REPRESENTATIONS OF QUANTIZATIONS

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1. Modules over quantizations

1.1. **Algebra case.** Let $\mathcal{A}$ be a filtered quantization of a $\mathbb{Z}$-graded finitely generated Poisson algebra A. By $\mathcal{A}$-mod we denote the category of finitely generated $\mathcal{A}$-modules.

A basic tool to study such modules is to reduce them to finitely generated A-modules that can be studied by means of Commutative algebra/ Algebraic geometry. Given an $\mathcal{A}$-module M, one introduces the notion of a *good filtration* $M = \bigcup_{i\in\mathbb{Z}} M_{\leqslant i}$: this is a complete and separated $\mathcal{A}$-module filtration on M such that $\operatorname{gr} M$ is a finitely generated A-module. Note that if $\mathcal{A}$ is $\mathbb{Z}_{\geqslant 0}$-filtered, then any good filtration on M is bounded from below.

A good filtration exists if and only if the module is finitely generated. That a module with a good filtration is finitely generated is an exercise. Let us produce a good filtration on a finitely generated module. Choose generators $m_1,\ldots,m_k$ and integers $d_1,\ldots,d_k$. Set $M_{\leqslant n} := \sum_{j=1}^k \mathcal{A}_{\leqslant n-d_k} m_k$. This is a good filtration. Indeed, we have an epimorphism $\mathcal{A}^{\oplus k} \twoheadrightarrow M$ defined by the generators. We equip each summand $\mathcal{A}$ with the original filtration shifted by d_i. Then the filtration on M is the induced filtration on the quotient, which easily shows that the filtration is good. In fact, any good filtration on M has this form.

The construction also implies that there are many good filtrations. However, despite this fact any two of them are "not very far from one another".

Lemma 1.1. *Let $M = \bigcup_{i\in\mathbb{Z}} M_{\leqslant i} = \bigcup_{i\in\mathbb{Z}} M_{\preceq i}$ be two good filtrations. Then there are integers a, b such that $M_{\leqslant i+a} \subset M_{\preceq i} \subset M_{\leqslant i+b}$ for all i.*

This lemma allows to prove that some invariants of $\operatorname{gr} M$ are independent of the choice of a good filtration. For example, the support of $\operatorname{gr} M$, a closed subvariety of $\operatorname{Spec}(A)$ (defined by the annihilator of $\operatorname{gr} M$ in A), is independent of the choice of a good filtration. This will be denoted by $\operatorname{Supp} M$. Now fix a closed subvariety $Y \subset \operatorname{Spec}(A)$. Consider the category $\mathcal{A}\text{-mod}_Y$ of all modules M with $\operatorname{Supp} M \subset Y$. Then the assignment $M \to \operatorname{gr} M$ gives rise to a well-defined map $K_0(\mathcal{A}\text{-mod}_Y) \to K_0(A\text{-mod}_Y)$ (we would like to emphasize that although the associated graded module is graded, the class in the K_0 of graded modules is not well-defined), compare to [CG, Section 2.3]. This has a formal consequence: the characteristic cycle (a refinement of the support) of an $\mathcal{A}$-module is well-defined. Namely, let M' be a finitely generated A-module. Let Y' be its support and $Y'_1,\ldots,Y'_k$ be the irreducible components of Y'. Then we set

$$\mathsf{CC}(M') = \sum_{\ell=1}^{k} \operatorname{rk}_{Y'_\ell} M' \cdot Y'_\ell,$$

where $\operatorname{rk}_{Y'_\ell} M'$ denote the rank of M' in the generic point of Y'_ℓ. From the claim that $M \mapsto \operatorname{gr} M$ is well-defined on the level of K_0's, we conclude that $\mathsf{CC}(\operatorname{gr} M)$ is independent of the choice of a good filtration. We write $\mathsf{CC}(M)$ for $\mathsf{CC}(\operatorname{gr} M)$.

1.2. **Sheaf case.** Now let X be a normal Poisson variety (with a $\mathbb{C}^\times$-action rescaling the symplectic structure) and $\mathcal{D}$ be its filtered quantization. We are going to define the notion of a coherent $\mathcal{D}$-module (that will be a sheaf in the conical topology).

As was explained in the previous section, our basic tool to study modules over a quantization $\mathcal{A}$ of an algebra A is to reduce them to finitely generated A-modules by means of a good filtration. We give a definition of a coherent $\mathcal{D}$-module in such a way that this reduction becomes possible.

Definition 1.2. We say that a $\mathcal{D}$-module M is coherent if it is equipped with a global complete and separated filtration such that $\operatorname{gr} M$ is a coherent $\mathcal{O}_X$-module (this filtration is called good).

The category of coherent $\mathcal{D}$-modules (where morphisms are the morphisms of sheaves of $\mathcal{D}$-modules) will be denoted by $\operatorname{Coh}(\mathcal{D})$.

The following lemma establishes basic properties of coherent $\mathcal{D}$-modules (that mirror properties of coherent sheaves in Algebraic geometry). To state the lemma we need the notion of a morphism $(f,\iota):(X,\mathcal{D}^X)\to(Y,\mathcal{D}^Y)$. Here f is a $\mathbb{C}^\times$-equivariant morphism $X\to Y$ of algebraic varieties and ι is a morphism $\mathcal{D}^Y\to f_\bullet\mathcal{D}^X$ of sheaves of filtered algebras on Y (where we write $f_\bullet$ for the sheaf-theoretic push-forward) whose associated graded is the morphism $\mathcal{O}_Y\to f_\bullet\mathcal{O}_X$ that is a part of the morphism f.

