# GEOMETRIC REPRESENTATION THEORY OF THE HILBERT SCHEMES PART III

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Abstract. Realizing the fixed point basis in the equivariant cohomology of $(\mathbb{C}^2)^{[n]}$ as the Jack polynomials, we prove an equivariant version of the Lehn theorem for $X=\mathbb{C}^2$.

## 1. The first Chern class of the tautological bundle

Let $\mathcal{Z}_n\subset X^{[n]}\times X$ be the universal family over $X^{[n]}$ and $p$ denote its projection to $X^{[n]}$. Then $\mathcal{T}_n:=p_*\mathcal{O}(\mathcal{Z}_n)$ is a rank $n$ vector bundle over $X^{[n]}$, called the *tautological bundle.*[1] In this section we compute the cup product operator $c_1(\mathcal{T}_n)\cup\bullet: H^*_T(X^{[n]})\to H^*_T(X^{[n]})$. This operator was first studied in [L] (in the non-equivariant setting). Our exposition follows [N].

### 1.1. Eigenvectors of $c_1(\mathcal{T}_n)\cup$.

We start from a straightforward computation of $c_1(\mathcal{T}_n)\cup\bullet$ in the fixed point basis.

**Lemma 1.1.** *The operator $c_1(\mathcal{T}_n)\cup\bullet$ is diagonalizable in the fixed point basis:*

$$c_1(\mathcal{T}_n)\cup[\xi_\lambda]=-(n(\lambda)\epsilon_1+n(\lambda^*)\epsilon_2)[\xi_\lambda],$$

*where $n(\lambda):=\sum_i(i-1)\lambda_i$.*

*Proof.* By definition, we have $c_1(\mathcal{T}_n)\cup[\xi_\lambda]=c_1(\mathcal{T}_{n|\xi_\lambda})[\xi_\lambda]$. It remains to notice that $c_1(\mathcal{T}_{n|\xi_\lambda})=\sum_{i=1}^{l(\lambda)}\sum_{j=1}^{\lambda_i}(-(i-1)\epsilon_1-(j-1)\epsilon_2)=-n(\lambda)\epsilon_1-n(\lambda^*)\epsilon_2$. □

### 1.2. Laplace-Beltrami operator.

**Definition 1.1.** The linear operator $\square^k_N:\Lambda_N\to\Lambda_N$, defined by

$$\square^k_N(f)=\left(\frac{k}{2}\sum_{i=1}^N x_i^2\partial^2_{x_i}+\sum_{i\neq j}\frac{x_i^2}{x_i-x_j}\partial_{x_i}-r(N-1)\right),\ f\in\Lambda^r_N,$$

is called the *Laplace-Beltrami operator.*

**Exercise 1.2.** *Check $\rho_{N+1,N}\circ\square^k_{N+1}=\square^k_N\circ\rho_{N+1,N}$.*

Hence, we can define a linear operator

$$\square^k:\Lambda\to\Lambda,\ \square^k:=\lim_{\longleftarrow}\square^k_N.$$

Those operators are actually diagonalizable in the basis of Jack polynomials:

**Proposition 1.3.** [M, Exercise VI.4.3(b)] *We have: $\square^k(P^{(k)}_\lambda)=(n(\lambda^*)k-n(\lambda))\cdot P^{(k)}_\lambda$.*

[1] The fiber of $\mathcal{T}_n$ at the codimension $n$ ideal $I\subset\mathbb{C}[x,y]$ is identified with $\mathbb{C}[x,y]/I$. Moreover, its determinant $\wedge^n\mathcal{T}_n$ is actually the line bundle $\mathcal{O}_{(\mathbb{C}^2)^{[n]}}(1)$ arising from the Proj-construction of $(\mathbb{C}^2)^{[n]}$.

1.3. **Geometric interpretation of $\Box^k$.**

Let $\theta^T:\Lambda_{\mathbb{F}}\xrightarrow{\sim}M^T_{\mathrm{loc}}=\oplus H^{T,BM}_*(X^{[n]})_{\mathrm{loc}}$ be the isomorphism from the last talk. Identifying $H^{T,BM}_i(X^{[n]})$ with $H^{4n-i}_T(X^{[n]})$, consider a linear operator $D:\Lambda_{\mathbb{F}}\to\Lambda_{\mathbb{F}}$ which corresponds to $c_1(\mathcal{T}_n)\cup\bullet: H^*_T(X^{[n]})\to H^*_T(X^{[n]})$ under this isomorphism.

