

# Grothendieck's simultaneous resolution and the Springer correspondence: Addendum

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September 20, 2013

## 1 An interpretation in terms of perverse sheaves

### 1.1 Small and semismall maps

Let  $f : X \rightarrow Y$  be a proper surjective map of varieties with  $X$  nonsingular. We say that a decomposition of  $Y = \bigsqcup_k S_k$  into finitely many non-singular locally closed subvarieties  $S_k$  is a stratification relative to  $f$  if each restriction  $f : f^{-1}(S_k) \rightarrow S_k$  is a topological locally trivial fibration.

**Proposition 1.1.** The following conditions are equivalent.

- (a)  $Rf_*\underline{\mathbb{C}}_X[\dim X]$  is a perverse sheaf on  $Y$ ,
- (b)  $\dim X \times_X X \leq \dim X$ , or
- (c)  $\dim S_k + 2(\dim f^{-1}(S_k) - \dim S_k) \leq \dim X$  for all  $k$ .

We say that  $f$  is *semismall* if any of the conditions in Proposition 1.1 hold and *small* if the inequality in (c) is strict for all strata  $S_k$  which are not dense in  $Y$ . Call a stratum *relevant* if equality holds in (c).

**Proposition 1.2.** If  $f$  is small, then

$$Rf_*\underline{\mathbb{C}}_X[n] = IC(Y, \mathcal{L}),$$

where  $\mathcal{L} = Rf_*\underline{\mathbb{C}}_X|_{Y_0}$  is the restriction of the (derived) push-forward to the open stratum.

For small and semismall maps, the Decomposition theorem takes on the following especially nice form.

**Theorem 1.3** (Decomposition theorem). If  $f : X \rightarrow Y$  is semismall, then

$$Rf_*\underline{\mathbb{C}}_X[\dim X] = \bigoplus_k IC(\bar{S}_k, \mathcal{L}_k),$$

where the sum is over relevant strata  $S_k$  and local systems  $\mathcal{L}_k$  on  $S_k$ .

### 1.2 Semismallness of the Springer resolution

Let us see how these results apply to our situation. Depending on the context, the map  $\pi$  will be either small or semismall.

**Proposition 1.4.** The Springer map  $\pi : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$  admits a stratification by  $G$ -orbits under the adjoint action. Relative to this stratification,  $\pi$  is semismall, and every orbit is a relevant stratum.

*Proof.* Let  $\mathcal{O}$  be a nilpotent orbit in  $\mathcal{N}$  and  $\xi \in \mathcal{O}$  an element. It is clear that  $\tilde{\mathcal{O}} \rightarrow \mathcal{O}$  is a fibration with fiber  $\mathcal{B}_\xi$ , so nilpotent orbits give a valid stratification. Now,  $\tilde{\mathcal{N}}$  is smooth, and  $\pi$  is evidently proper and surjective. Some dimension estimates from our first talk together imply that

$$\dim \mathcal{O} + 2 \dim \mathcal{B}_x = \dim Z_{\mathcal{O}} = \dim \mathcal{N},$$

so we conclude that  $\pi$  is semi-small and that each stratum is relevant. □

**Proposition 1.5.** The simultaneous resolution  $\pi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$  is small.

*Proof.* Again,  $\pi$  is proper surjective with  $\tilde{\mathfrak{g}}$  smooth, so it suffices to verify the dimension condition (c) for some stratification. Let  $\mathfrak{g}_n \subset \mathfrak{g}$  be

$$\mathfrak{g}_n = \{x \in \mathfrak{g} \mid \dim \mathcal{B}_x = n\}.$$

Take a stratification of  $\mathfrak{g}$  which is a possible refinement of

$$\mathfrak{g} = \mathfrak{g}_{\text{sr}} \cup (\mathfrak{g}_0 - \mathfrak{g}_{\text{sr}}) \cup \bigcup_{n \geq 1} \mathfrak{g}_n$$

