QUANTUM HAMILTONIAN REDUCTION IN CHARACTERISTIC p

IVAN LOSEV

Contents

1. p-centers

Here we basically review the stuff that has appeared in Kostya's talk last term.

1.1. **Notation.** Our base field is an algebraically closed field $\mathbb{F}$ of characteristic p. Below we will always assume that p is large enough. By X we denote a symplectic variety over $\mathbb{F}$ and by $X^{(1)}$ its Frobenius twist. Recall that when X is affine, we have $\mathbb{F}[X^{(1)}] = \{f^p | f \in \mathbb{F}[X]\} \subset \mathbb{F}[X]$. In general we have a dominant morphism $\mathrm{Fr} : X \to X^{(1)}$ that becomes a morphism of $\mathbb{F}$-varieties when we twist the product on $\mathcal{O}_{X^{(1)}}$ with the inverse of the Frobenius of $\mathbb{F}$.

We remark that if X is an $\mathbb{F}$-vector space ($\mathbb{F}$-algebra), then so is $X^{(1)}$ (the only thing that differs is the product with elements of $\mathbb{F}$). We also remark that if X is defined over $\mathbb{F}_p$, then $X \cong X^{(1)}$ as $\mathbb{F}$-varieties.

1.2. **Frobenius twist for symplectic varieties.** If f is a local section of $\mathcal{O}_X$, then $\{f^p, \cdot\} = 0$. An intelligent way to phrase this is to say that $\mathrm{Fr}_* \mathcal{O}_X$ becomes a sheaf of Poisson $\mathcal{O}_{X^{(1)}}$-algebras (which means that the sheaf $\mathrm{Fr}_* \mathcal{O}_X$ of algebras on $X^{(1)}$ carries a natural Poisson bracket that is $\mathcal{O}_{X^{(1)}}$-bilinear).

We also would like to point out that $\mathrm{Fr}_* \mathcal{O}_X$ is a vector bundle of rank $p^{\dim X}$.

These notes would have never appeared without inspiration from Mitya Vaintrob.

1.3. **Differential operators.** Now let X_0 be a smooth variety over $\mathbb{F}$. We consider the sheaf of (crystalline) differential operators. This is a quasicoherent sheaf D_{X_0} of algebras on X_0 that is generated by $\mathcal{O}_{X_0}$ and the sheaf $\operatorname{Vect}_{X_0}$ of vector fields on X_0 subject to the following relations:

$$f\cdot g = fg, f\cdot\xi = f\xi, \xi\cdot f = f\xi+\xi(f), \xi\cdot\eta-\eta\cdot\xi=[\xi,\eta].$$

Here f,g are local sections of $\mathcal{O}_{X_0}$ and ξ,η are local sections of $\operatorname{Vect}_{X_0}$. The product $\cdot$ in the left hand side of the relations is that in D_{X_0}, while on the right hand side we have usual operations on functions and vector fields.

The sheaf of algebras D_{X_0} is filtered, the associated graded is $p_*\mathcal{O}_{T^*X_0}$, where $p: T^*X_0\to X_0$ is a natural projection.

For simplicity of exposition, assume now that X_0 is affine and consider the algebra $D(X_0)$ of global differential operators. Set $X=T^*X_0$. We have an embedding $\mathbb{F}[X]\hookrightarrow D(X_0)$ defined as follows: it sends $f\in\mathbb{F}[X_0]$ to f^p and $\xi\in\operatorname{Vect}(X_0)$ to $\xi^p-\xi^{[p]}$. Recall that $\xi^{[p]}$ stands for the pth power of ξ viewed as a derivation of $\mathbb{F}[X]$ (this makes sense because the pth power of a derivation in characteristic p is again a derivation). This is embedding is not $\mathbb{F}$-linear but it induces an $\mathbb{F}$-linear embedding $\mathbb{F}[X^{(1)}]\to D(X_0)$. The image is called the *p-center* of $D(X_0)$ (it actually coincides with the whole center).