Lemma 1.3. *The following is true.*

- (a) *Let X be affine. Then the functors $M\mapsto M^{loc}:=\mathcal{D}\otimes_{\mathcal{A}}M$ and $N\mapsto\Gamma(N)$ are mutually inverse equivalences between $\mathcal{A}$-mod and $\operatorname{Coh}(\mathcal{D})$.*
- (b) *A submodule and a quotient of a coherent $\mathcal{D}$-module are coherent.*
- (c) *Let f be a morphism $(X,\mathcal{D}^X)\to(Y,\mathcal{D}^Y)$. Then there is a pull-back functor $f^*:\operatorname{Coh}(\mathcal{D}^Y)\to\operatorname{Coh}(\mathcal{D}^X)$ given by $M\mapsto\mathcal{D}^X\otimes_{f^\bullet\mathcal{D}^Y}f^\bullet M$, where $f^\bullet$ is the sheaf theoretic pull-back.*

Note that (c) will be the main source of coherent modules over quantizations of non-affine varieties. We will use it when $\mathbb{C}[X]$ is finitely generated, $Y=\operatorname{Spec}(\mathbb{C}[X])$ and $\mathcal{D}^Y=\Gamma(\mathcal{D}^X)$.

Proof. Let us prove (a). Note that $\operatorname{gr}(M^{loc})$ is the coherent sheaf on X associated to $\operatorname{gr} M$ and $\operatorname{gr}\Gamma(N)=\Gamma(\operatorname{gr} N)$, the latter is true because $H^1(X,\operatorname{gr} N)=0$. This shows that the natural homomorphisms $M\mapsto\Gamma(M^{loc}),\Gamma(N)^{loc}\to N$ are isomorphisms after passing to the associated graded modules. Hence these natural homomorphisms are isomorphisms themselves because all the filtrations involved are complete and separated.

Let us prove (b). Let $M'\subset M$ be a submodule and M be coherent. Then we can restrict the filtration from M to M'. For an open affine subspace U, we have $\Gamma(U,M')\subset\Gamma(U,M)$ and $\Gamma(U,M)$ is a finitely generated $\Gamma(U,\mathcal{D})$-module with a good filtration. It follows that $\Gamma(U,M')$ is closed (compare to the case of left ideals from the previous lecture, Exercise 2.3 there) and from here one deduces that the filtration on $\Gamma(U,M')$ is complete and separated. So the filtration on M' is complete and separated. Besides, $\operatorname{gr} M'\subset\operatorname{gr} M$ and so $\operatorname{gr} M'$ is coherent. So M' is coherent. To show that M/M' is coherent we notice that it inherits a (global) filtration and, by (a), $M/M'|_U$ is coherent for every open affine U. From here we deduce that M/M' is coherent.

To prove (c), notice that f^*M comes with a natural global filtration. Locally, it is a quotient of a free finitely generated module with induced filtration. So the filtration on f^*M is complete and separated and the associated graded is coherent. $\square$

Let us proceed to quasi-coherent $\mathcal{D}$-modules. By definitions, those are unions of their coherent submodules. Here are their basic properties.

Lemma 1.4. *The following is true.*

1. *The direct analogs of (a)-(c) of Lemma 1.3 hold.*
2. *In the notation of (c) of Lemma 1.3, we have the push-forward functor* f_* : $\operatorname{QCoh}(\mathcal{D}^X) \to \operatorname{QCoh}(\mathcal{D}^Y)$ *(that coincides with the sheaf theoretic push-forward). If f is proper, then this functor restricts to* $\operatorname{Coh}(\mathcal{D}^X) \to \operatorname{Coh}(\mathcal{D}^Y)$.
3. *The category* $\operatorname{QCoh}(\mathcal{D})$ *contains enough injectives.*
4. *The natural morphism* $D^?(\operatorname{Coh}(\mathcal{D})) \to D^?(\operatorname{QCoh}(\mathcal{D}))$ *(where ? is either a + or a −) is a full embedding.*

Proof. Let us prove (1). The analog of (a) of Lemma 1.3 holds because the localization and global section functors commute with taking unions. The analog of (b) is straightforward and (c) follows because tensor products commute with direct limits.

Let us prove (2). The push-forward commutes with taking unions. So it is enough to show that f_*M is quasi-coherent if M is coherent. We can cover X with open subsets X_i such that $f^i := f|_{X_i} : X_i \to Y$ is affine. Then f_*M is the kernel of $\bigoplus_i f^i_* M \to \bigoplus_{i\neq j} f^{ij}_* M$, where f^{ij} is the restriction of f to $X_i \cap X_j$. So in the proof it is enough to assume that f is affine. Moreover, we can assume that $f = \iota \circ g$, where g is a morphism of affine varieties and ι is an open embedding of an affine variety. It is clear that g_* maps quasi-coherent sheaves to quasi-coherent ones. It remains to show that ι_* maps coherent sheaves to quasi-coherent ones. We note that the sheaves $\iota_* M$ are generated by their global sections, hence are quotients of $(\mathcal{D}^Y)^{\oplus ?}$ and hence are quasi-coherent. This completes the proof of the claim that f_* maps quasi-coherent sheaves to quasi-coherent ones.

