Reconstructing double categories from End-indexings

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Functor Categories, Model Theory, and Constructive Category Theory. Almería

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Plan for the talk

1. Review on double categories
2. Globularly generated double categories
3. Length of double categories
4. End-indexings

Bicategories and double categories

Review on double categories

Double categories

Definition A double category is a category internal to categories, functors and natural transformations. Models for second order categories/categorified categories: Same as a category, where the word class is replaced by the word category and the word function is replaced by the word functor. Same procedure which we define Lie groups, algebras, topological algebras, etc. Explicitly: A double category C has:

1. **Category of objects:** C_0
2. **Category of morphisms:** C_1
3. **Left and right frame:** $L, R : C_1 \to C_0$
4. **Horizontal identity functor:** $U : C_0 \to C_1$
5. **Composition functor:** $\odot : C_1 \times_{C_0} C_1 \to C_1$

1-5 satisfy the functorial versions of the usual identities defining a category. Ex. $\mathbb{M}\mathbf{od}$, $\mathbb{P}\mathbf{rof}$, $\mathbb{B}\mathbf{ord}_n$, $\mathbb{A}\mathbf{dj}$, $\rho^{\square}(X_2, X_1, X_0)$, etc. What do double categories look like?

The data of a double category

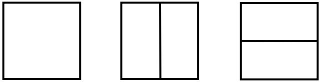
Let C be a double category. C_0 category of objects of C, C_1 category of squares of C. $\odot$ horizontal composition in C, compositions in C_0, C_1 vertical compositions in C, U horizontal identity in C.

1. **Objects of C_0**: Objects/vertices of C $\circ$, $a, b, c, x, y, x, \ldots$
2. **Morphisms of C_0**: Vertical morphisms/edges of C, $\downarrow$, $f, g, h, ..$
3. **Objects of C_1**: Horizontal morphisms/edges of C, $\circ \to \circ$, $\alpha, \beta, \gamma, M, N, L, \ldots$
4. **Morphisms of C_1**: Squares of C, $\varphi, \psi, \ldots$
5. **Vertical identites**: If a is an object of C, id_a in C_0 verical identity of a, $\downarrow$. Identify with a.
6. **Horizontal identities**: If a is an object in C, U_a horizontal identity of a, $a \to a$.

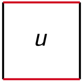
Read everything from top to bottom and from left to right. Forget arrow tips. Horizontal composition along left and right frame and vertical composition along vertical domain and codomain.

Notation

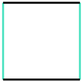
We draw squares and their horizontal, vertical composition in a double category C as:



Given a vertical edge f, the horizontal identity U_f



Globular squares are squares of the form:



Globular squares are closed under horizontal and vertical compositions. The collection of globular squares inherits from C the structure of a bicategory (collapse blue vert. arrows). HC horizontalization of C.

Globularly generated double categories

Globularly generated double categories

Decorated bicategories

Observation: $\mathbf{Mod} = H\mathbb{M}\mathbf{od}$, $\mathbf{Prof} = H\mathbb{P}\mathbf{rof}$, $\mathbf{Bord}_n = H\mathbb{B}\mathbf{ord}_n$, $\mathbf{Adj} = H\mathbb{A}\mathbf{dj}$, etc. 2-categories and bicategories are almost ubiquious well known objects of which we can say a lot, double categories not so much. Difference: Enrichment vs internalization, Shape of 2-cells: Globes vs squares.

A decorated bicategory is a pair (B^*, B) where B^* is a category and B is a bicategory such that the objects of B^* and B are the same. Represent a decorated bicategory as a bunch of diagrams of the form:

$$\begin{tikzcd}
\circ \arrow[d] & & \\
\circ & \circ \arrow[r, bend left] \arrow[r, bend right] & \circ
\end{tikzcd}$$

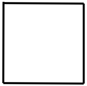
where the sets of vertices of the two types of diagrams are the same.

Example: Let C be a double category. The pair (C_0, HC) is a decorated bicategory. Write H^*C and call it the decorated horizontalization of C.

Internalizations

Problem: Given a decorated bicategory (B^*, B). Find double categories C such that $H^*C = (B^*, B)$. We call any such C an internalization of (B^*, B). Can we understand the internalizations of (B^*, B)?