which makes  $\pi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$  a fibration above each stratum. On  $\mathfrak{g}_{\text{sr}}$ ,  $\pi$  is a  $|W|$ -to-1 cover, so (c) follows trivially. For any other stratum  $S \subset \mathfrak{g}_n$ , we see that

$$\dim S + n = \dim \pi^{-1}(S) = \dim \mathcal{B} + \dim(S \cap \mathfrak{b}),$$

hence

$$\dim S + 2n = \dim \mathcal{B} + \dim(S \cap \mathfrak{b}) + n.$$

On the other hand, we see that

$$\dim(S \cap \mathfrak{b}) + n = \dim(S \cap \mathfrak{b}) \times_{\mathfrak{g}} \tilde{\mathfrak{g}} < \dim \mathfrak{b} \times_{\mathfrak{g}} \tilde{\mathfrak{g}},$$

where the strict inequality is because  $S$  is not the dense stratum. Thus, to show smallness, it suffices to check that

$$\dim \mathfrak{b} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} = \dim \mathfrak{b}.$$

For this, observe that

$$\mathfrak{b} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} = \{(x, \mathfrak{b}') \mid x \in \mathfrak{b}, x \in \mathfrak{b}'\},$$

so the second projection equips it with a map to  $\mathcal{B}$ . Its pullback over each Schubert cell  $X(w) = BwB/B \subset \mathcal{B}$  is given by

$$\{(x, gB) \mid x \in \mathfrak{b} \cap \text{ad}_g(\mathfrak{b}), gB \in BwB\},$$

so it suffices to check that  $\dim \mathfrak{b} \cap \text{ad}_g(\mathfrak{b}) + \dim X(w) = \dim \mathfrak{b}$ , which is true because  $\mathfrak{b} \cap \text{ad}_g(\mathfrak{b})$  is the Lie algebra of the stabilizer of the  $B$ -action on  $X(w)$ .  $\square$

### 1.3 Action on the derived pushforward

Because  $\pi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$  is small, we obtain that

$$R\pi_* \underline{\mathbb{C}}_{\tilde{\mathfrak{g}}}[\dim \mathfrak{g}] = IC(\mathfrak{g}, \mathcal{L}),$$

where  $\mathcal{L}$  is the local system

$$\mathcal{L} = R\pi_* \underline{\mathbb{C}}_{\tilde{\mathfrak{g}}}[\dim \mathfrak{g}] \Big|_{\mathfrak{g}_{\text{sr}}} = R\pi_* \underline{\mathbb{C}}_{\tilde{\mathfrak{g}}_{\text{sr}}}[\dim \mathfrak{g}].$$

Recall that  $\tilde{\mathfrak{g}}_{\text{sr}}$  is a  $|W|$ -fold covering of  $\mathfrak{g}_{\text{sr}}$  and therefore  $\underline{\mathbb{C}}_{\tilde{\mathfrak{g}}_{\text{sr}}}$  comes equipped with a  $W$ -action. By functoriality, this yields a  $W$ -action on  $\mathcal{L}$  and hence  $R\pi_* \underline{\mathbb{C}}_{\tilde{\mathfrak{g}}}[\dim \mathfrak{g}]$ . We now degenerate this action to  $\pi : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$ . Because  $\pi$  is proper, by base change on the inclusions  $i : \mathcal{N} \rightarrow \mathfrak{g}$  and  $\tilde{i} : \tilde{\mathcal{N}} \rightarrow \tilde{\mathfrak{g}}$ , we have that

$$i^* R\pi_* \underline{\mathbb{C}}_{\tilde{\mathfrak{g}}}[\dim \mathfrak{g}] = R\pi_* \tilde{i}^* \underline{\mathbb{C}}_{\tilde{\mathfrak{g}}}[\dim \mathfrak{g}] = R\pi_* \underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[\dim \mathfrak{g}],$$

so again by functoriality and dimension shift, we obtain a  $W$ -action on  $R\pi_* \underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}]$ .