Now assume that f is proper. Note that a choice of a filtration on $M \in \operatorname{Coh}(\mathcal{D}^X)$ gives rise to a filtration on f_*M. Moreover, $\operatorname{gr}(f_*M) \subset f_*(\operatorname{gr} M)$. The latter is a coherent sheaf. From here one deduces that f_*M is coherent.

Let us prove (3). Recall that the category of modules over a ring contains enough injectives. Now we can cover X with an open affine $\mathbb{C}^\times$-stable subsets, $X = \bigcup_k X^k$, let ι_k denote the inclusion $X^k \hookrightarrow X$. Let $\mathcal{I}^k$ be an injective hull of $\Gamma(M|_{X^i})$. Then $\bigoplus_k \iota_{k*}\mathcal{I}^k$ is an injective hull of M.

(4) is a formal corollary of the claim that every quasi-coherent module is the union of its coherent submodules (and so in every complex with coherent homology we can produce a quasi-isomorphic subcomplex with coherent terms). □

Let us discuss supports. If $M \in \operatorname{Coh}(\mathcal{D})$, then the notion of the support still makes sense and it is a closed $\mathbb{C}^\times$-stable subvariety of X (characteristic cycles makes sense as well). The following result, known as the Gabber involutivity theorem (see [Ga] or [Gi, Section 1.2]), is of fundamental importance. Recall that a subvariety Y in a symplectic variety X is called coisotropic if T_yY contains its orthogonal complement for every smooth point $y \in Y$.

Theorem 1.5. $\operatorname{Supp} M$ *is a coisotropic subvariety in* X.

A module $M \in \operatorname{Coh}(\mathcal{D})$ is called *holonomic* if its support is lagrangian (=coisotropic of dimension $\frac{1}{2}\dim X$).

1.3. **Hamiltonian reductions.** We now concentrate on the categories of coherent modules over Hamiltonian reductions $\mathcal{A}^0_\lambda := D(R)///_\lambda G$, $\mathcal{A}^\theta_\lambda := D_R///^\theta_\lambda G$, where θ is generic (recall that this means that the G-action on $\mu^{-1}(0)^{\theta-ss}$ is free). We will relate these categories to the category of (G,λ)-equivariant finitely generated $D(R)$-modules. The equivariance condition means the following. Suppose that we have a $D(R)$-module M equipped with a rational (a.k.a. algebraic) G-action. This gives rise to a map $\xi \mapsto \xi_M : \mathfrak{g} \to \operatorname{End}(M)$. On the other hand, $\mathfrak{g}$ acts on M by left multiplications by the elements ξ_R. We say that M is (G,λ)-equivariant if $\xi_M = \xi_R - \langle \lambda, \xi\rangle$ (for $\lambda = 0$ we get the usual notion of an equivariant D-module). The category of all (G,λ)-equivariant finitely generated $D(R)$-modules will be denoted by $D(R)\operatorname{-mod}^{G,\lambda}$.

Note that the category $D(R)\operatorname{-mod}^{G,\lambda}$ can be thought as a quantum analog of the category of G-equivariant coherent sheaves on $\mu^{-1}(0)$. Indeed, on a (G,λ)-equivariant module M we can pick a G-stable good filtration. The multiplication by ξ_R preserves the filtration degree and so $\operatorname{gr} M$ is a $\mathbb{C}[T^*R]/\mathbb{C}[T^*R]\mu^*(\mathfrak{g}) = \mathbb{C}[\mu^{-1}(0)]$-module ($G$-equivariant by the construction).

Let us produce a quotient functor $D(R)\operatorname{-mod}^{G,\lambda} \to \mathcal{A}^0_\lambda\operatorname{-mod}$. The functor is $M \mapsto M^G$. Let us check that M^G is a module over $D(R)///_\lambda G$ (a priori, it is only a $D(R)^G$-module, and $\mathcal{A}^0_\lambda = D(R)^G/(D(R)\mathcal{I}_\lambda)^G$, where, recall, $\mathcal{I}_\lambda := D(R)\{\xi_R - \langle\lambda,\xi\rangle\}$). If $m \in M^G$, then $\xi_M m = 0$. This means that $(\xi_R - \langle \lambda,\xi\rangle)m = 0$ so the $D(R)^G$-action on m factors through $\mathcal{A}^0_\lambda$. The claim that M^G is finitely generated is established as follows. It is enough to prove that $(\operatorname{gr} M)^G$ is finitely generated over $\mathbb{C}[T^*R]^G$, that follows from GIT.