Problem of existence of internalizations: Is the decorated horizontalization construction generic? We think of the above problem as a problem of coherently 'filling' 'hollow' squares of the form:



which we form with the 1-dimensional data provided to us by (B^*, B) in such a way that the 1-dimensional and the globular data we started with is fixed. Problems of filling squares with globular data classically appear in Kelly and Street's formulation of the mates correspondence [Kelly, Street '74], in the definition of the 2-dimensional homotopy groups and the proof of the higher HHSvK theorem [Brown, Spencer '76], and in providing symmetric monoidal structures on bicategories via Shulman's theory of framed bicategories [Shulman '08].

The globularly generated piece

Let C be a duble category. Write γC for the minimal sub-double category of C containing U_f for every vertical morphisms f of C and all globular squares of C.

Lemma (O 18')

Let C be a double category.

1. $H^* C = H^* \gamma C$.
2. *If D is a sub-double category of C satisfying the equation $H^* C = H^* D$ then γC is a sub-double category of D.*

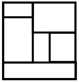
C is a solution to internalization for $H^* C$. (1) says that so is γC. (2) says that γC is the minimal solution in C. We call γC the globularly generated piece of C. **Question:** Can we understand these 'minimal' solutions outside of the context of the ambient double category C?

Globularly generated double categories

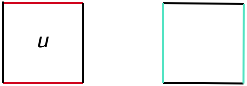
We say that a double category C is globularly generated if any of the following three equivalent conditions is satisfied:

1. $\gamma C = C$.
2. C is generated, as a double category, by its globular squares.
3. C contains no proper sub-double categories D such that $H^* C = H^* D$.

C is globularly generated if every square in C admits a subdivision, say as:



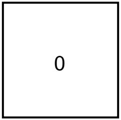
where every smaller square is of one of the two forms:



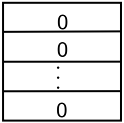
Length of double categories

Length

Let C be a globularly generated double category. We think of globular and horizontal identity squares as the simplest possible squares in C. Draw them as:



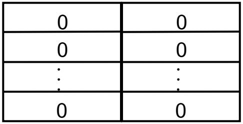
If a square φ in C is as above we say that φ is of length 0 and write $\ell\varphi = 0$. Consider the squares of C that can be subdivided as vertical composition of squares of length 0. Geometrically, squares of the form:



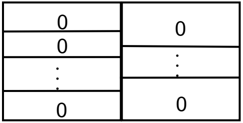
We say that a square φ in C that is as above and is not of length 0 is of length 1. In that case we write $\ell\varphi = 1$. We draw squares of length 1 as above, only marked with 1.

Length

Given two horizontally compatible squares φ, ψ of length 1, it might be the case the we can find compatible length 1 presentations of φ and ψ, i.e. the composition $\varphi \odot \psi$ can be made to look like:



In that case $\ell(\varphi \odot \psi) \leq 1$. This doesn't happen in general. $\varphi \odot \psi$ can look like:



for every length 1 subdivision of φ and ψ. In that case $\ell(\varphi \odot \psi) = 1 + 1/2$. Squares φ such that $\ell\varphi = 2$ are vertical compositions of squares of length $\leq 1 + 1/2$, etc. $\ell\varphi$ is defined for every square in a globularly generated double category.

Length of a double category

Let C be a globularly generated double category. Write ℓC for $Sup\{\ell\varphi : \varphi \in C_1\}$. Call ℓC the vertical length of C. For general C we define the vertical lenght of C, ℓC, as $\ell\gamma C$. **Intuition:** ℓC measures the complexity of mixed compositions of horizontal identity and globular squares in C.

Examples: $\ell\mathbb{H}B = 1$ for any bicategory B, $\ell\mathbf{Q}B = 1$ for any 2-category B. $\ell\mathbb{M}mbox\mathbf{od} = 1$, $\ell\mathbb{P}rof = 1$. $\ell\mathbb{S}\mathbf{pan}(C) = 1$, $\ell\mathbf{co}\mathbb{S}\mathbf{pan}(C) = 1$ for any category C with pullbacks/pushouts. $\ell coMon_{Poly} = 1$, $\ell Open(Petri) = 1$, $\ell\rho_2^{\square}(X_*) = 1$

Question: Is ℓ trivial? Answer: No. There are examples of globularly generated double categories C_n with $\ell C_n = n$ for every $n \geq 1$. $\ell \varinjlim C_n = \infty$.

