Lawvere distributions, pseudolinear algebra and factorization structures

M. Jibladze

“Functor Categories, Model Theory and Constructive Category Theory”
Almeria, July 11, 2022

Research supported by the Shota Rustaveli National Science Foundation of Georgia Grant FR-18-10849

Overview (or, tl;dl)

There are several kinds of peculiar algebraic structures involved in mathematical approaches to the two-dimensional conformal field theory (CFT).

One of them in particular, based on the concept of pseudolinear algebra (Bojko Bakalov, Alessandro D'Andrea, Victor G. Kac, "Theory of finite pseudoalgebras", Advances in Math. **162** (2001) 1–140) seems to be related to a categorification of the Radon-Nicodym derivative invented by Bill Lawvere.

I will define both and then try to say something about this relationship.

Operator-valued distributions

It seems that the basic notion underlying most algebraic formalisms for CFT is that of the formal operator-valued distribution.

This is an expression of the form

$$A(z) = \sum_{n \in \mathbb{Z}} A_n z^n,$$

where z is a formal variable and each A_n is a linear operator on some vector space.

One can form a composite of two such "at different points", $A(z)B(w)$, obtaining an expression in two variables, but it cannot be given sense for $z = w$.

Operator-valued distributions as fields

Rather, physicists, who encounter such distributions as quantizations of classical fields, are interested in computing operator product expansions

$$A(z)B(w) \sim \frac{C_1(w)}{z-w} + \frac{C_2(w)}{(z-w)^2} + \dots$$

describing how exactly does the composite diverge when nearby points tend to coincidence.

(Slightly) more precisely, they work with fields — those $A(z) = \sum_{n\in\mathbb{Z}} A_n z^n$ which satisfy the condition that for every v there is an n_0 with $A_n v = 0$ for $n < n_0$.

Algebras of mutually local fields

As a substitute for the notion of commutativity, fields A and B are said to be mutually local if

$$(z-w)^N (A(z)B(w) - B(w)A(z)) = 0$$

for $N \gg 0$.

There have been several proposals for an algebraic formalism to describe what does it mean that a collections of mutually local fields is closed under composition and so forms an algebra.

Beilinson and Drinfeld introduced factorization algebras. Roughly speaking, these are sheaves A over the Ran space $\mathscr{R}(X)$ of all finite subsets of an algebraic curve X, together with canonical identifications $A_{S\sqcup T} = A_S \otimes A_T$ of stalks at disjoint finite subsets $S, T \subset X$, $S \cap T = \varnothing$.

Factorization algebras

State-of-the-art version that works for X a manifold of arbitrary dimension belongs to Costello and Gwilliam.

Their factorization algebra is a collection of chain complexes $\mathscr{A}(U)$, with U any open subset of X, together with maps

$$\mathscr{A}(U_1) \otimes \cdots \otimes \mathscr{A}(U_n) \to A(V)$$

for disjoint open subsets U_1, ..., U_n of V.

Factorization algebras

These are required to satisfy certain compatibility conditions and a homotopy version of the cosheaf condition, asserting that for a cover $\mathscr{U}$ of an open set U the canonical map $\check{C}(\mathscr{U}, \mathscr{A}) \to \mathscr{A}(U)$ from the Čech complex of $\mathscr{U}$ to $\mathscr{A}(U)$ is a quasi-isomorphism, if the cover $\mathscr{U}$ is factorizing (by definition this means that any finite subset of U is covered by a finite family of disjoint opens from $\mathscr{U}$).

The crucial consequence is that for any disjoint open subsets U_1, ..., U_n the map

$$\mathscr{A}(U_1) \otimes \cdots \otimes \mathscr{A}(U_n) \to \mathscr{A}(U_1 \cup \cdots \cup U_n)$$

is a quasi-isomorphism.

Factorization algebras and chiral algebras

Returning to the Beilinson-Drinfeld version, in the algebro-geometric setting, when X is an algebraic curve, their requirements on a factorization algebra A provide in particular a D-module structure on A (module over the sheaf $\mathscr{D}_X$ of algebras of differential operators on the structure sheaf $\mathscr{O}_X$ of X).

