

# Research Statement

Gabor Lippner

## Introduction

My research is focused on the boundary between discrete and continuous mathematics. In the recent years many surprising connections have been discovered between the two worlds, often leading to a better understanding of both. Exciting new results were obtained, for example in additive combinatorics, extremal graph theory, or computer science, by applying analytic techniques in a discrete setting. At the same time finite approximations of continuous objects proved to be a successful approach in fields ranging from ergodic theory to functional analysis.

Part of my research is in graph limits, the emerging field bridging the gap between combinatorics and measure theory. I'm particularly interested in the theoretical aspects of property testing, developing a measurable version of graph theory, and using the finite approximation technique to analyze groups and group actions.

I am also interested in geometric/topological aspects of graph theory, like understanding the behavior of Laplace type operators and their eigenfunctions of graphs and simplicial complexes, embedding graphs into surfaces, or approximating manifolds by graphs.

The two features I find most exciting about my research is that it brings together combinatorial ideas with theory building, and that its multidisciplinary nature allows for collaboration with mathematicians working in a wide range of fields from abstract algebra to applied mathematics.

**Applications of the Borel Oracle method.** The theory of graph limits connects the world of large finite graphs with measure preserving group actions on a standard Borel measure space. Developed jointly with Gabor Elek, the Borel Oracle method uses this connection to translate results and constructions in measure theory and ergodic theory into statements about finite graphs. The first theme of my research is to expose the full strength of this general method.

**Measurable graph theory.** A measurable graph is simply a graph on the unit interval that respects the underlying Borel structure. Many notions of classical graph theory - ranging from colorings to spectra and expansion - have their analogues in the measurable case. Part of my research is to understand, to what extent do results in classical graph theory remain true in the measurable setting. This often requires a mixture of techniques from combinatorics, ergodic theory, and functional analysis. The main motivation for this investigation is the hope to shed some light on the famous Aldous-Lyons conjecture: can all measurable graphs be approximated by finite graphs?

**Geometry of graph eigenfunctions.** The study of Laplace type operators on graphs, together with their eigenvalues and eigenfunctions, has a broad range of applications in discrete mathematics and computer science. These operators behave somewhat similarly to their analogues on manifolds, hence geometric ideas can be brought to bear on graph theory problems. Such methods have already had a few great successes, like relating expansion to spectral gap. However the analogy has so far been only exploited on a very basic level. In this direction I am studying how deeper geometric ideas, for example the notion of curvature, could be used in graph theory.

**Discrete models of quantum mechanics.** The Laplace operator analogy also turns graphs into good discrete models for quantum phenomena. Discrete models are often easier to handle than their continuous counterparts and hence allow for a more general treatment, and one hopes that the general discrete results might even suggest new directions in the classical case. Motivation for understanding such models also comes from quantum computing, where quantum walks on graphs have already been used extensively. I am currently interested in describing long term behavior of solutions to the discrete Schrödinger equation in the presence of a potential.

**Network analysis using spectral methods on simplicial complexes.** Networks arise naturally in almost all sciences. In recent years the access to large amounts of data prompted the development of fast and robust mathematical techniques to analyze networks. Spectral graph methods have proven to be a very successful approach in many areas, including importance ranking (e.g. Google’s Page Rank algorithm) and clustering. However a graph is not always the best model for the data, a simplicial complex often describes the network more accurately. My research in this direction aims to extend spectral methods to simplicial complexes based on the Hodge Laplacian, in order to provide finer analysis for such data.

## Graph limits

Various notions of graph-convergence have emerged in the past two decades. Borgs et al, and Lovasz and Szegedy (see e.g. [4, 25]) investigated sequences of dense graphs, building a theory that has already led to numerous breakthroughs in extremal combinatorics as well as a better understanding of Szemerédi’s regularity lemma. Independently, Benjamini and Schramm (in [3]) pioneered limits of bounded degree graph sequences, leading to applications ranging from group theory to constant time algorithms.

