in \mathcal{C}$ we have $(A \otimes B, C) \simeq (A, [B, C])$.

For instance $\mathcal{C} = \text{Mod-}R$ with R commutative, equipped with the usual $\otimes$ for modules over R, is closed, with $[-, -]$ being just the usual hom bifunctor.

The monoidal structure on $\mathcal{C}^{\mathrm{fp}}$ induces a monoidal structure on $(\mathcal{C}^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}$ by Day convolution.

The results in this section come from:

• R. Wagstaffe, A monoidal analogue of the 2-category anti-equivalence between $\mathbb{ABEX}$ and $\mathbb{DEF}$, J. Pure Appl. Algebra, to appear, arXiv:2010:12029

Say that a definable subcategory $\mathcal{D}$ of $\mathcal{C}$ is **fp-hom-closed** if, for every $A \in \mathcal{C}^{\mathrm{fp}}$ and $X \in \mathcal{D}$, we have $[A, X] \in \mathcal{D}$ (that is, $\mathcal{D}$ is closed under internal-hom sorts).

Theorem 1.2. *Suppose that $\mathcal{C}$ is an additive finitely accessible category with products and that there is a closed symmetric monoidal structure $\otimes$ on $\mathcal{C}$ such that $\mathcal{C}^{\mathrm{fp}}$ is closed under $\otimes$. Suppose that $\mathcal{D}$ is a definable subcategory of $\mathcal{C}$ and let $\mathcal{S}_{\mathcal{D}}$ be its annihilator in $(\mathcal{C}^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}$. Then $\mathcal{S}$ is a tensor-ideal in $(\mathcal{C}^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}$ iff $\mathcal{D}$ is fp-hom-closed.*

It then follows that $\mathrm{fun}(\mathcal{D}) = (\mathcal{C}^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}/\mathcal{S}_{\mathcal{D}}$ has an induced monoidal structure.

Indeed there are natural bijections between
- the fp-hom-closed definable subcategories of $\mathcal{C}$,
- the Serre tensor-ideals of $\mathcal{C}^{\mathrm{fp}}$-mod and
- the closed subsets of a corresponding coarser version of the Ziegler spectrum (whose closed subsets are the intersections of such definable subcategories with the set of isomorphism types of indecomposable pure-injectives in $\mathcal{C}$).

For the analogue of Theorem 1.1 we require more, namely that the tensor product on the functor category be exact.

Define the 2-category $\mathbb{ABEX}^{\otimes}$ to have objects the skeletally small abelian categories equipped with a symmetric monoidal structure which is exact in each variable, morphisms the exact monoidal functors and 2-arrows the natural transformations.

The exactness requirement on $\otimes$ is needed to ensure that $\mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ is fp-hom-closed in $(\mathcal{A}, \mathbf{Ab})$.

Reflecting the exactness of $\otimes$ on the functor category, we need the following further condition on $\mathcal{D}$. A definable subcategory $\mathcal{D}$ of an additive finitely accessible category with products $\mathcal{C}$ satisfies the **exactness condition** if, given morphisms $f : A \to B$ and $g : C \to D$ in $\mathcal{C}^{\mathrm{fp}}$ and a morphism $h : A \otimes C \to X \in \mathcal{D}$, if $(f \otimes C)|h$ and $(A \otimes g)|h$, then $(f \otimes g)|h$. Here $f|g$ means that there is some h such that $g = hf$. In the alternative notation $f \geq g$, the condition becomes $(f \otimes C) \geq h$ and $(A \otimes g) \geq h$ implies $(f \otimes g) \geq h$, that is, $f \otimes g$ is the meet of $f \otimes C$ and $A \otimes g$, at least with respect to $\mathcal{D}$.

$$\begin{tikzcd} C \arrow[r, "g"] & D \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd}
\quad
\begin{tikzcd} A \otimes C \arrow[r, "A\otimes g"] \arrow[d, "f\otimes C"'] \arrow[dr, "f\otimes g"] & A \otimes D \arrow[d, "f\otimes D"] \\ B \otimes C \arrow[r, "B\otimes g"'] & B \otimes D \end{tikzcd}
\quad
\begin{tikzcd} A \otimes C \arrow[r, "A\otimes g"] \arrow[d, "f\otimes C"'] \arrow[drr, "h"] & A \otimes D \arrow[d, "f\otimes D"] \arrow[dr, bend left] & \\ B \otimes C \arrow[r, "B\otimes g"'] \arrow[rr, bend right] & B \otimes D & X \end{tikzcd}
\quad
\begin{tikzcd} A \otimes C \arrow[r, "A\otimes g"] \arrow[d, "f\otimes C"'] \arrow[dr, "f\otimes g"] \arrow[drr, "h"'] & A \otimes D \arrow[d, "f\otimes D"] & \\ B \otimes C \arrow[r, "B\otimes g"'] & B \otimes D \arrow[r, dotted] & X \end{tikzcd}$$

The exactness condition on $\mathcal{D}$ ensures that the induced monoidal structure on $\mathrm{fun}(\mathcal{D})$ is exact.