Using peculiar interplay between the categories of left and right D-modules they are able to define structures essentially equivalent to factorization algebras that they call chiral algebras, as monoids in certain pseudo-tensor category.

Factorization algebras and chiral algebras

Roughly speaking, they rigorously formalize the following idea.

Let $\Delta : X \hookrightarrow X \times X$ be the diagonal embedding, and let $j : U \hookrightarrow X \times X$ be its complement.

Then, for a factorization algebra A based on X one has a canonical map $j_* j^*(A \boxtimes A) \to \Delta_*(A)$.

Here $A \boxtimes A$ is the sheaf on $X \times X$ with the stalk $A_{x_1} \otimes A_{x_2}$ over (x_1, x_2).

Thus, although one cannot produce maps $A_x \otimes A_x \to A_x$, one has data describing how do $A_{x_1} \otimes A_{x_2}$ and A_x come together when gluing $A \boxtimes A$ on U and A on X along the diagonal.

Pseudolinear algebra of Bakalov - D'Andrea - Kac

Singling out those properties of D-modules that Beilinson and Drinfeld needed to construct their pseudo-tensor category, Bakalov, D'Andrea and Kac come up with the notion of pseudo-linear algebra that makes sense for modules over arbitrary Hopf algebra.

For left modules M, N over a Hopf algebra H, they define H-pseudolinear maps $\phi : M \to (H \otimes H) \otimes_H N$ as those linear maps satisfying

$$\phi(hm) = ((1 \otimes h) \otimes_H 1)\phi(m)$$

for $h \in H$, $m \in M$. The set of all such maps that they denote by $\mathrm{Chom}(M, N)$, is a left H-module with respect to the action

$$(h\phi)(m) = ((h \otimes 1) \otimes_H 1)\phi(m).$$

Pseudolinear algebra of Bakalov - D'Andrea - Kac

Pseudolinear maps cannot be composed directly, but they possess composition structure of the form

$$* \in \mathrm{Lin}(\{\mathrm{Chom}(Y,Z), \mathrm{Chom}(X,Y)\}, \mathrm{Chom}(X,Z))$$

for left H-modules X, Y, Z, with respect to the pseudotensor category given by

$$\mathrm{Lin}(\{M_1, ..., M_n\}, N) := \mathrm{Hom}_{H^{\otimes n}}(M_1 \boxtimes \cdots \boxtimes M_n, H^{\otimes n} \otimes_H N).$$

Pseudolinear algebra of Bakalov - D'Andrea - Kac

They establish the above passing back and forth between pseudolinear maps and their Fourier transforms that are maps of the form $\phi_x \in \mathrm{Hom}(M, N)_{x \in X}$, indexed by the dual space $X = H^*$ of H, which satisfy

$$\forall\ m \in M\ \mathrm{codim}_X\{x \in X \mid \phi_x m = 0\} < \infty$$

and

$$\phi_x(hm) = h^{(2)}(\phi_{h^{(-1)}x} m),$$

where the Sweedler-like notation has been used,

$$(S \otimes \text{identity})\Delta(h) = h^{(-1)} \otimes h^{(2)}.$$

Here X is viewed as a left H-module with

$$(hx)(h') = x(S(h)h').$$

Pseudolinear algebra of Bakalov - D'Andrea - Kac

For a pseudolinear map $\phi : M \to H^{\otimes 2} \otimes_H N$ given by

$$\phi(m) = \sum_i (f_i \otimes g_i) \otimes_H n_i$$

they define

$$\phi_x(m) = \sum_i x(S(f_i g_i^{(-1)}))g_i^{(2)} n_i,$$

while for $\phi_x \in \mathrm{Hom}(M, N)$, $x \in X$ as above,

$$\phi(m) = \sum_i (S(h_i) \otimes 1) \otimes_H \phi_{x_i} m,$$

where (h_i), (x_i) are the dual bases of H and X, respectively.