In both theories, one defines a metric on the space of graphs, and then constructs a compactification of the resulting metric space. The most difficult part in both cases is finding natural “limit objects” to represent the limits of graph sequences. In the dense setting symmetric measurable functions play this role, in the bounded degree case unimodular random rooted graphs (probability distributions on rooted graphs that satisfy a certain involution invariance property). Both theories are relatively new and have many open questions, however our understanding of bounded degree graph limits is much less complete compared to the dense version. In particular it is known that any symmetric measurable function is the limit of a (dense) graph sequence. On the other hand the most important open question in the bounded degree case is (see Aldous and Lyons, [1]) whether every unimodular random rooted graph arises as a limit.

Elek (see [11]) noticed that for bounded degree graphs there is a more convenient choice for the limit objects which we now explain. Let  $(X, \mathcal{B}, \mu)$  be a standard Borel probability measure space and  $E \subset X \times X$  a symmetric Borel relation. We say that  $E$  is measure preserving if for any  $f : A \rightarrow B$  measurable bijection between subsets  $A, B \subset X$  such that  $\{(x, f(x)) : x \in A\} \subset E$ , we have  $\mu(A) = \mu(B)$ . The pair  $\mathcal{G} = (X, E)$  is called a *graphing*. Alternatively a graphing can be defined as the Schreier graph of a group action. The degree of  $x \in X$  in the graphing  $\mathcal{G}$  is the number of vertices that are in relation with  $x$ .

Let us fix a  $d \in \mathbb{N}$  and denote by  $G_d$  the set of connected graphs with uniform degree bound  $d$ . For any  $r \in \mathbb{N}$  let  $RG_d(r)$  denote the set of connected graphs with a marked vertex called the root, degrees bounded uniformly by  $d$  and such that the distance of any vertex from the root is at most  $r$ . These graphs are often called rooted  $r$ -balls. Given any finite graph  $G \in G_d$  we can define a probability distribution  $\rho_G(r)$  on  $RG_d(r)$  as follows. Pick a vertex  $x$  of  $G$  uniformly at random, and take its  $r$ -neighborhood in  $G$ . Together with  $x$  as the root, this is a random element of  $RG_d(r)$ . Similarly, for a bounded degree graphing  $\mathcal{G}$ , we can define a probability distribution  $\rho_{\mathcal{G}}(r)$  on  $RG_d(r)$  by choosing a random  $x \in X$  according to  $\mu$  and then taking its  $r$ -neighborhood.

Finally we say that the graphing  $\mathcal{G}$  is the limit of the sequence of finite graphs  $G_n$  if  $\rho_{G_n}(r) \rightarrow \rho_{\mathcal{G}}(r)$  for every  $r$ . In this terminology the question of Aldous and Lyons is equivalent to asking whether every bounded degree graphing is the limit of a sequence of finite graphs.

## Borel Oracles

With Elek we have devised a general method that can be used in conjunction with the graph limit machinery. Intuitively we do the following: given a graphing  $\mathcal{G}$  and a "structure"  $S$  supported in  $\mathcal{G}$ , then for any sequence of graphs  $G_n$  that converge to  $\mathcal{G}$  we can "pull back"  $S$  to the graph sequence and construct structures  $S_n$  supported on  $G_n$  such that  $S$  is the limit of  $S_n$ . The structure  $S$  is referred to as a "Borel Oracle".

This method is already implicit in [16] where we proved an analogue of the Szemerédi regularity lemma for bounded degree graphs, stating that any such graph can be cut into a constant number of homogeneous parts. In this case the structure is the ergodic decomposition of a measure, and it can be pulled back to a partition of graphs.

In [18] we prove that there is a constant time approximation algorithm for finding the size of the maximum matching in a bounded degree graph. In this case the Borel oracle is a measurable matching in the limit graphing that has no short augmenting paths. This can be pulled back to a convergent graph sequence to exhibit almost-maximal matchings. In [12] Elek uses the oracle technique to show that if a sequence converges to an amenable graphing then the sequence is hyperfinite.