Definition A double category C is a framed bicategory if $L \times R$ is a Grothendieck bifibration. Formal restriction and extension of scalars Ex. all the above. **Conjecture:** If C is a framed bicategory then $\ell C = 1$.

End-Indexings

End-indexings

Double categories of length 1

Problem: For any decorated bicategory (B^*, B) together with some extra data associate an internalization C such that $\ell C = 1$.

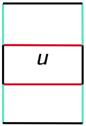
Observation: Let C be a globularly generated double category. $\ell C = 1$ iff for every φ and ψ squares of C of length 1, such that $\varphi \odot \psi$ is well defined, we can find compatible vertical subdivisions:

0	0
0	0
$\vdots$	$\vdots$
0	0

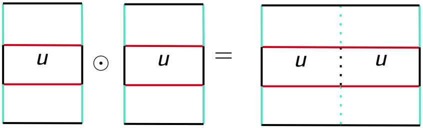
of φ and ψ. We wish to re-create this. **Strategy:** Use the extra data to construct globularly generated double categories in which every square admits a canonical vertical subdivision into squares of complexity 0, such that canonical subdivisions are closed under horizontal composition.

An example

Let C be a globularly generated double category. Suppose every square φ of C of length 1 admits a vertical subdivision into squares of complexity 0 of the form:



The horizontal composition of any two squares of the above form is of the above form:

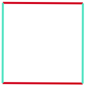


Thus if every square of length 1 in C is of the above form, then $\ell C = 1$. This is how we prove $\ell\mathbb{M}\mathbf{od} = 1$. Choose this as canonical form.

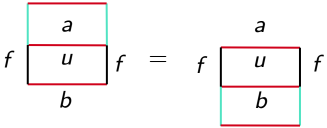
The extra data

Definition: Let (B^*, B) be a decorated bicategory. A End-indexing of B^* with respect to (B^*, B) is a functor $\Phi : B^{*op} \to$**Cat** such that for every object a in B^*, the category $\Phi(a)$ is $\Omega End(id_a)$. Organize End-indexings into a category $\mathbf{EndCat}_{(B^*,B)}$.

What do End-indexings buy us? Observe that if (B^*, B) is a decorated bicategory, the diagrams of shape:



form a commutitve monoid (Use Eckmann-Hilton for commutativity) for every a. An End-indexing allows us to formally an coherently slide horizontal identity squares as:



End-indexings and internalization

Given a decorated bicategory (B^*, B) and and End-indexing $\Phi \in$**EndCat**$_{(B^*,B)}$, vertically stacking diagrams of (B^*, B) (shaped like squares) we can prove that new composite square diagrams are of the canonical form described before, and thus the collection of such diagrams forms a globularly generated double category of length 1. Formally:

Let (B^*, B) be a decorated bicategory. Write $\mathbf{gCat}^{\ell=1}_{(B^*,B)}$ for the category of globularly generated double categories C such that $H^*C = (B^*, B)$ and such that $\ell C = 1$.

Theorem

Let (B^*, B) be a decorated bicategory. There exists an embedding $C : \mathbf{EndCat}_{(B^*,B)} \to \mathbf{gCat}^{\ell=1}_{(B^*,B)}$. Given an End-indexing Φ of B^* with respect to (B^*, B) the category of horizontal endomorphism squares of C^Φ under vertical composition is $\int_{B^*} \Phi$.

Corollaries, questions and conjectures

Lemma: Let G be a group. Let A be an abelian group. Every decorated bicategory of the form $(\Omega G, \Omega^2 A)$ admits internalizations, and all internalizations of $(\Omega G, \Omega^2 A)$ are of length 1.
Question: Let G be a group. Let A be an abelian group. Are all internalizations of $(\Omega G, \Omega^2 A)$ of the form C^Φ?

Lemma: Let (B^*, B) be a decorated bicategory. If (B^*, B) has a single object then (B^*, B) admits internalizations of length 1.
Question: Does every decorated bicategory admit an internalization? Does every decorated bicategory admit an internalization of length 1? Does every single object decorated bicategory admit an internalization C such that C is a framed bicategory? What is the relation of the construction C^Φ and Shulman's construction of framed bicategories via monoidal fibrations?

Question: What formal properties does ℓ satisfy? Can we categorify ℓ?
[Paré] Study general numerical invariants of double categories.

References

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Gracias!