The embedding is compatible with the gradings/filtrations. Namely, $\mathbb{F}[X^{(1)}]\subset\mathbb{F}[X]$ is a graded subalgebra (all degrees in $\mathbb{F}[X^{(1)}]$ are divisible by p). On the other hand, $\mathbb{F}[X^{(1)}]\subset D(X_0)$ inherits a filtration from $D(X_0)$ and this filtration coincides with the one induced from grading.

So we can consider $D(X_0)$ as an algebra over $\mathbb{F}[X^{(1)}]$. This algebra is finitely generated and so we can view D_{X_0} as a coherent sheaf of algebras over $X^{(1)}$. This sheaf is a vector bundle of rank $p^{\dim X}$. Moreover, it is *Azumaya*, meaning that all geometric algebras are matrix algebras over $\mathbb{F}$. We will see below that D_{X_0} viewed as a sheaf on $X^{(1)}$ is a quantization of $\operatorname{Fr}_*\mathcal{O}_X$.

We also would like to point out that the assumption that X_0 is affine is not necessary: in general D_{X_0} is still an Azumaya sheaf of algebras on $X^{(1)}$. This can be seen using gluing of affine pieces.

1.4. **Universal enveloping algebras.** Now let G be an algebraic group over $\mathbb{F}$. Its Lie algebra, $\mathfrak{g}$, comes equipped with an additional *restricted* structure induced from the Frobenius morphism for G: a Lie algebra homomorphism $x\mapsto x^{[p]}:\mathfrak{g}\to\mathfrak{g}^{(1)}$. For example, if G is a matrix group, then $\mathfrak{g}$ is closed with respect to taking pth powers of matrices and $x^{[p]}$ is this pth power.

The element $x^p-x^{[p]}$ is central in the universal enveloping algebra $U(\mathfrak{g})$. Similarly to the previous subsection, the map $x\to x^p-x^{[p]}$ induces a central embedding $\mathbb{F}[\mathfrak{g}^{(1)}]\hookrightarrow U(\mathfrak{g})$. Its image is called the p-center of $U(\mathfrak{g})$. We remark that it does not need to coincide with the whole center and $U(\mathfrak{g})$ is almost never Azumaya over the p-center.

Also let us recall that this construction is compatible with that from the previous subsection: $U(\mathfrak{g})=D(G)^G$.

2. Classical Hamiltonian reduction

2.1. **GIT in positive characteristic.** One difficulty that arises in dealing with reductive (=trivial unipotent radical) algebraic groups in characteristic p: their representations are almost never completely reducible (an algebraic torus is an exception). In this notes, we need to deal with GIT for GL_n and so we need to explain how this works in positive characteristic.

It turns out however that reductive groups satisfy a weaker condition than being linearly reductive, they are *geometrically reductive.* This was conjectured by Mumford and proved by Haboush, [H]. To state the condition of being geometrically reductive, let us reformulate the linear reductivity first: a group G is called linearly reductive, if, for any linear G-action on a vector space V and any fixed point $v \in V$, there is $f \in (V^*)^G$ with $f(v) \neq 0$. A group G is called geometrically reductive if instead of $f \in (V^*)^G$, one can find $f \in S^r(V^*)^G$ (for some $r > 0$) with $f(v) \neq 0$.

This condition is enough for many applications. For example, if X is an affine algebraic variety acted on by a reductive (and hence geometrically reductive) group G, then $\mathbb{F}[X]^G$ is finitely generated. So we can consider the quotient morphism $X \to X /\!/ G$. This morphism is surjective and separates the closed orbits. Moreover, if $X' \subset X$ is a G-stable subvariety, then the natural morphism $X' /\!/ G \to X /\!/ G$ is injective with closed image.

The claim about the properties of the quotient morphism in the previous paragraph can be deduced from the following lemma, [MFK, Lemma A.1.2].