Let us produce a right inverse functor, this will show that $M \mapsto M^G$ is a quotient functor. Note that $B := D(R)/\mathcal{I}_\lambda$ is a $D(R)$-$\mathcal{A}^0_\lambda$-bimodule. So we have a functor $\mathcal{A}^0_\lambda\operatorname{-mod} \to D(R)\operatorname{-Mod}$ given by

$$\kappa : N \mapsto B \otimes_{\mathcal{A}^0_\lambda} N$$

Note that $\kappa(N)$ carries a natural rational G-action (on the first factor). Moreover, it is easy to see that $B \in D(R)\operatorname{-mod}^{G,\lambda}$ (the operator ξ_B is induced from $[\xi_R,\cdot] = [\xi_R - \langle\lambda,\xi\rangle,\cdot]$ on $D(R)$). It follows that $\kappa(N) \in D(R)\operatorname{-mod}^{G,\lambda}$. Also note that

$$\kappa(N)^G = B^G \otimes_{\mathcal{A}^0_\lambda} N = N.$$

So we see that $\mathcal{A}^0_\lambda\operatorname{-mod}$ is the quotient category of $D(R)\operatorname{-mod}^{G,\lambda}$ (by the Serre subcategory of all modules without G-invariants).

Let us proceed to $\operatorname{Coh}(\mathcal{A}^\theta_\lambda)$. Here we consider the case when $R = R(Q,v,w)$ is a coframed representation space of a quiver Q.

Lemma 1.6 (Proposition 2.8 in [BL]). *The category $\operatorname{Coh}(\mathcal{A}^\theta_\lambda)$ is naturally identified with the quotient of $D(R)\operatorname{-mod}^{G,\lambda}$ by the full subcategory of all modules with θ-unstable support (meaning that the support does not intersect $(T^*R)^{\theta-ss}$).*

To understand the claim of Lemma 1.6 better, consider the commutative situation. We have the restriction functor $\operatorname{Coh}^G(\mu^{-1}(0)) \to \operatorname{Coh}^G(\mu^{-1}(0)^{\theta-ss})$, which is a quotient functor. Since the G-action on $\mu^{-1}(0)^{\theta-ss}$ is free, we see that the functor $M \mapsto \pi_*(M)^G$, where $\pi : \mu^{-1}(0)^{\theta-ss} \to T^*R///^\theta_0 G$ is the quotient morphism, gives a category equivalence $\operatorname{Coh}^G(\mu^{-1}(0)^{\theta-ss}) \xrightarrow{\sim} \operatorname{Coh}(\mu^{-1}(0)^{\theta-ss}/G)$.

On the non-commutative level, consider the category $\operatorname{Coh}^{G,\lambda}(D_R|_{(T^*R)^{\theta-ss}})$ of (G,λ)-equivariant objects in $\operatorname{Coh}(D_R|_{(T^*R)^{\theta-ss}})$. We have the exact restriction functor

$$D(R)\operatorname{-mod}^{G,\lambda} \to \operatorname{Coh}^{G,\lambda}(D_R|_{(T^*R)^{\theta-ss}}).$$

Note that the support of an object in $\mathrm{Coh}^{G,\lambda}(D_R|_{(T^*R)^{\theta-ss}})$ lies in $\mu^{-1}(0)^{\theta-ss}$, by the same reasons as above. Now we have an equivalence $\mathrm{Coh}^{G,\lambda}(D_R|_{(T^*R)^{\theta-ss}}) \to \mathrm{Coh}(\mathcal{A}^\theta_\lambda)$ that sends an object M to $\pi_*(M)^G$.

1.4. **Translation equivalences.** The description of $\mathrm{Coh}(\mathcal{A}^\theta_\lambda)$ as a quotient of $D(R)\text{-mod}^{G,\lambda}$ has an important corollary. Let χ be a character of G. Then the categories $D(R)\text{-mod}^{G,\lambda}$ and $D(R)\text{-mod}^{G,\lambda+\chi}$ are equivalent via the twist of a G-action by χ (the $D(R)$-action stays the same). This clearly does not change the support and so induces an equivalence $\mathrm{Coh}(\mathcal{A}^\theta_\lambda) \xrightarrow{\sim} \mathrm{Coh}(\mathcal{A}^\theta_{\lambda+\chi})$. This functor can equivalently be described as follows. There is a natural $\mathcal{A}^\theta_{\lambda+\chi}$-$\mathcal{A}^\theta_\lambda$-bimodule to be denoted by $\mathcal{A}^\theta_{\lambda,\chi}$ that quantizes the line bundle $\mathcal{O}(\chi)$. Before we discuss this bimodule, let us describe its global analog, the $\mathcal{A}^0_{\lambda+\chi}$-$\mathcal{A}^0_\lambda$-bimodule $\mathcal{A}^0_{\lambda,\chi}$. It is defined by $\mathcal{A}^0_{\lambda,\chi} := [D(R)/\mathcal{I}_\lambda]^{G,\chi}$ (where the superscript G,χ indicates that we take χ-semiinvariants for the G-action). It is clearly a right $\mathcal{A}^0_\lambda$-module. To check that it is a left $\mathcal{A}_{\lambda+\chi}$-module we need to show that $(\xi - \langle\lambda+\chi,\xi\rangle)a \in \mathcal{I}_\lambda$ for $a+\mathcal{I}_\lambda \in [D(R)/\mathcal{I}_\lambda]^{G,\chi}$. The inclusion implies $[\xi,a]-\langle\chi,\xi\rangle a \in \mathcal{I}_\lambda$. So $(\xi-\langle\lambda+\chi,\xi\rangle)a = [\xi,a]-\langle\chi,\xi\rangle a + a(\xi-\langle\lambda,\xi\rangle)$ definitely lies in $\mathcal{I}_\lambda$.

