

# HIGHEST WEIGHT CATEGORIES

IVAN LOSEV

Let  $\mathbb{F}$  be a field,  $\Lambda$  be a finite poset and  $\mathcal{C}$  be an  $\mathbb{F}$ -linear artinian category with simple objects parameterized by  $\Lambda$ . By a highest weight category structure on  $\mathcal{C}$  with respect to  $\Lambda$  we mean a collection of *standard* objects  $\Delta(\lambda)$  in  $\mathcal{C}$ , one for each  $\lambda \in \Lambda$  such that

- $\text{Hom}(\Delta(\lambda), \Delta(\mu)) \neq 0 \Rightarrow \lambda \leq \mu$ .
- $\text{End}(\Delta(\lambda)) = \mathbb{F}$  for any  $\lambda$ .
- $\mathcal{C}$  has enough projectives. The indecomposable projectives are parameterized by  $\Lambda$ ,  $\lambda \mapsto P(\lambda)$ , where  $P(\lambda)$  is the projective cover of  $L(\lambda)$ . The object  $P(\lambda)$  surjects onto  $\Delta(\lambda)$  and the kernel admits a filtration whose quotients are of the form  $\Delta(\mu)$  with  $\mu > \lambda$ .

We write  $\mathcal{C}^\Delta$  for the full subcategory of all standardly filtered objects in  $\mathcal{C}$ , i.e., all objects that admit a filtration with standard quotients.

**0.1. Simple constituents of standards.** Let  $L(\lambda)$  denote the simple corresponding to  $\lambda \in \Lambda$ . Show that if  $L(\lambda)$  occurs in  $\Delta(\mu)$ , then  $\lambda \leq \mu$ . Moreover, show that the multiplicity of  $L(\lambda)$  in  $\Delta(\lambda)$  is 1, and  $\Delta(\lambda) \twoheadrightarrow L(\lambda)$ .

**0.2. Ext's between standards.** Show that if  $\text{Ext}^i(\Delta(\lambda), \Delta(\mu)) \neq 0$  for some  $i > 0$ , then  $\lambda < \mu$ .

**0.3. Subcategories.** Let  $\Lambda' \subset \Lambda$  be an ideal in the sense that if  $\lambda \in \Lambda'$  and  $\mu \leq \lambda$ , then  $\mu \in \Lambda'$ . Consider the Serre subcategory  $\mathcal{C}' \subset \mathcal{C}$  spanned by the simples  $L(\lambda)$  with  $\lambda \in \Lambda'$ .

- (1) Show that  $\mathcal{C}'$  is a highest weight category with standard objects  $\Delta(\lambda)$ ,  $\lambda \in \Lambda'$ .
- (2) Show that the left adjoint functor  $\iota^!$  to the embedding  $\mathcal{C}' \hookrightarrow \mathcal{C}$  is exact on  $\mathcal{C}^\Delta$ .
- (3) Deduce that there is a projective resolution  $P^\bullet$  in  $\mathcal{C}$  of  $M \in \mathcal{C}'^\Delta$  such that  $\iota^!(P^\bullet)$  is a projective resolution of  $M$  in  $\mathcal{C}'$  (in fact, this is true for any projective resolution).

**0.4. Quotients.** Now let  $\mathcal{C}'' := \mathcal{C}/\mathcal{C}'$  and  $\pi$  be the quotient functor  $\mathcal{C} \rightarrow \mathcal{C}''$ . Let  $\pi^!$  denote its left adjoint functor.

- (1) Show that  $\mathcal{C}''$  is a highest weight category with standard objects  $\pi(\Delta(\lambda))$ .
- (2) Moreover, show that the natural morphism  $\pi^!(\pi(M)) \rightarrow M$  is an isomorphism provided  $M$  admits a filtration whose quotients are  $\Delta(\mu)$  with  $\mu \notin \Lambda'$ .
- (3) Deduce that  $\pi^!$  gives rise to an equivalence between  $\mathcal{C}''^\Delta$  and a full subcategory in  $\mathcal{C}^\Delta$  consisting of all objects whose filtration quotients are  $\Delta(\mu)$  with  $\mu \notin \Lambda'$ .

**0.5. Characterization of projectives.** Show that a standardly filtered object  $P \in \mathcal{C}$  is projective if and only if  $\text{Ext}^1(P, \Delta(\lambda)) = 0$  for all  $\lambda \in \Lambda$ .