I would like to continue investigating possible applications and extensions of this technique. In particular I hope to be able to use it in the context of constant time approximation algorithms of various graph parameters concerning spanning trees, factors and network flows. Hatami, Lovász, and Szegedy (see [15]) have recently defined a stronger notion of convergence for bounded degree graphs and proved that graphings still can be used as limit objects. It would be interesting to work out a stronger version of the oracle method for this setting.

The strongest applications of the oracle method are so far ones that are complemented by a reverse oracle, or push-forward method. In certain cases it seems possible to take a convergent graph sequence satisfying a certain property, and use it to force a certain structure on the limit graphing. This is typically more difficult than the pull-back of structures, and is often easiest done via non-standard methods like ultra products. I plan to study problems that could be pursued by developing push-forward methods, including constant time algorithms for testing the existence of certain types of subgraphs, questions related to the "combinatorial cost" (the finitary version of Gaboriau's Fixed Price Conjecture, see [13] and [14]), and the mod- $p$  version of the Lück approximation theorem (see [20]).

## Measurable graph theory

Graphings can be thought of as measure theoretic generalizations of graphs. As such they provide a new, natural context to extend the study of classical graph theory. This new direction is called "measurable graph theory". While it is interesting in its own right, it also has deep connections to ergodic theory and probability.

The connected components of a graphing are just countable graphs, but the measurability condition can lead to surprising behavior. Theorems that are true "component wise" are not necessarily true in the measurable sense. A simple example exhibiting this phenomenon is the graphing generated by an irrational rotation of the unit circle. In this graphing each component is isomorphic to the bi-infinite path, hence can be colored with two colors. However, since an irrational rotation is an ergodic transformation on the circle, there is no global measurable 2-coloring of the graphing.

The systematic study of chromatic numbers of graphings was initiated by Kechris, Solecki and Todorcevic in [19]. In particular they showed that a graphing with maximum degree  $d$  can be colored by  $d + 1$  colors in the measurable sense. Brooks' theorem states that a finite graph can be colored by  $d$  colors unless it contains a  $d + 1$  vertex complete subgraph, or an odd cycle if  $d = 2$ . As indicated by the irrational rotation on the circle, this theorem is false in the measurable case. However Conley and Kechris ([6]) proved that if a graphing  $\mathcal{G}$  comes from the free action of a group  $\Gamma$  that has finitely many ends but is not isomorphic to  $\mathbb{Z}$  or  $\mathbb{Z}_2 \star \mathbb{Z}_2$  then Brooks' theorem is valid for  $\mathcal{G}$ . Whether Brooks' theorem holds for group actions where the group has infinitely many ends is unknown.

The marriage theorem states that any  $d$ -regular bipartite graph has a perfect matching and its edges can

be colored with  $d$  colors. The first result about edge colorings of graphings is due to Laczkovich ([21]) who constructed a measurable, 2-regular bipartite graph that does not have a measurable perfect matching, and hence is not 2 edge-colorable. This construction has later been generalized to  $2d$  regular graphs, but both the existence of a perfect matching and the existence of edge-coloring is unresolved for odd degrees.

On the positive side, motivated by questions about random processes on groups, Lyons and Nazarov ([26]) showed the existence of perfect matchings in regular bipartite expander graphs. In a joint work with Csoka ([9]) we have generalized this result to non-bipartite graphs. If  $\Gamma$  is a non-amenable group, then the Bernoulli shift of  $\Gamma$  can be represented by a graphing to which our result applies. This implies that there is a translation invariant probability distribution on the set of perfect matchings of  $\Gamma$ .