Lemma 2.1. *Let G be a geometrically reductive group acting on a finitely generated commutative $\mathbb{F}$-algebra R rationally and by algebra automorphisms. Let $I \subset R$ be a G-stable ideal and $f \in (R/I)^G$. Then there is n such that f^{p^n} lies in the image of R^G in $(R/I)^G$.*

In characteristic p, we can still speak about unstable and semistable points for reductive group actions on vector spaces, about GIT quotients etc.

Another very useful and powerful result of Invariant theory in characteristic 0 is Luna's étale slice theorem. There is a version of this theorem in characteristic p due to Bardsley and Richardson, see [BR]. We will need a consequence of this theorem dealing with free actions.

Recall that in characteristic 0, an action of an algebraic group G on a variety X is called free if the stabilizers of all points are trivial. In characteristic p one should give this definition more carefully: the stabilizer may be a nontrivial finite group scheme with a single point. An example is provided by the left action of G on $G^{(1)}$, we will discuss a closely related question in the next subsection. We have the following three equivalent definitions of a free action.

- For every $x \in X$, the stabilizer G_x equals $\{1\}$ as a group scheme.
- For every $x \in X$, the orbit map $G \to X$ corresponding to x is an isomorphism of algebraic varieties.
- For every $x \in X$, G_x coincides with $\{1\}$ as a set and the stabilizer of x in $\mathfrak{g}$ is trivial.

The following is a weak version of the slice theorem that we need.

Lemma 2.2. *Let X be a smooth affine variety equipped with a free action of a reductive algebraic group G. Then the quotient morphism $X \to X/G$ is a principal G-bundle in étale topology.*

2.2. **Frobenius kernel.** Now we want to investigate the simplest nonreduced one-point group subscheme of G, the 1st *Frobenius kernel.* In this subsection, G denotes a connected algebraic group over $\mathbb{F}$. A reference for this subsection is [J, Chapter 9].

Let us notice that $\mathrm{Fr}: G \to G^{(1)}$ is a group homomorphism. The schematic fiber of $1 \in G^{(1)}$ is denoted by G_1 and called the Frobenius kernel. This is a one-point nonreduced group subscheme of G of length $p^{\dim G}$. This is a pretty formal exercise to check that $\mathbb{F}[G_1]$ is a Hopf quotient of the Hopf algebra $\mathbb{F}[G]$, hence G_1 is indeed a group scheme. The algebra $\mathbb{F}[G_1]$, of course, is $\mathbb{F}[G]/\mathfrak{m}_1^p$, where $\mathfrak{m}_1$ denotes the maximal ideal of 1.

We want to describe the dual Hopf algebra $\mathbb{F}G_1 = \mathbb{F}[G_1]^*$ (to be thought as the group algebra of G_1). We have $\mathbb{F}G_1 = \mathbb{F} \cdot 1 \oplus \mathbb{F}(\mathfrak{m}_1/\mathfrak{m}_1^p)^*$. So we have a natural inclusion $\mathfrak{g} = (\mathfrak{m}_1/\mathfrak{m}_1^2)^* \hookrightarrow \mathbb{F}G_1$. One can check that this is a homomorphism of Lie algebras. So it extends to a homomorphism $U(\mathfrak{g}) \to \mathbb{F}G_1$ of associative algebras. This homomorphism is surjective and its kernel can be described as follows. Recall the p-center $\mathbb{F}[\mathfrak{g}^{(1)}] \subset U(\mathfrak{g})$. Let $U(\mathfrak{g})^0$ denote the quotient of $U(\mathfrak{g})$ by the two-sided ideal $U(\mathfrak{g})\mathbb{F}[\mathfrak{g}^{(1)}]_+$ generated by the ideal $\mathbb{F}[\mathfrak{g}^{(1)}]_+$ of 0 in $\mathbb{F}[\mathfrak{g}^{(1)}]$. This two-sided ideal of $U(\mathfrak{g})$ coincides with the kernel of $U(\mathfrak{g}) \to \mathbb{F}G_1$ and so $\mathbb{F}G_1$ is identified with $U(\mathfrak{g})^0$.