Recall now that  $\pi : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$  is semismall, so by the Decomposition theorem we have that

$$R\pi_* \underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}] = \bigoplus_k IC(\bar{S}_k, \mathcal{L}_k) \tag{1}$$

for some local systems  $\mathcal{L}_k$  on strata  $S_k$  for which  $\dim S_k + 2(\dim \pi^{-1}(S_k) - \dim S_k) = \dim \mathcal{N}$ . By Proposition 1.4,  $\mathcal{N}$  admits a stratification by nilpotent orbits  $\mathcal{O}_\xi$ ; further, each simple local system  $\mathcal{L}_{\xi, \chi}$  on  $\mathcal{O}_\xi$  corresponds to a monodromy representation  $\chi$  of  $\pi_1(\mathcal{O}_\xi)$ . Letting  $V_{\xi, \chi}$  be the multiplicity space of  $\chi$  in the monodromy representation corresponding to  $\mathcal{L}_\xi$ , we obtain the decomposition

$$R\pi_* \underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}] = \bigoplus_{\xi, \chi \in \text{Irr}(\pi_1(\mathcal{O}_\xi))} IC(\overline{\mathcal{O}}_\xi, \mathcal{L}_{\xi, \chi}) \otimes V_{\xi, \chi}.$$

We may characterize the multiplicity spaces more explicitly by examining the fibers of  $R\pi_* \underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}]$ .

**Proposition 1.6.** For any nilpotent orbit  $\mathcal{O}_\xi$ , we have

$$H_{2d_\xi}^{BM}(\mathcal{B}_\xi) = \bigoplus_{\chi} \mathbb{C}_\chi \otimes V_{\xi, \chi},$$

where  $d_\xi = \dim \pi^{-1}(\mathcal{O}_\xi) - \dim \mathcal{O}_\xi$ .

*Proof.* Choose  $\xi \in \mathcal{O}_\xi$ ; taking the cohomology in degree  $-\dim \mathcal{O}_\xi$  of the stalks at  $\xi$  of (1) yields

$$H^{-\dim \mathcal{O}_\xi}(\mathcal{B}_\xi, \underline{\mathbb{C}}_{\mathcal{B}_\xi}[\dim \mathcal{N}]) = H^{-\dim \mathcal{O}_\xi}(R\pi_* \underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}]_\xi) = \bigoplus_{\chi} (IC(\overline{\mathcal{O}}_\xi, \mathcal{L}_{\xi, \chi}) \otimes V_{\xi, \chi})_\xi = \bigoplus_{\chi} \mathbb{C}_\chi \otimes V_{\xi, \chi},$$

where we note that  $(\mathcal{L}_{\xi', \chi})_\xi = 0$  unless  $\mathcal{O}_\xi \subset \overline{\mathcal{O}_{\xi'}}$  and that

$$H^{-\dim \mathcal{O}_\xi}(\mathcal{O}_\xi, IC(\overline{\mathcal{O}_{\xi'}}, \mathcal{L}_{\xi', \chi'})) = 0$$

for  $\mathcal{O}_\xi \subset \overline{\mathcal{O}_{\xi'}}$  by the support conditions on IC sheaves.

Observe now that  $i_\xi : \mathcal{B}_\xi \rightarrow \tilde{\mathcal{N}}$  is the inclusion of a fiber into the total space of the fibration  $\mathcal{B}_\xi \rightarrow \mathcal{O}_\xi$ , hence we see that  $i_\xi^! = i_\xi^*[-2 \dim \mathcal{O}_\xi]$ . Thus, we find that

$$\begin{aligned} H_{2d_\xi}^{BM}(\mathcal{B}_\xi) &= H^{-2d_\xi}(\mathcal{B}_\xi, i_\xi^! \underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[2 \dim \tilde{\mathcal{N}}]) \\ &= H^{-2d_\xi}(\mathcal{B}_\xi, \underline{\mathbb{C}}_{\mathcal{B}_\xi}[2 \dim \tilde{\mathcal{N}} - 2 \dim \mathcal{O}_\xi]) = H^{-\dim \mathcal{O}_\xi}(\mathcal{B}_\xi, \underline{\mathbb{C}}_{\mathcal{B}_\xi}[\dim \mathcal{N}]). \quad \square \end{aligned}$$