I would like to systematically study measurable graph theory, in particular the measurable edge- and vertex-chromatic numbers of graphings, as well as the existence of measurable factors. Brooks' theorem and Vizing's theorem are examples where the answer in general is unknown, but where our methods developed in [9] are likely to be applicable.

So far all known negative results involve amenable constructions, while classical theorems seem to hold verbatim for non-amenable actions. It would be very interesting to find the least set of assumptions under which most theorems remain valid, and the known cases strongly suggest that amenability will play an important role in these investigations.

## Laplace operators

Graphs are in many aspects good discrete analogues of manifolds. The Laplace operator is perhaps the strongest manifestation of this analogy.

On a Riemannian manifold  $(M, g)$  the exterior derivative  $d : \Omega^0(M) \rightarrow \Omega^1(M)$  takes smooth functions to 1-forms. It is possible to construct inner products on  $\Omega^0$  and  $\Omega^1$  from  $g$  via Hodge-theory, and use them to define an adjoint  $d^* : \Omega^1(M) \rightarrow \Omega^0(M)$ . The Laplace operator is then given by  $\Delta(f) = d^*d(f)$ .

There is an analogue of this construction for graphs. Given a graph  $G(V, E)$  one can define  $d : \mathbb{R}^V \rightarrow \mathbb{R}^E$  by the formula  $df(xy) = f(y) - f(x)$ . Here  $\mathbb{R}^V$  is the space of functions on the vertices,  $\mathbb{R}^E$  is the space of functions on edges (the discrete version of 1-forms), and  $d$  can be naturally considered as a discrete derivation. Choosing the trivial inner products one gets an adjoint  $d^* : \mathbb{R}^E \rightarrow \mathbb{R}^V$  and the composition  $\Delta f = d^*df$  is the classical graph Laplacian.

Besides the similarity of their construction, there is an additional fact that connects the Laplace operators in the discrete and the continuous case: Courant et. al. [7] showed that solutions to PDEs given by Laplace type operators on a Euclidean domain can be obtained as limits of solutions to the corresponding problems about graph Laplacians on lattices inscribed in the domain. This motivates the study of discrete analogues of geometric questions concerning Laplace operators. On one hand geometric intuition can be used to solve problems on graphs, on the other hand one can hope to obtain geometric results by discrete approximations.

### 0.0.1 Curvature

One of the most fundamental concepts in differential geometry is curvature. Often much stronger results can be proved for manifolds whose curvature is controlled in some way than in general. An important example of such results, and one that lies at the heart of many others, is the Li-Yau [22] inequality. It gives a rather precise control on how fast temperature can decay given a lower bound on the curvature.

In recent years there have been many attempts to generalize the notion of curvature (or at least the notion of curvature bounded from below, say) to various non-smooth settings, including graphs. The standard method is to pick a simple consequence of the curvature bound on a manifold that can also be expressed in the non-smooth case, and use it as the definition. One example of such a consequence that has been used before is the so-called Curvature Dimension inequality, which states that if the dimension of a manifold is  $n$ ,

and its curvature is bounded from below by  $K$ , then for any smooth function  $f$  one has

$$\Delta|\nabla f|^2 \geq K|\nabla f|^2 + n(\Delta f)^2 + 2\langle f, \Delta f \rangle.$$

Trying to recover the Li-Yau inequality is a good test to see if one has found the "right" generalization. In certain cases, for example when the Laplace operator generates a Markov diffusion semigroup, it has been shown that the Curvature Dimension inequality implies the Li-Yau estimate. Graphs, however, do not satisfy this property, and it seems that the Curvature Dimension inequality is insufficient for proving the Li-Yau inequality.

In [2] we found a way to modify the Curvature Dimension inequality in such a way that it does imply the Li-Yau estimate. It turns out that this approach already leads to previously unknown properties of the simple random walk on the lattice. Nevertheless we feel that the notion of curvature one gets out of our variant of the Curvature Dimension inequality is only a first approximation to what it should finally be. In particular our notion of curvature at a vertex depends only on the 2-step neighborhood of it.