This discussion has the following corollary.

Corollary 2.3. *Let V be a rational representation of G. Then we have a natural $U(\mathfrak{g})$-action on V that factors through $U(\mathfrak{g})^0$. Further, the $\mathfrak{g}$-invariants in V coincide with G_1-invariants.*

Corollary 2.4. *Let G act on an affine variety X. This induced G-action on $X^{(1)}$ factors through $G^{(1)}$.*

The proof is an exercise on the previous corollary.

2.3. **Hamiltonian actions and reduction.** We again assume that X is a symplectic variety. Let an algebraic group G act on X by symplectomorphisms. We still have G-equivariant Lie algebra homomorphisms $\mathfrak{g} \to \mathrm{Vect}(X), \mathbb{F}[X] \to \mathrm{Vect}(X)$. The notion of a classical comoment map $\mu^*: \mathfrak{g} \to \mathbb{F}[X]$ (a G-equivariant map that intertwines the two homomorphisms above) still makes sense. The dual map, $\mu: X \to \mathfrak{g}^*$, is called the moment map. An action equipped with a moment map is called Hamiltonian.

The Hamiltonian reduction can still be defined as well. We will consider two kinds of reduction: the categorical reduction $X /\!/\!/ G$ for an action of a reductive group G on a symplectic affine variety X and a GIT reduction $X /\!/\!/^\theta G$ of the same action for a character $\theta: G \to \mathbb{F}^\times$. We remark that if a G-action on X^{ss} is free, then the quotient morphisms $X^{ss} \to X /\!/\!/^\theta G, \mu^{-1}(0)^{ss} \to \mu^{-1}(0)^{ss}/G$ are principal G-bundle in étale topology, and the reduction is still a symplectic variety.

Let us characterize the Frobenius twist of a reduction under a free action.

Lemma 2.5. *Assume that X is equipped with a free Hamiltonian G-action such that the quotient X/G exists and $X \to X/G$ is a principal bundle in étale topology. Then $(X /\!/\!/ G)^{(1)} = (\mu^{(1)})^{-1}(0)/G^{(1)}$, where $\mu^{(1)}: X^{(1)} \to \mathfrak{g}^{*(1)}$ is a morphism induced by μ.*

We remark that $G^{(1)}$ acts freely on $X^{(1)}$ so that the right hand side makes sense. The proof is left as an exercise.

3. Frobenius constant quantizations

3.1. **Definition.** Now let us proceed to defining Frobenius constant quantizations of symplectic varieties as well as G-equivariant Frobenius constant quantizations of symplectic varieties equipped with Hamiltonian G-actions. Below in this section X always denotes a symplectic variety. As we will see, the main difference (and simplification) compared to zero characteristic is as follows. In characteristic 0 in order to consider sheaf quantizations we had to deal with microlocal sheaves (and hence with completions). However, in characteristic p this is not necessary: nice quantizations can be viewed as coherent sheaves of algebras on $X^{(1)}$.

Since we want to consider filtered quantizations we will need a suitable $\mathbb{F}^\times$-action on X (if we were dealing with quantizations over $\mathbb{F}[\hbar]$ we would not need that additional assumption). We assume that there is a $\mathbb{F}^\times$-action on X that rescales the symplectic form: $t.\omega = t\omega$ for all $t \in \mathbb{F}^\times$.

By definition, a Frobenius constant quantization of X is a coherent sheaf of Azumaya algebras $\mathcal{D}$ on $X^{(1)}$ equipped with a filtration (when viewed as a sheaf in conical topology) and an identification $\operatorname{gr}\mathcal{D} \cong \operatorname{Fr}_*(\mathcal{O}_X)$ (of sheaves of graded Poisson algebras; here the grading on $\operatorname{Fr}_*(\mathcal{O}_X)$ is induced by the $\mathbb{F}^\times$-action on X). Moreover, we require that the filtration on $\mathcal{O}_{X^{(1)}} \subset \mathcal{D}$ is induced by the grading.