**Theorem 1.4.** *We have: $D=\epsilon_1\cdot\Box^k$.*

*Proof.* According to the main result from the last time, we have:

$$\theta^T: P^{(k)}_\lambda\mapsto \epsilon_1^{-|\lambda|}c_\lambda(k)^{-1}\cdot[\xi_\lambda],\ k=-\epsilon_2/\epsilon_1.$$

Therefore $D$ is determined by the condition $D(P^{(k)}_\lambda)=\epsilon_1(n(\lambda^*)k-n(\lambda))P^{(k)}_\lambda$. Combining with Proposition 1.3, we get the result. □

The following is straightforward (see Appendix for the proof):

**Corollary 1.5.** *Identifying $\Lambda_{\mathbb{C}}\simeq\mathbb{C}[p_1,p_2,\ldots]$, the operator $\Box^k$ is given by*

$$\Box^k=\frac{k}{2}\sum_{m,n>0}mnp_{m+n}\partial_{p_m}\partial_{p_n}+\frac{k-1}{2}\sum_{m>0}m(m-1)p_m\partial_{p_m}+\frac{1}{2}\sum_{m,n>0}(m+n)p_mp_n\partial_{p_{m+n}}.$$

1.4. **Lehn's formula.**

In this section we reformulate Corollary 1.5 in a more standard form.

Recall that under the isomorphism $\theta^T:\Lambda_{\mathbb{F}}\xrightarrow{\sim}M^T_{\mathrm{loc}}$, the operators $p_m$ and $-m\partial_{p_m}$ correspond to $\mathfrak{q}_{\epsilon_2}[-m]=Z_{\epsilon_2}[-m]$ and $\mathfrak{q}_{\epsilon_1}[m]=\frac{(-1)^m}{k}Z_{\epsilon_2}[m]=(-1)^{m-1}Z_{\epsilon_1}[m]$, respectively.

Hence, the operator $c_1(\mathcal{T}_n)\cup\bullet$ is given by the following formula:

$$c_1(\mathcal{T}_n)\cup\bullet=\frac{\epsilon_1+\epsilon_2}{2}\sum_{m>0}(m-1)\mathfrak{q}_{\epsilon_2}[-m]\mathfrak{q}_{\epsilon_1}[m]-$$

$$\sum_{m,n>0}\left(\frac{\epsilon_2}{2}\mathfrak{q}_{\epsilon_2}[-m-n]\mathfrak{q}_{\epsilon_1}[m]\mathfrak{q}_{\epsilon_1}[n]+\frac{\epsilon_1}{2}\mathfrak{q}_{\epsilon_2}[-m]\mathfrak{q}_{\epsilon_2}[-n]\mathfrak{q}_{\epsilon_1}[m+n]\right).$$

Let us now introduce $\delta_T:H^*_T(X)\to H^*_T(X)\otimes H^*_T(X)$ as the adjoint of the cup product $\cup: H^*_T(X)\otimes H^*_T(X)\to H^*_T(X)$ with respect to the intersection pairing. In other words, $\delta_T$ is a push-forward along the diagonal embedding $X\to X\times X$. This is a $H^*_T(\mathrm{pt})$-linear map with $\delta_T(1)=1\otimes[X]=\epsilon_1\epsilon_2\cdot 1\otimes 1$. Iterating $\delta_T$, we get $\delta^r_T(1)=(\epsilon_1\epsilon_2)^r\cdot 1\otimes\cdots\otimes 1$.

For $\alpha\in H^*_T(X)$ with $\delta_T(\alpha)=\sum_i\alpha^1_i\otimes\alpha^2_i$, we set:

$$(\mathfrak{q}_m\mathfrak{q}_n)(\alpha):=\sum\mathfrak{q}_{\alpha^1_i}[m]\mathfrak{q}_{\alpha^2_i}[n].$$

Using this notation together with $K_X=-\epsilon_1-\epsilon_2$ ($K_{\mathbb{C}^2}$ is generated by $dx\wedge dy$), we get:

**Theorem 1.6.** [L] *We have*

$$c_1(\mathcal{T}_n)\cup\bullet=-\frac{1}{6}\sum_{m_1+m_2+m_3=0}:\mathfrak{q}_{m_1}\mathfrak{q}_{m_2}\mathfrak{q}_{m_3}:(1)-\frac{1}{4}\sum_m(|m|-1):\mathfrak{q}_{-m}\mathfrak{q}_m:(K_X),$$

*where* $:\ :$ *denotes the normal ordering.*

This beautiful result was first proved by Lehn ([L]) in the non-equivariant setting for any $X$. The key observation of [L] was a geometric action of Vir on $M$ discussed in the next section.

1.5. **Virasoro action on $M$.**

Let us first introduce another important Lie algebra:

**Definition 1.2.** The complex Lie algebra Vir with a basis $\{L_n, n \in \mathbb{Z}, c\}$ and a Lie bracket

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta^0_{n+m},\ [c, L_n] = 0,\ n, m \in \mathbb{Z},$$

is called the *Virasoro algebra*. Its representation $V$ is of *central charge* $c_0 \in \mathbb{C}$ if $c_{|V} = c_0 \cdot \mathrm{Id}_V$.