I feel that [2] opens up a whole new line of very interesting research. The first step would be refining this curvature notion so it "sees" more than just 2-step neighborhoods. While extending the notion to weighted graphs is straightforward, our current method of proving the Li-Yau inequality doesn't yet work in the generality we believe it should.

Finally, the most important application of the Li-Yau inequality in the past decade has been it's relevance to the Ricci flow. It would be very interesting to find a discrete analogue of the Ricci flow - a natural, intrinsic evolution of the graph that makes it more homogeneous. So far there has been little to no progress in this direction, but our result gives hope that through the Li-Yau inequality one might be able to come up with a meaningful discrete version.

## Nodal geometry

Let  $(M, g)$  be a smooth Riemannian manifold and  $f : M \rightarrow \mathbb{R}$  a smooth function. The *nodal set* of  $f$  is  $\{x \in M : f(x) = 0\}$ . The connected components of the nodal set are called nodal domains. The Laplace operator of  $(M, g)$  is a positive self-adjoint operator with discrete spectrum  $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \dots$ . Let  $\varphi_k$  denote the eigenfunction corresponding to the eigenvalue  $\lambda_k$ . The Courant Nodal Domain Theorem [8] states that  $\varphi_k$  has at most  $k$  nodal domains. Cheng [5] used this result to bound the multiplicity of  $\lambda_k$  on a surface of genus  $g$  in terms of  $k$  and  $g$ .

Special cases of both of these results are very important theorems in graph theory. The Perron-Frobenius Theorem can be thought of as a discrete analogue of the Courant Nodal Domain Theorem for  $k = 2$ . The fact that planar graphs have Colin de Verdiere invariant (multiplicity of the second smallest Laplacian eigenvalue) at most 3 is a discrete manifestation of Cheng's result for  $k = 2, g = 0$ . In [23], jointly with Lin, Mangoubi and Yau, we have provided discrete versions for both Courant's and the Cheng's theorem for general  $k$  and  $g$ , under certain mild conditions about the graph. In Courant's theorem these extra assumptions are known to be necessary, but we believe that Cheng's results can be further extended to more general graphs. To improve our version of Cheng's result we would need to control the "thickness" of the nodal sets.

I would like to continue this line of research by studying the structure of nodal sets of Laplacian eigenfunctions. On a manifold an eigenfunction cannot vanish on an open set. I would like to find the right discrete analogue of this fact, in particular since it should shed light on Cheng's result.

Many classical results in this area, including the non-vanishing of eigenfunctions, is proved by bounding the local growth of eigenfunctions, like in Carleman-type inequalities and log-convexity estimates. Unfortunately these estimates are usually proved by methods that do not adapt easily to a discrete setting, for example using spherical coordinates, or chain rule type arguments. My immediate goal is to work out proofs for these analytic inequalities using the language of Brownian motion, and then find discrete versions of them.

## Tunneling

In one of the standard models of quantum mechanics the state space of a quantum particle is represented by a manifold  $M$ . The state at time  $t$  is described by a complex valued function  $\psi_t : M \rightarrow \mathbb{C}$  such that  $\int_M |\psi_t|^2 = 1$ . The value  $|\psi_t(x)|^2$  is interpreted as the probability density that at time  $t$  the particle is in  $x$ . The movement is governed in part by the geometry of  $M$  (represented via the Laplace operator) and in part by a fixed potential function  $V : M \rightarrow \mathbb{R}$ . To be more precise, the time evolution of  $\psi_t$  is given by Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi_t = \Delta \psi_t - V \cdot \psi_t.$$

This naturally yields itself to discretization, since every piece of the model has an appropriate analogue on a graph. In a joint work with Lin and Yau [24] we have analyzed solutions of the continuous time discrete Schrödinger equation. In particular we have studied the so called tunneling effect, in which there is a positive probability that the quantum particle crosses a potential barrier that is higher than it's kinetic energy. In the continuous case, in certain symmetric scenarios, the higher the barrier the larger is the probability of this unlikely crossing.