Define $\mathbb{DEF}^{\otimes}$ to be the 2-category with objects the triples $(\mathcal{D}, \mathcal{C}, \otimes)$ where $\mathcal{C}$ is an additive finitely accessible category with products, $\otimes$ is a symmetric closed monoidal structure on $\mathcal{C}$ such that $\mathcal{C}^{\mathrm{fp}}$ is a monoidal subcategory, and $\mathcal{D}$ is an fp-hom-closed definable subcategory which satisfies the exactness condition. The morphisms from $(\mathcal{D}, \mathcal{C}, \otimes)$ to $(\mathcal{E}, \mathcal{C}', \otimes')$ are the functors $I : \mathcal{D} \to \mathcal{E}$ which commute with direct products and directed colimits and which are such that the induced exact morphism $I_0 : \mathrm{fun}(\mathcal{E}) \to \mathrm{fun}(\mathcal{D})$ is monoidal. The 2-arrows are, as always, just the natural transformations.

Theorem 1.3. *There is an anti-equivalence of 2-categories:*

$$\mathbb{DEF}^{\otimes} \simeq^{\mathrm{op}} \mathbb{ABEX}^{\otimes}.$$

A couple more results illustrate that these seemingly technical conditions are actually natural. If $\mathcal{K}$ is a closed symmetric monoidal category with tensor-unit $\mathbb{1}$, then $\mathcal{K}$ is **rigid** if, for every $A, B \in \mathcal{K}$, the natural map $A^\vee \otimes B \to [A, B]$ is an isomorphism, where $A^\vee = [A, \mathbb{1}]$ is the **dual** of A.

Theorem 1.4. *Suppose that $\mathcal{C}$ is a finitely accessible category with products equipped with a closed symmetric monoidal structure such that $\mathcal{C}^{\mathrm{fp}}$ is rigid. Suppose that $\mathcal{D}$ is a definable subcategory of $\mathcal{C}$. Then $\mathcal{S}_{\mathcal{D}}$ is a tensor-ideal of $\mathcal{C}^{\mathrm{fp}}$-*mod *iff $\mathcal{D}$ is a tensor-ideal of $\mathcal{C}$.*

Theorem 1.5. *Suppose that R is a small preadditive category with a symmetric rigid monoidal structure and induce monoidal structures on* Mod-R *and* R-Mod *by Day convolution. Then a definable subcategory $\mathcal{D}$ of* Mod-R *is fp-hom-closed in* Mod-R *iff its dual definable category $\mathcal{D}^{\mathrm{d}}$ is a tensor-ideal in* R-Mod.

2 ((Tensor-)Triangulated categories (briefly)

Suppose that $\mathcal{T}$ is a triangulated category with coproducts. An object $A \in \mathcal{T}$ is **compact** if $(A, -)$ commutes with direct sums. Let $\mathcal{T}^{\mathrm{c}}$ denote the subcategory of compact objects. Say that $\mathcal{T}$ is **compactly generated** if, for $X \in \mathcal{T}$, if $(A, X) = 0$ for every $A \in \mathcal{T}^{\mathrm{c}}$, then $X = 0$.

We have the restricted Yoneda functor $y : \mathcal{T} \to ((\mathcal{T}^{\mathrm{c}})^{\mathrm{op}}, \mathbf{Ab}) = \text{Mod-}\mathcal{T}^{\mathrm{c}}$ given on objects by $X \mapsto (-, X) \restriction \mathcal{T}^{\mathrm{c}}$. This functor does wonderful things, very analogous to the Gruson-Jensen functor Mod-$R \to (R\text{-mod}, \mathbf{Ab})$ given by $M \mapsto M \otimes -$. For instance it is an equivalence between the subcategory of pure-injective objects of $\mathcal{T}$ and the subcategory of injective objects of Mod-$\mathcal{T}^{\mathrm{c}}$. And the definable subcategory generated by the image of $\mathcal{T}$ is the subcategory of flat = absolutely pure $\mathcal{T}^{\mathrm{c}}$-modules. If we base the model theory of $\mathcal{T}$ on the idea that an element of $X \in \mathcal{T}$ of sort $A \in \mathcal{T}^{\mathrm{c}}$ is a morphism $A \to X$, then the model theory of $\mathcal{T}$ is identical, *via* the functor y, with the model theory of flat $\mathcal{T}^{\mathrm{c}}$-modules. Furthermore, the corresponding functor category fun$(\mathcal{T})$ is $\mathcal{T}^{\mathrm{c}}$-mod.

References:

for purity and definability

• H. Krause, Smashing subcatgories and the telescope conjecture - an algebraic approach, Invent. Math., 139(1) (2000), 99-133.