Define operators $\mathcal{L}_n : H^*(X) \to \mathrm{End}(M)$ by $\mathcal{L}_n(\alpha) := \frac{1}{2}\sum_{l \in \mathbb{Z}} : \mathfrak{q}_l \mathfrak{q}_{n-l} : (\alpha)$. According to [L, Theorem 3.3], those operators satisfy the following commutator relation:

$$[\mathcal{L}_n(\alpha), \mathcal{L}_m(\beta)] = (n-m)\mathcal{L}_{n+m}(\alpha \cup \beta) - \frac{n^3 - n}{12}\delta^0_{n+m} \cdot \langle c_2(X), \alpha\beta\rangle \cdot \mathrm{Id}_M\,. \tag{1}$$

**Corollary 1.7.** *The operators $\{\mathcal{L}_n(1)\}$ define an action of the Virasoro algebra* Vir *on $M$ of central charge $-e(X)$ ($e(X)$ is the Euler number of $X$).*

*Remark* 1.1. This result can be considered as a slight update of the classical Vir-action on the Fock space over the Heisenberg algebra $\mathcal{H}$ (see [KR, Proposition 2.3]).

In [L], Theorem 1.6 is derived from the following commutator formula:

$$[c_1(\mathcal{T}_n) \cup \bullet, \mathfrak{q}_\alpha[n]] = n \cdot \mathcal{L}_n(\alpha) + \frac{n(|n|-1)}{2}\mathfrak{q}_{K_X \cup \alpha}[n]. \tag{2}$$

We refer the reader to [L] for more details on this elegant result.

## Appendix A. Proof of Corollary 1.5

In this section we prove Corollary 1.5, that is

$$\square^k = \frac{k}{2}\sum_{m,n>0} mnp_{m+n}\partial_{p_m}\partial_{p_n} + \frac{k-1}{2}\sum_{m>0} m(m-1)p_m\partial_{p_m} + \frac{1}{2}\sum_{m,n>0}(m+n)p_m p_n \partial_{p_{m+n}}.$$

It suffices to check this on the basis element $p_\lambda = p_{\lambda_1}p_{\lambda_2}\dots p_{\lambda_s}$. We also work with $\Lambda_N, N \gg 1$, so that the equality in $\Lambda$ is obtained as the limit. Applying the differential operator on the right hand side to $p_\lambda$ we obtain:

$$k\sum_{1\le i<j\le s}\lambda_i\lambda_j p_{\lambda_i+\lambda_j}p_{\lambda_1}\dots\widehat{p_{\lambda_i}}\dots\widehat{p_{\lambda_j}}\dots p_{\lambda_s} + \frac{k-1}{2}\sum_{1\le i\le s}\lambda_i(\lambda_i-1)p_{\lambda_1}\dots p_{\lambda_s} + \sum_{1\le i\le s}\frac{\lambda_i}{2}\sum_{c,d>0}^{c+d=\lambda_i} p_c p_d p_{\lambda_1}\dots\widehat{p_{\lambda_i}}\dots p_{\lambda_s}. \tag{3}$$

Let us now compute $\square^k_N(p_\lambda)$, where we expand $p_\lambda$ as $p_\lambda = (\sum_{j_1} x_{j_1}^{\lambda_1})\cdot\ldots\cdot(\sum_{j_s} x_{j_s}^{\lambda_s})$:

$$\left(\frac{k}{2}\sum_{1\le r\le s}\lambda_r(\lambda_r-1)p_\lambda + k\sum_{1\le r_1<r_2\le s}\lambda_{r_1}\lambda_{r_2}p_{\lambda_{r_1+r_2}}p_{\lambda_1}\dots\widehat{p_{\lambda_{r_1}}}\dots\widehat{p_{\lambda_{r_2}}}\dots p_{\lambda_s}\right) + \sum_{1\le r\le s}\lambda_r\sum_{1\le i\le N}\sum_{1\le j\le N}^{j\ne i}\frac{x_i^{\lambda_r+1}}{x_i - x_j}p_{\lambda_1}\dots\widehat{p_{\lambda_r}}\dots p_{\lambda_s} - (\lambda_1+\ldots+\lambda_s)(N-1)p_\lambda. \tag{4}$$

To see that (4) simplifies to (3), use the following identity:

$$\sum_{1\le i\ne j\le N}\frac{x_i^{t+1}}{x_i - x_j} = \sum_{1\le i<j\le N}(x_i^t + x_i^{t-1}x_j + \ldots + x_i x_j^{t-1} + x_j^t) = (N-1)p_t + \frac{1}{2}\sum_{c,d>0}^{c+d=t}p_c p_d - \frac{t-1}{2}p_t.$$

## References


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