A basic example is D_{X_0} (viewed as a sheaf on $X^{(1)}$, where X_0 is a smooth variety and $X = T^*X_0$): it is a Frobenius constant quantization of X.

Let us describe the G-equivariant quantization. Let X be equipped with a Hamiltonian G-action with comoment map μ^*. We require the G-action to commute with $\mathbb{F}^\times$ and the map μ^* to be $\mathbb{F}^\times$-equivariant (for the usual dilation action on $\mathfrak{g}$). By a G-equivariant Frobenius constant quantization $\mathcal{D}$ of X we mean the following data:

(a) A Frobenius constant quantization $\mathcal{D}$ of X.
(b) A filtration preserving G-action on $\mathcal{D}$ by algebra automorphisms that gives rise to the action map $x \mapsto x_{\mathcal{D}}$ for $\mathfrak{g}$, i.e., to $\mathfrak{g} \to \operatorname{Der}(\mathcal{D})$ (which seems always to be the case if there is a G-action but we are not going to discuss that).
(c) A quantum comoment map $\Phi : \mathfrak{g} \to \Gamma(X^{(1)}, \mathcal{D})$ with image in filtration degree 1 (meaning that Φ is G-equivariant and $[\Phi(x), \cdot] = x_{\mathcal{D}}$).

These data are supposed to satisfy the following additional axioms:

(1) The associated graded of Φ is μ^*.
(2) The image of the p-center $\mathbb{F}[\mathfrak{g}^{(1)}] \subset U(\mathfrak{g})$ under Φ is contained in $\mathbb{F}[X^{(1)}] \subset \Gamma(X^{(1)}, \mathcal{D})$. The resulting map $\mathbb{F}[\mathfrak{g}^{(1)}] \to \mathbb{F}[X^{(1)}]$ coincides with $\mu^{(1)*}$.

Here is our main example. Let G act on a smooth variety X_0. Then D_{X_0} satisfies (b) and also has a quantum comoment map given by $x \mapsto x_{X_0}$. Axiom (1) is clear and axiom (2) follows from an observation made in Kostya's talk: that $x \mapsto x_{X_0}$ intertwines the restricted pth power maps.

We want to finish this subsection with a discussion on the choice of Φ. The condition of Φ being a quantum comoment map is preserved when we replace Φ with $\Phi - \lambda$ where $\lambda \in \mathfrak{g}^{*G}$. So is axiom (1). However, axiom (2) only holds if λ is integral, i.e., comes from a character of G.

3.2. **Quantum Hamiltonian reduction.** Here we prove a quantization commutes with reduction claim. This is the most important result in these notes.

Theorem 3.1. *Let X be a symplectic variety equipped with an $\mathbb{F}^\times$-action as in the previous subsection. Suppose that a connected algebraic group G acts freely on X in a Hamiltonian way and that there is a quotient $X \to X/G$ that is a principal G-bundle in étale topology. Let $\mathcal{D}$ be a G-equivariantly Frobenius constant quantization of X. Then $\mathcal{D} /\!/\!/_0\, G = R(\mathcal{D}, G, 0)$ is a Frobenius constant quantization of $X /\!/\!/\, G$.*