• H. Krause, Cohomological quotients and smashing localizations, Amer. J. Math., 127(6) (2005), 1191-1246.

• A. Beligiannis, Relative homological algebra and purity in triangulated categories, J. Algebra, 227(1) (2000), 268-361.

for model theory in compactly generated (tensor-)triangulated categories

• G. Garkusha and M. Prest, Triangulated categories and the Ziegler spectrum, Algebras and Representation Theory, 8(4) (2005), 499-523.

• M. Prest and R. Wagstaffe, Model theory in compactly generated (tensor-)triangulated categories, arXiv:XX

Krause established a fundamental correspondence:

Theorem 2.1. *Suppose that $\mathcal{T}$ is a compactly generated triangulated category. Then there are natural bijections between:*

the definable subcategories of $\mathcal{T}$;

the Serre subcategories of mod-$\mathcal{T}^{\mathrm{c}}$*;*

the cohomological = annihilator ideals of morphisms in $\mathcal{T}^{\mathrm{c}}$;

the closed subsets of the Ziegler spectrum $\mathrm{Zg}(\mathcal{T})$ *of $\mathcal{T}$*

In

• R. Wagstaffe, Definability in tensor triangulated categories, arXiv:XX
which is based on

• R. Wagstaffe, Definability in Monoidal Additive and Tensor Triangulated Categories, PhD Thesis, University of Manchester, 2021, *available at* `https://personalpages.manchester.ac.uk/staff/mike.prest/publications.html`

there are variants of Theorem 2.1 where conditions are imposed on the definable categories - being shift-closed, or triangulated, or, in the case that $\mathcal{T}$ has a suitable monoidal structure, being tensor-closed, or being a tensor-ideal.

In particular, suppose that $\mathcal{T}$ is a **tensor triangulated** category - triangulated and with a symmetric monoidal structure $(\otimes, \mathbb{1})$ such that $\otimes$ is triangulated in each variable. Suppose also that $\mathcal{T}$ is **rigidly compactly generated**, also referred to as a **big tt-category** - that is, $\mathcal{T}$ is compactly generated, $\mathcal{T}^{\mathrm{c}}$ is a rigid monoidal subcategory and the duality functor $(-)^{\vee} : (\mathcal{T}^{\mathrm{c}})^{\mathrm{op}} \to \mathcal{T}^{\mathrm{c}}$ is triangulated. Then there is the following restriction of the fundamental correspondence.

Theorem 2.2. *Suppose that $\mathcal{T}$ is a rigidly compactly generated tensor-triangulated category. Then there are natural bijections between:*

the definable tensor-closed subcategories of $\mathcal{T}$;

the Serre tensor-ideals of mod-$\mathcal{T}^{\mathrm{c}}$*;*

the tensor-closed cohomological = annihilator ideals of morphisms in $\mathcal{T}^{\mathrm{c}}$;

the closed subsets of the tensor Ziegler spectrum $\mathrm{Zg}^{\otimes}(\mathcal{T})$ *of $\mathcal{T}$*

Further, in this case we have an internal duality of pp-pairs and definable subcategories. In particular, given a definable subcategory $\mathcal{D}$ of $\mathcal{T}$, let $\mathcal{A} = \{A \xrightarrow{f} B \in \mathcal{T}^{\mathrm{c}} : \forall X \in \mathcal{D}\ Xf = 0\}$ denote the **annihilator** ideal of $\mathcal{D}$, where by $Xf = 0$ we mean that for every morphism $b : B \to X$ we have $bf = 0$. Set $\mathcal{A}^{\vee} = \{f^{\vee} : f \in \mathcal{A}\}$ and define $\mathcal{D}^{\vee} = \{X \in \mathcal{T} : \forall\ A \xrightarrow{g} B \in \mathcal{A}\ \ Xg = 0\}$ to be the annihilator in $\mathcal{T}$ of $\mathcal{A}^{\vee}$. Call this the **internal dual** of $\mathcal{D}$.

Theorem 2.3. *Suppose that $\mathcal{T}$ is a rigidly compactly generated tensor-triangulated category. Let $\mathcal{D}$ be a definable subcategory of $\mathcal{T}$. Then $\mathcal{D}^{\vee}$ is a definable subcategory and $(\mathcal{D}^{\vee})^{\vee} = \mathcal{D}$.*

External dualities of triangulated, in particular derived, categories are considered in

• G. Garkusha and M. Prest, Triangulated categories and the Ziegler spectrum, Algebras and Representation Theory, 8(4) (2005), 499-523.

and

• L. Angeleri Hügel and M. Hrbek, Parametrizing torsion pairs in derived categories, Represent. Theory, 25 (2021), 679-731.

A general approach to dualities of triangulated categories, with specific examples, is in

• I. Bird and J. Williamson, Duality pairs, phantom maps, and definability in triangulated categories, 2022, arXiv:2202.08113.