The bimodule $\mathcal{A}^\theta_{\lambda,\chi}$ is defined similarly:

$$\mathcal{A}^\theta_{\lambda,\chi} := \pi_*(D_R/\mathcal{I}_\lambda|_{(T^*R)^{\theta-ss}})^{G,\chi}.$$

It is a $\mathcal{A}^\theta_{\lambda+\chi}$-$\mathcal{A}^\theta_\lambda$-bimodule for the same reason as before. Also it quantizes $\mathcal{O}(\chi)$. Note that we have a natural homomorphism $\mathcal{A}^\theta_{\lambda+\chi,\chi'} \otimes_{\mathcal{A}^\theta_{\lambda+\chi}} \mathcal{A}^\theta_{\lambda,\chi} \to \mathcal{A}^\theta_{\lambda,\chi+\chi'}$ induced by the multiplication in $D_R|_{T^*R^{\theta-ss}}$. On the level of the associated graded modules, it is the natural isomorphism $\mathcal{O}(\chi')\otimes\mathcal{O}(\chi) \xrightarrow{\sim} \mathcal{O}(\chi+\chi')$. So our initial homomorphism is an isomorphism as well. This implies that $\mathcal{A}^\theta_{\lambda,\chi}$ is an invertible bimodule, the inverse is $\mathcal{A}^\theta_{\lambda+\chi,-\chi}$. The equivalence $\mathcal{A}^\theta_{\lambda,\chi}\otimes_{\mathcal{A}^\theta_\lambda}\bullet : \mathrm{Coh}(\mathcal{A}^\theta_\lambda) \to \mathrm{Coh}(\mathcal{A}^\theta_{\lambda+\chi})$ coincides with the equivalence via quotients explained above.

2. Localization theorems

Again, we deal with $R = R(Q,v,w)$ and $G = \mathrm{GL}(v)$.

Let us write $\mathcal{A}^\theta_\lambda$ for $D_R///^\theta_\lambda G$ and $\mathcal{A}_\lambda$ for $\Gamma(\mathcal{A}^\theta_\lambda)$. Recall that the higher cohomology of $\mathcal{A}^\theta_\lambda$ vanish and $\mathcal{A}_\lambda$ is a quantization of $\mathcal{M}(v) := \mathrm{Spec}(\mathbb{C}[\mathcal{M}^\theta(v)])$. Let us write $\varphi : \mathcal{M}^\theta(v) \to \mathcal{M}(v)$ for the resolution of singularities morphism.

We would like to compare the categories $\mathrm{Coh}(\mathcal{A}^\theta_\lambda)$ and $\mathcal{A}_\lambda\text{-mod}$. Note that we have a functor $\Gamma : \mathrm{Coh}(\mathcal{A}^\theta_\lambda) \to \mathcal{A}_\lambda\text{-mod}$ of taking global sections (we will write Γ^θ_λ if we want to indicate the dependence on λ,θ), this is the same as the pushforward φ_*. Equivalently, the functor can be given by $\mathrm{Hom}_{\mathrm{Coh}(\mathcal{A}^\theta_\lambda)}(\mathcal{A}^\theta_\lambda,\bullet)$. It follows the functor Γ^θ_λ has a left adjoint functor $\mathrm{Loc}^\theta_\lambda := \mathcal{A}^\theta_\lambda \otimes_{\mathcal{A}_\lambda} \bullet$ (the pullback functor φ^*).

If Γ^θ_λ is an equivalence (in which case, $\mathrm{Loc}^\theta_\lambda$ is automatically a quasi-inverse equivalence) we will say that *abelian localization holds for* (λ,θ).

We also have the derived versions of the functors $\Gamma^\theta_\lambda, \mathrm{Loc}^\theta_\lambda$. The derived functor $R\Gamma^\theta_\lambda : D^+(\mathrm{Coh}(\mathcal{A}^\theta_\lambda)) \to D^+(\mathcal{A}_\lambda\text{-mod})$ can be defined using an injective resolution in $\mathrm{QCoh}(\mathcal{A}^\theta_\lambda)$. But also it can be given by taking the Čech complex (as in Algebraic geometry). In particular, it restricts to $D^b(\mathrm{Coh}(\mathcal{A}^\theta_\lambda)) \to D^b(\mathcal{A}_\lambda\text{-mod})$. Also we have the derived localization functor $L\,\mathrm{Loc}^\theta_\lambda : \mathcal{A}^\theta_\lambda \otimes^L_{\mathcal{A}_\lambda} \bullet : D^-(\mathcal{A}_\lambda\text{-mod}) \to D^-(\mathrm{Coh}(\mathcal{A}^\theta_\lambda))$ (computed using a free resolution). We say that *derived localization holds for* (λ,θ) if $R\Gamma^\theta_\lambda$ and $L\operatorname{Loc}^\theta_\lambda$ are quasi-inverse equivalences (between the bounded derived categories).

2.1. **Abelian localization.** Here we want to produce a sufficient condition for the abelian localization to hold. Our result should be thought as a weaker analog of the Beilinson-Bernstein localization theorem from the representation theory of semisimple Lie algebras.

First of all, let us explain what it means for the abelian localization to hold in more pedestrian terms.