Lawvere — categorifying the Radon-Nikodym derivative

Bill Lawvere in early eighties came up with what in modern terms can be called categorification of the Radon-Nikodym derivative.

Recall that given two measures μ, ν on a measurable space (X, Σ) such that ν is absolutely continuous with respect to μ, the Radon-Nikodym derivative of ν with respect to μ, denoted $\frac{d\nu}{d\mu}$, is a unique, up to a set of μ-measure zero, Σ-measurable function f on X satisfying

$$\nu(A) = \int_A f \, d\mu$$

for any $A \in \Sigma$.

Lawvere — categorifying the Radon-Nikodym derivative

In the simplest case, for a small category $\mathbf{C}$, consider the category of set-valued presheaves $\mathrm{Set}^{\mathbf{C}^{\mathrm{op}}}$ on $\mathbf{C}$.

For Lawvere this is the analog of the algebra of measurable functions.

The analog of the space of distributions — the dual of the algebra of functions — is for him the category of cocontinuous (colimit preserving) functors to the category of sets.

In this particular case it comes out as

$$\mathrm{Cocont.}(\mathrm{Set}^{\mathbf{C}^{\mathrm{op}}}, \mathrm{Set}) \cong \mathrm{Set}^{\mathbf{C}};$$

we thus get the category of set-valued copresheaves (covariant functors) on $\mathbf{C}$.

Lawvere — categorifying the Radon-Nikodym derivative

Lawvere constructs on $\mathrm{Set}^{\mathbf{C}}$ a structure of a tensored cotensored $\mathrm{Set}^{\mathbf{C}^{\mathrm{op}}}$-enriched category.

Easiest to begin with for that is the cotensor structure. For $P : \mathbf{C}^{\mathrm{op}} \to \mathrm{Set}$, $F : \mathbf{C} \to \mathrm{Set}$ define $F^P : \mathbf{C} \to \mathrm{Set}$ via

$$(F^P)(C) = F(C)^{P(C)}.$$

Lawvere — categorifying the Radon-Nikodym derivative

For this to be a cotensor with respect to an enrichment, given another $F' : \mathbf{C} \to \mathrm{Set}$ one must have $\mathbf{Hom}(F', F) : \mathbf{C}^{\mathrm{op}} \to \mathrm{Set}$ admitting natural isomorphisms

$$\mathrm{Nat}(F', F^P) \approx \mathrm{Nat}(P, \mathbf{Hom}(F', F)).$$

Taking here for P a representable functor, Yoneda lemma tells us that we then have essentially unique choice for such $\mathbf{Hom}$:

$$\mathbf{Hom}(F', F)(c) = \mathrm{Nat}(F', F^{\hom_{\mathbf{C}}(-,c)}).$$

One checks easily that this indeed defines an enrichment with required properties.

Lawvere — categorifying the Radon-Nikodym derivative

Lawvere then extends this more generally with a category of sheaves on $\mathbf{C}$ with respect to a Grothendieck topology in place of $\mathrm{Set}^{\mathbf{C}^{\mathrm{op}}}$, and its dual category of cosheaves in place of $\mathrm{Set}^{\mathbf{C}}$.

In a subsequent work Bunge and Funk related this dual enrichment to a certain duality between local homeomorphisms and complete spreads.

Lawvere — categorifying the Radon-Nikodym derivative

Local homeomorphisms can be viewed as maps determined by their sheaves of sections. More precisely, $f : X \to Y$ is a local homeomorphism if and only if for any $y \in Y$ the canonical map

$$\varinjlim_{U \ni y} \Gamma_U(f) \to f^{-1}(y)$$

is bijective.

As it turns out, there is an alternative definition of complete spread which is in a sense dual to this, although under certain restrictions. If X and Y are locally connected, then $f : X \to Y$ is a complete spread if and only if the canonical map

$$f^{-1}(y) \to \varprojlim_{U \ni y} \pi_0(f^{-1}(U))$$

is bijective.

That's all, unfortunately!