In our discrete model a graph  $G$  plays the role of the state space. We choose a potential  $V : G \rightarrow \mathbb{R}$  with two or more global minima, the so called potential wells. Let  $x$  and  $y$  be two such wells. We study the evolution of a particle starting from  $x$  at time 0, and measure the maximum probability of ever being observed in  $y$ , as a function of the given potential. Finally we use a scaled potential  $c \cdot V$  and let  $c \rightarrow \infty$ . We say that the system exhibits tunneling if the probability of wall-crossing (observing the particle in  $y$ , the "other side of the wall") tends to 1 in this limit.

So far we have considered potentials with two and three wells, and found a rather surprising contrast between the two cases. In the double well case we have found the necessary and sufficient conditions under which tunneling occurs, and this condition depends only on a bounded neighborhood of the two wells within the graph. In this sense tunneling is a local effect. We have only studied the triple well case using the simplest potentials. Even this is not completely understood, there are certain special configurations where the precise behavior still needs to be determined. Nevertheless we have been able to construct tripe well systems in which tunneling is not local: arbitrarily small perturbations of the graph at arbitrarily far locations can completely turn the tunneling effect around.

Our first goal is to complete the characterization of the triple well case. We then plan to extended our analysis to more complex systems. We believe that the local/global contrast is not the only interesting phenomenon that is hidden in this model. In particular we think that it is possible to construct configurations that exhibit chaotic behavior.

Another direction that would be interesting to pursue is a quantitative analysis in the case of a fixed, not very large potential. It is likely to have applications in quantum computing, where a potential-free, discrete-time version of this process, the so-called quantum walk has already been applied with success.

## Ranking

Laplace operators and spectral techniques have been successfully applied in a broad range of problems in computer science, in particular for clustering and vertex ranking (e. g. the PageRank algorithm of Google is based on such methods). Real world networks have been so far almost exclusively modeled with graphs. However in many concrete cases a simplicial complex is a more accurate model: the higher dimensional faces contain information that is often lost by passing to the underlying graph. It is very likely that more accurate ranking and clustering methods can be developed using simplicial data.

In a new project with Horn and Jadbabaie, we plan to develop and analyze such methods based on higher order Laplace operators. Given a simplicial complex  $X$ , one can associate to it a chain complex

$$C_0(X) \xrightarrow{\partial_1} C_1(X) \xrightarrow{\partial_2} C_2(X) \xrightarrow{\partial_3} C_3(X) \leftarrow \dots$$

where  $C_k(X)$  is the  $\mathbb{R}$ -vectorspace generated by the  $k$ -dimensional faces of  $X$ , and  $\partial_k : C_k \rightarrow C_{k-1}$  is the boundary operator. The higher dimensional analogues of the Laplace operator are then defined by the formula

$$\Delta_k = \partial_k^* \partial_k + \partial_{k+1} \partial_{k+1}^*,$$

and in particular  $\Delta = \Delta_0$  is the classical Laplace operator of the underlying graph. This is a standard construction from topology where it is used to compute the homology groups of  $X$ . It is a well-known fact that  $\dim H_k(X, \mathbb{R}) = \dim \ker \Delta_k$ .

The combinatorial meaning of higher order Laplacians is much less understood, than that of the original. We plan to develop a combinatorial theory for  $\Delta_k$ . In particular we would like to investigate the relation of  $\Delta_k$  to random walks on  $X$ . We would like to study boundary value problems associated to  $\Delta_k$  in terms of existence and uniqueness of solutions, and how random walk methods can be used to find these.

We plan to apply this theory to edge (or face) ranking in simplicial complexes using analogues of the PageRank function via higher order Laplacians. We also plan to investigate how spectral properties and eigenfunctions of  $\Delta_k$  are related to clustering problems.

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