Lemma 2.1. *The following conditions are equivalent.*

(i) *Abelian localization holds for* (λ,θ).
(ii) *The functor* $\operatorname{Loc}^\theta_\lambda$ *is essentially surjective and* Γ^θ_λ *is exact.*
(iii) *Any* $M\in\operatorname{Coh}(\mathcal{A}^\theta_\lambda)$ *is generated by its global sections and has vanishing higher cohomology.*

(ii) and (iii) are tautologically equivalent and (i) implies (ii). The claim that (ii) implies (i) is left as an exercise.

Recall that for any cohererent sheaf N on X and any ample line bundle $\mathcal{L}$, there is an integer k such that $N\otimes\mathcal{L}^n$ is generated by its global sections and has no higher cohomology for all $n\geqslant k$. But, obviously, one cannot find one value of k that will serve all N. An advantage of the quantum setting is that the situation is different, [BPW, Section 5]. Here is a bit stronger result, [BL, Proposition 5.27].

Proposition 2.2. *Let* χ *lie in the interior of the chamber* C *of* θ *(so that* $\mathcal{O}(\chi)$ *is ample). Then for any* λ *there is* $k\in\mathbb{Z}$ *such that abelian localization holds for* (λ',θ) *with* $\lambda'\in\lambda+k\chi+(C\cap\mathbb{Z}^{Q_0})$.

The locus of (λ,θ) satisfying the abelian localization is known in some cases. There is a conjecture due to Bezrukavnikov and the author, [BL, Section 9.1], describing the locus, where the abelian localization holds.

2.2. **Derived localization.** Here we are going to obtain a criterium for the derived localization to hold. The category $\operatorname{Coh}(\mathcal{A}^\theta_\lambda)$ has *finite homological dimension*: there is $d\in\mathbb{Z}_{>0}$ such that $\operatorname{Ext}^i(M,N)=0$ for $i>d$ and any $M,N\in\operatorname{Coh}(\mathcal{A}^\theta_\lambda)$. We can take $d:=\dim X$ (that is a bound for the homological dimension of $\operatorname{Coh}(X)$, the homological dimension of $\operatorname{Coh}(\mathcal{A}^\theta_\lambda)$ does not exceed that because $\operatorname{gr}\operatorname{Ext}^i(M,N)\hookrightarrow\operatorname{Ext}^i(\operatorname{gr}M,\operatorname{gr}N)$). Obviously, if $R\Gamma^\theta_\lambda$ is an equivalence $D^b(\operatorname{Coh}(\mathcal{A}^\theta_\lambda))\xrightarrow{\sim}D^b(\mathcal{A}_\lambda\text{-mod})$, then the category $\mathcal{A}_\lambda$-mod has finite homological dimension as well (in this case we say that the algebra $\mathcal{A}_\lambda$ has finite homological dimension and say that the parameter λ is *regular*, otherwise the parameter λ is called singular).

The following result is due to McGerty and Nevins, [MN, Theorem 1.1].

Proposition 2.3. *The following are equivalent.*

(1) $\mathcal{A}_\lambda$ *has finite homological dimension.*
(2) $R\Gamma^\theta_\lambda$ *and* $L\operatorname{Loc}^\theta_\lambda$ *are mutually (quasi-)inverse equivalences.*

Note that (1) is completely independent of θ.

Conjecturally, for quantized quiver varieties, the locus of singular parameters is an explicit finite union of hyperplanes, see [BL, Section 9.1]. This is known in some cases but not in general.

2.3. **Interpretation via quotients of twisted equivariant D-modules.** Here we suppose, in addition, that the natural homomorphism $\mathcal{A}^0_{\lambda+n\chi} \to \mathcal{A}_{\lambda+n\chi}$ is an isomorphism for all $n \geqslant 0$ (where we assume that χ is in the chamber of θ, one can show, [BL, Proposition 2.7], that this is true if we replace λ with $\lambda + m\chi$ with m large enough).

Recall that the categories $\mathcal{A}_\lambda$-mod, $\mathrm{Coh}(\mathcal{A}^\theta_\lambda)$ are quotients of $D(R)$-$\mathrm{mod}^{G,\lambda}$, let $\pi^0_\lambda, \pi^\theta_\lambda$ denote the corresponding functors. Recall that π^0_λ has a left adjoint $(\pi^0_\lambda)^! := D(R)/\mathcal{I}_\lambda \otimes_{\mathcal{A}_\lambda} \bullet$ (the functor π^θ_λ almost has a right adjoint, but its image consists of quasi-coherent modules in general, so we are not going to consider that). The following claim is [BL, Lemma 2.11].

Lemma 2.4. *We have* $L\,\mathrm{Loc}^\theta_\lambda = \pi^\theta_\lambda \circ L(\pi^0_\lambda)^!$.

The following lemma gives a criterium for the abelian localization to hold. Note that (1)⇒(2) is a direct consequence of the previous lemma, while the opposite implication is more subtle.