Given this decomposition, the  $W$ -action on  $R\pi_* \underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}]$  becomes a map

$$\mathbb{C}[W] \rightarrow \text{End}(R\pi_* \underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}]) = \bigoplus_{\xi, \chi} \text{End}(V_{\xi, \chi}), \quad (2)$$

where the equality follows because

$$\text{End}(IC(\overline{\mathcal{O}}_\xi, \mathcal{L}_{\xi, \chi}), IC(\overline{\mathcal{O}}_{\xi'}, \mathcal{L}_{\xi', \chi'})) = \begin{cases} 0 & (\xi, \chi) \neq (\xi', \chi') \\ \mathbb{C} & (\xi, \chi) = (\xi', \chi'). \end{cases}$$

**Theorem 1.7.** The map (2) is an isomorphism of algebras. In particular, the multiplicity spaces  $V_{\xi, \chi}$  form the complete set of irreducible representations of  $W$ .

**Remark.** Together, Proposition 1.6 and Theorem 1.7 imply that every irreducible representation of  $W$  is a direct summand of  $H_{2d_\xi}^{BM}(\mathcal{B}_\xi)$  for some  $\xi \in \mathcal{N}$ . Further, if  $\mathcal{O}_\xi$  is simply connected (which is the case for  $W = S_n$ ), then each  $H_{2d_\xi}^{BM}(\mathcal{B}_\xi)$  is itself irreducible.

*Proof of Theorem 1.7.* The proof consists of two steps. First, we show that (2) is injective. For this, observe that taking stalks at  $0 \in \mathcal{N}$  and applying proper base change yields a map

$$\mathbb{C}[W] \rightarrow \text{End}((R\pi_* \underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}])_0) = \text{End}(R\pi_* \underline{\mathbb{C}}_{\mathcal{B}}) \rightarrow \text{End}(H^*(\mathcal{B}, \underline{\mathbb{C}}_{\mathcal{B}}))$$

where we note that  $\mathcal{B} = \mathcal{B}_0 = \pi^{-1}(0)$ . Recall now that  $H^*(\mathcal{B}, \mathbb{C}_{\mathcal{B}})$  is isomorphic to the regular representation of  $W$ . We claim without proof that this action coincides with the action given by the map above, which would show that the composition is injective.<sup>1</sup>

To finish, we note that  $\pi_1(\mathcal{O}_{\xi}) \simeq C(\xi)$ . Further, because  $H_{2d_{\xi}}^{BM}(\mathcal{B}_{\xi})$  is top dimensional, it is generated by the fundamental classes of the irreducible components of  $\mathcal{B}_{\xi}$ , and the orbits of the monodromy representation of  $\pi_1(\mathcal{O}_{\xi})$  are exactly the orbits of the  $C(\xi)$  action on irreducible components. Recall by Proposition 1.6 that the irreducible components of  $Z_{\mathcal{O}_{\xi}}$  are in bijection with  $C(\xi)$ -orbits on pairs of irreducible components of  $\mathcal{B}_{\xi}$ . By the orbit stabilizer theorem, the number of such pairs is

$$\frac{1}{|C(\xi)|} \sum_{c \in C(\xi)} \text{Fix}_c(\text{Irred}(\mathcal{B}_{\xi}))^2 = \frac{1}{|C(\xi)|} \sum_{c \in C(\xi)} \text{Tr}|_{H_{2d_{\xi}}^{BM}(\mathcal{B}_{\xi})}(c)^2 = \|\text{ch}_{H_{2d_{\xi}}^{BM}(\mathcal{B}_{\xi})}\|^2 = \sum_{\chi} (\dim V_{\xi, \chi})^2.$$