**0.6. Axiomatic characterization of standards.** Show that the objects  $\Delta(\lambda)$  are uniquely recovered from the poset structure on  $\Lambda$  as follows. Pick  $\lambda \in \Lambda$ . Let  $\Lambda_{\leq \lambda} = \{\mu \in \Lambda \mid \mu \leq \lambda\}$ . Consider the Serre subcategory  $\mathcal{C}_{\leq \lambda}$  spanned by the simples  $L(\mu)$  with  $\mu \in \Lambda_{\leq \lambda}$ . Then  $\Delta(\lambda)$  is the projective cover of  $L(\lambda)$  in  $\mathcal{C}_{\leq \lambda}$ .

0.7. **Costandard objects.** Let  $\nabla(\lambda)$  stand for the injective hull of  $L(\lambda)$  in  $\mathcal{C}_{\leq \lambda}$ . This is a so called *costandard object*. Show that  $\dim \operatorname{Hom}(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda, \mu}$ , while  $\operatorname{Ext}^i(\Delta(\lambda), \nabla(\mu)) = 0$  for  $i > 0$ .

0.8. **Highest weight structure on  $\mathcal{C}^{opp}$ .** Prove that the collection of costandard objects  $\nabla(\lambda)$  makes the opposite category  $\mathcal{C}^{opp}$  into a highest weight category with respect to the poset  $\Lambda$ .

0.9. **Characterization of (co)standardly filtered objects.** Prove the object  $M \in \mathcal{C}$  is  $\Delta$ -filtered if and only if  $\operatorname{Ext}^i(M, \nabla(\lambda)) = 0$  for all  $\lambda \in \Lambda, i > 0$ , if and only if  $\operatorname{Ext}^1(M, \nabla(\lambda)) = 0$  for all  $\lambda \in \Lambda$ . Deduce the kernel of an epimorphism of standardly filtered objects is standardly filtered. State and prove the dual statements.

0.10. **BGG reciprocity.** Show that the multiplicity of  $\Delta(\lambda)$  in  $P(\mu)$  coincides with the multiplicity of  $L(\mu)$  in  $\nabla(\lambda)$ .

0.11. **Tilting objects.** An object in  $\mathcal{C}$  is called *tilting* if it is both standardly and costandardly filtered.

- (1) For  $\lambda \in \Lambda$  consider an object  $T(\lambda)$  constructed as follows. Order linearly elements of  $\{\mu \in \Lambda \mid \mu \leq \lambda\}$  refining the original poset structure, say  $\lambda = \lambda_1 > \lambda_2 > \dots > \lambda_k$ . Construct the object  $T^i(\lambda), i = 1, \dots, k$  inductively as follows. Set  $T^1(\lambda) = \Delta(\lambda)$ . Further, if  $T^{i-1}(\lambda)$  is already defined let  $T^i(\lambda)$  be the extension of  $\operatorname{Ext}^1(\Delta(\lambda_i), T^{i-1}(\lambda)) \otimes \Delta(\lambda_i)$  by  $T^{i-1}(\lambda)$  corresponding to the unit endomorphism of  $\operatorname{Ext}^1(\Delta(\lambda_i), T^{i-1}(\lambda))$ . Show that  $T(\lambda) := T^k(\lambda)$  is an indecomposable tilting.
- (2) Prove that any other tilting in  $\mathcal{C}$  is isomorphic to the direct sum of the objects  $T(\lambda)$ .

0.12. **Ringel duality.** Set  $T := \bigoplus_{\lambda \in \Lambda} T(\lambda)$ . Let  $\mathcal{C}^\vee$  be the category of finitely generated  $\operatorname{End}(T)$ -modules. Show that this category is highest weight with respect to the opposite poset  $\Lambda^{opp}$  with standard objects  $\operatorname{Hom}(\Delta(\lambda), T)$ . Show that  $(\mathcal{C}^\vee)^\Delta \cong (\mathcal{C}^\Delta)^{opp}$  and that, under this identification, the projective objects in  $\mathcal{C}$  correspond to tilting objects in  $\mathcal{C}^\vee$ , while tilting objects in  $\mathcal{C}$  correspond to projective objects in  $\mathcal{C}^\vee$ . Finally, identify  $(\mathcal{C}^\vee)^\vee$  with  $\mathcal{C}^{opp}$ .