Lemma 2.5 (Lemma 2.14 in [BL])**.** *The following are equivalent.*

(1) *Abelian localization holds for* (λ, θ).
(2) $\ker \pi^0_\lambda = \ker \pi^\theta_\lambda$, *i.e., a* (G, λ)*-equivariant* $D(R)$*-module has no nonzero* G*-invariants if and only if its support is* θ*-unstable.*

3. Counting result

3.1. **Geometric construction of representations of $\mathfrak{g}(Q)$.** The main reason why the Nakajima quiver varieties are of importance for the Geometric Representation theory is that one can construct representations of the Kac-Moody algebra $\mathfrak{g}(Q)$ (or the corresponding quantum group) in the various geometric invariants such as (co)homology or K-theory associated to the smooth quiver varieties $\mathcal{M}^\theta_0(v, w)$. Let us recall the most basic construction: of the irreducible integrable highest weight module L_ω, where ω is a highest weight.

We will do this under the assumption that Q has no loops. In this case we can define the Kac-Moody algebra $\mathfrak{g}(Q)$ for Q: it is generated by elements e_k, h_k, f_k subject to the usual relations read from Q viewed as a symmetric Dynkin diagram (we ignore orientations of arrows; we also may need to include more elements to Cartan such as the grading element d in the affine case). We consider the highest weight integrable representations of $\mathfrak{g}(Q)$ (this just amounts to finite dimensional ones when Q is of finite type). Irreducible representations of this kind are still classified by the dominant weights $\omega = \sum_{k\in Q_0} w_i \pi_i$, where π_i is given by $\pi_i(h_j) = \delta_{ij}$. It is not a coincidence that we denote the coefficients of ω in the same way as the coefficients of the framing vector: we will assign the dominant weight ω to the framing vector $w = (w_k)_{k\in Q_0}$. To the dimension vector v we will assign a (non-necessarily dominant) weight $\nu := \omega - \sum_{k\in Q_0} v_k \alpha_k$, where we write α_k for the simple root indexed by k.

Theorem 3.1. *There is a representation of* $\mathfrak{g}(Q)$ *in* $\bigoplus_v H_{mid}(\mathcal{M}^\theta(v))$*, where* θ *is generic, that makes the latter space isomorphic to* L_ω *in such a way that* $H_{mid}(\mathcal{M}^\theta(v)) = L_\omega[\nu]$*, the weight space of weight* ν.

When we write "mid", we mean the middle dimensional homology, the degree is $\dim_\mathbb{C} \mathcal{M}^\theta(v)$.

We will need a somewhat different interpretation of $H_{mid}(\mathcal{M}^\theta(v))$. Recall $H_{mid}(\mathcal{M}^\theta(v)) = H_{top}(\varphi^{-1}(0))$. The right hand side has a basis naturally indexed by the irreducible components of $\varphi^{-1}(0)$.

We will briefly recall the construction of the action in the next lecture.

3.2. **Characteristic cycle map.** Set $\mathcal{A}_\lambda (= \mathcal{A}_\lambda(v,w)) := \Gamma(\mathcal{A}_\lambda^\theta(v,w))$. Let $\mathcal{A}_\lambda\text{-mod}_{fin}$ denote the category of finite dimensional $\mathcal{A}_\lambda$-modules. We are going to compute the number of the irreducible modules in this category. In other words, we need to compute the dimension of $K_0(\mathcal{A}_\lambda\text{-mod}_{fin})$ (for simplicity, all K_0 groups we consider will be vector spaces over $\mathbb{C}$). We will do this in the case when λ is regular (meaning that the homological dimension of $\mathcal{A}_\lambda$ is finite). For this we will produce an injective map $K_0(\mathcal{A}_\lambda\text{-mod}_{fin}) \to H_{mid}(\mathcal{M}^\theta(v,w)) = L_\omega[\nu]$.

Since the homological dimension of $\mathcal{A}_\lambda$ is finite, the functor $L\operatorname{Loc}_\lambda^\theta : D^b(\mathcal{A}_\lambda\text{-mod}) \to D^b(\operatorname{Coh}(\mathcal{A}_\lambda^\theta))$ is an equivalence. As with the pull-back in Algebraic geometry, the supports of all homology of $L\operatorname{Loc}_\lambda^\theta(M)$ are contained in $\varphi^{-1}(\operatorname{Supp} M)$. The finite dimensional $\mathcal{A}_\lambda$-modules are precisely those supported at 0. So $L\operatorname{Loc}_\lambda^\theta$ maps the full subcategory $D^b_{fin}(\mathcal{A}_\lambda\text{-mod})$ of all complexes with finite dimensional homology to the full subcategory $D^b_{\varphi^{-1}(0)}(\operatorname{Coh}(\mathcal{A}_\lambda^\theta)) \subset D^b(\operatorname{Coh}(\mathcal{A}_\lambda^\theta))$ of all complexes whose homology are supported on $\varphi^{-1}(0)$. The functor $R\Gamma_\lambda^\theta$ also respects supports in a suitable sense and so maps $D^b_{\varphi^{-1}(0)}(\operatorname{Coh}(\mathcal{A}_\lambda^\theta))$ to $D^b_{fin}(\mathcal{A}_\lambda\text{-mod})$. We conclude that $D^b_{fin}(\mathcal{A}_\lambda\text{-mod}) \xrightarrow{\sim} D^b_{\varphi^{-1}(0)}(\operatorname{Coh}(\mathcal{A}_\lambda^\theta))$ and, in particular, the K_0's of these two categories are identified. We have $K_0(D^b_{fin}(\mathcal{A}_\lambda\text{-mod})) = K_0(\mathcal{A}_\lambda\text{-mod}_{fin})$. On the other hand, $K_0(D^b_{\varphi^{-1}(0)}(\operatorname{Coh}(\mathcal{A}_\lambda^\theta)))$ admits the characteristic cycle map to the space with basis indexed by the irreducible components of $\varphi^{-1}(0)$ defined by $\mathsf{CC}(M) := \sum_i (-1)^i \mathsf{CC}(H_i(M))$. The target space is $H_{mid}(\mathcal{M}_0^\theta(v))$ and hence is $L_\omega[\nu]$. So we indeed get a map $K_0(\mathcal{A}_\lambda\text{-mod}_{fin}) \to L_\omega[\nu]$ to be also denoted by CC_v, one can show that it is independent of θ, this is nontrivial.