Combining this with our classification of irreducible components of  $Z$ , we then have

$$|W| = \sum_{\xi, \chi} (\dim V_{\xi, \chi})^2,$$

which implies that  $\mathbb{C}[W]$  and  $\text{End}(R\pi_* \mathbb{C}_{\mathcal{N}}[\dim \mathcal{N}])$  have the same dimension, so (2) is an isomorphism.  $\square$

**Proposition 1.8.** The isomorphism of (2) is compatible with the isomorphism  $\mathbb{C}[W] \simeq H_{\dim \mathcal{N}}^{BM}(Z)$ .

*Proof.* We will give an isomorphism of vector spaces

$$H_{\dim \tilde{\mathcal{N}}}^{BM}(Z) \simeq \text{End}(R\pi_* \mathbb{C}_{\tilde{\mathcal{N}}}[\dim \tilde{\mathcal{N}}])$$

but omit the proof that it is compatible with the algebra structure (which may be found in Chriss-Ginzburg Chapter 8). Consider the Cartesian square

$$\begin{array}{ccc} Z & \xrightarrow{i} & \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \\ \pi \downarrow & & \pi \times \pi \downarrow \\ \mathcal{N} & \xrightarrow{\Delta} & \mathcal{N} \times \mathcal{N} \end{array}$$

Applying base change on this square and using Verdier duality, we have the chain of isomorphisms

$$\begin{aligned} H_{\dim \tilde{\mathcal{N}}}^{BM}(Z) &\simeq H^{-\dim \tilde{\mathcal{N}}}(Z, \mathbb{D}(Z)) \\ &\simeq H^0(Z, i^!(\mathbb{C}_{\tilde{\mathcal{N}}}[\dim \tilde{\mathcal{N}}] \boxtimes \mathbb{C}_{\tilde{\mathcal{N}}}[\dim \tilde{\mathcal{N}}])) \\ &\simeq H^0(Z, i^!(\mathbb{D}(\mathbb{C}_{\tilde{\mathcal{N}}}[\dim \tilde{\mathcal{N}}]) \boxtimes \mathbb{C}_{\tilde{\mathcal{N}}}[\dim \tilde{\mathcal{N}}])) \\ &\simeq H^0(\mathcal{N}, R\pi_* i^!(\mathbb{D}(\mathbb{C}_{\tilde{\mathcal{N}}}[\dim \tilde{\mathcal{N}}]) \boxtimes \mathbb{C}_{\tilde{\mathcal{N}}}[\dim \tilde{\mathcal{N}}])) \\ &\simeq H^0(\mathcal{N}, \Delta^! R(\pi \times \pi)_*(\mathbb{D}(\mathbb{C}_{\tilde{\mathcal{N}}}[\dim \tilde{\mathcal{N}}]) \boxtimes \mathbb{C}_{\tilde{\mathcal{N}}}[\dim \tilde{\mathcal{N}}])) \\ &\simeq H^0(\mathcal{N}, \Delta^!(R\pi_*(\mathbb{D}(\mathbb{C}_{\tilde{\mathcal{N}}})) \boxtimes R\pi_*(\mathbb{C}_{\tilde{\mathcal{N}}})) \\ &\simeq H^0(\mathcal{N}, \mathbb{D}(R\pi_*(\mathbb{C}_{\tilde{\mathcal{N}}})) \otimes R\pi_*(\mathbb{C}_{\tilde{\mathcal{N}}})) \\ &\simeq \text{End}(R\pi_*(\mathbb{C}_{\tilde{\mathcal{N}}}), R\pi_*(\mathbb{C}_{\tilde{\mathcal{N}}})) \end{aligned} \quad \square$$

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<sup>1</sup>Strictly speaking, this is cheating!