The following result of Baranovsky and Ginzburg has not been published yet, but hopefully is true.

Theorem 3.2. *The map CC_v is injective.*

3.3. **Etingof type conjecture.** So what we need to do is to compute the rank of the map CC. We will actually compute its image (that should, of course, depend on λ). To state the result, we need more notation.

Recall that a root α for $\mathfrak{g}(Q)$ is called *real* if it is conjugate to a simple root under the $W(Q)$-action. We consider the subalgebra $\mathfrak{a}^\lambda \subset \mathfrak{g}(Q)$ generated by the Cartan subalgebra of $\mathfrak{g}(Q)$ (non-interesting part) and the root spaces $\mathfrak{g}(Q)_\beta$, where β runs over all real roots $\sum_{k\in Q_0} b_k\alpha_k$ with $\beta\cdot\lambda = \sum_{k\in Q_0} b_k\lambda_k \in \mathbb{Z}$. For example, when λ is (Weil) generic (e.g. "very irrational"), then $\mathfrak{a}$ is just the Cartan. The other extreme is when $\lambda \in \mathbb{Z}^{Q_0}$. Here $\mathfrak{a} = \mathfrak{g}(Q)$. So $\mathfrak{a}$ measures "how integral λ is". Note that $\mathfrak{a}$ is also a Kac-Moody algebra.

We consider the space $L_\omega^{\mathfrak{a}}$ that is the $\mathfrak{a}$-submodule in L_ω generated by the extremal weight spaces, $L_\omega[\sigma\omega]$, $\sigma \in W(Q)$, where $W(Q)$ stands for the Weyl group. In other words,

$$L_\omega^{\mathfrak{a}} = \bigoplus U(\mathfrak{a}) L_\omega[\sigma\omega],$$

where the sum is taken over all weights $\sigma\omega$ such that $\sigma\omega$ is dominant for $\mathfrak{a}$.

The following is the main result of [BL].

Theorem 3.3. *Let Q be of finite or affine type and λ be regular. Then* $\operatorname{im}\mathsf{CC}_v = L_\omega^{\mathfrak{a}}[\nu] := L_\omega^{\mathfrak{a}} \cap L_\omega[\nu]$.

The inclusion $\operatorname{im}\mathsf{CC}_v \supset L^{\mathfrak{a}}_\omega[\nu]$ is true without any additional assumptions on Q (we still require that Q has no loops, if Q has loops, then $\mathcal{A}_\lambda(v)$ has a tensor factor that is the differential operators on $\mathbb{C}$, this algebra has no finite dimensional representations).

There are also conjectures, [BL, Section 9.4], on the number of finite dimensional irreducibles without the assumption that $\mathcal{A}_\lambda(v)$ has finite homological dimension.

3.4. **Plan.** First of all, if $\nu = \sigma\omega$, then $\mathcal{M}^\theta(v) = \{pt\}$. This can be shown by computing the dimension of $\mathcal{M}^\theta(v)$, it equals $(\omega,\omega) - (\nu,\nu)$, a more general claim will be proved in the next lecture. So, obviously, $\mathcal{A}_\lambda = \mathbb{C}$ and $L_\omega[\sigma\omega] = \operatorname{im}\mathsf{CC}_v$. For a suitable system $\Pi^{\mathfrak{a}}$ of simple roots of $\mathfrak{a}$, we will produce (triangulated) functors

$$F_\alpha : D^b(\mathcal{A}^\theta_\lambda(v)) \rightleftarrows D^b(\mathcal{A}^\theta_\lambda(v+\alpha)) : E_\alpha$$

with the following two properties:

(i) The functors preserve the subcategories $D^b_{\varphi^{-1}(0)}(\ldots)$.
(ii) Under CC, the classes of F_α, E_α on K_0, become the operators $f_\alpha, e_\alpha \in \mathfrak{a}$ (up to a sign).

The functors F_α may be (to some extent) viewed as induction functors that allow to produce new finite dimensional $\mathcal{A}_\lambda$-modules (rather complexes with finite dimensional homology) from existing ones. Property (ii) shows that $L^{\mathfrak{a}}_\omega[\nu] \subset \operatorname{im}\mathsf{CC}$.

The opposite inclusion is much more subtle. This will require studying an interplay between t-structures on $D^b(\mathcal{A}_\lambda\text{-mod})$ coming from the identifications with $D^b(\operatorname{Coh}(\mathcal{A}^\theta_\lambda))$ for various θ.

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