Proof. Let us start by explaining how $\mathcal{D}\,///_0\,G$ is equipped with a structure of a sheaf of $\mathcal{O}_{(X\,///\,G)^{(0)}}$-modules. The quotient $\mathcal{D}/\mathcal{D}\Phi(\mathfrak{g})$ is a coherent sheaf of $\mathcal{O}_{(\mu^{(1)})^{-1}(0)}$-modules (here we use axiom (2)). The group G_1 acts on $\mathcal{D}/\mathcal{D}\Phi(\mathfrak{g})$ by $\mathcal{O}_{(\mu^{(1)})^{-1}(0)}$-linear automorphisms. The invariant sheaf $[\mathcal{D}/\mathcal{D}\Phi(\mathfrak{g})]^{G_1} = R(\mathcal{D}, G_1, 0)$ is a $G^{(1)}$-equivariant sheaf of algebras on $(\mu^{(1)})^{-1}(0)$. The sheaf $\mathcal{D}\,///_0\,G$ is, by definition, $\pi^{(1)}_*\left([\mathcal{D}/\mathcal{D}\Phi(\mathfrak{g})]^{G_1}\right)^{G^{(1)}}$, where $\pi^{(1)}$ stands for the quotient morphism $(\mu^{(1)})^{-1}(0) \to (\mu^{(1)})^{-1}(0)/G^{(1)} = (X\,///\,G)^{(1)}$. The sheaf is equipped with a filtration induced from $\mathcal{D}$. We need to check that $\mathcal{D}\,///_0\,G$ is a Frobenius constant quantization of $X\,///\,G$.

Step 1. Let us notice that $\operatorname{gr}\mathcal{D}/\mathcal{D}\Phi(\mathfrak{g}) = \operatorname{Fr}^{\mu^{-1}(0)}_*\mathcal{O}_{\mu^{-1}(0)}$ (here the supersript indicates the variety for which we take the Frobenius morphism). This is proved in the same way as the similar claim in characteristic 0 in Yi's talk.

In the subsequent steps we will show that $[\mathcal{D}/\mathcal{D}\Phi(\mathfrak{g})]^{G_1}$ is an Azumaya algebra on $(\mu^{(1)})^{-1}(0)$ and that $\operatorname{gr}([\mathcal{D}/\mathcal{D}\Phi(\mathfrak{g})]^{G_1}) = [\operatorname{Fr}^{\mu^{-1}(0)}_*\mathcal{O}_{\mu^{-1}(0)}]^{G_1}$. Then we will deduce analogous claims about $\mathcal{D}\,///_0\,G$.

Step 2. Let us show that $\mathcal{D}\,///_0\,G_1 := [\mathcal{D}/\mathcal{D}\Phi(\mathfrak{g})]^{G_1}$ is a sheaf of Azumaya algebras on $(\mu^{(1)})^{-1}(0)$ of rank $p^{\dim X - 2\dim G}$. We note that $\mathcal{D}/\mathcal{D}\Phi(\mathfrak{g})$ is a vector bundle on $(\mu^{(1)})^{-1}(0)$ of rank $p^{\dim X - \dim G}$, indeed $\mathcal{D}/\mathcal{D}\Phi(\mathfrak{g})$ is a deformation of the vector bundle $\operatorname{Fr}^{\mu^{-1}(0)}_*\mathcal{O}_{\mu^{-1}(0)}$ of that rank by the previous paragraph. Also note that $\mathcal{D}/\mathcal{D}\Phi(\mathfrak{g})$ is a module over the Azumaya algebra $\mathcal{D}|_{(\mu^{(1)})^{-1}(0)}$. Next, we have

$$\mathcal{D}\,///_0\,G_1 = [\mathcal{D}/\mathcal{D}\Phi(\mathfrak{g})]^{\mathfrak{g}} = \mathcal{E}nd_{\mathcal{D}}(\mathcal{D}/\mathcal{D}\Phi(\mathfrak{g})) = \mathcal{E}nd_{\mathcal{D}|_{(\mu^{(1)})^{-1}(0)}}(\mathcal{D}/\mathcal{D}\Phi(\mathfrak{g})).$$

The Azumaya algebra $\mathcal{D}|_{(\mu^{(1)})^{-1}(0)}$ has rank $p^{\dim X}$. It follows that $\mathcal{D}\,///\,G_1$ is an Azumaya algebra of rank $p^{\dim X - 2\dim G}$.

Step 3. We claim that $\operatorname{gr}[\mathcal{D}/\mathcal{D}\Phi(\mathfrak{g})]^{G_1} = [\operatorname{Fr}^{\mu^{-1}(0)}_*\mathcal{O}_{\mu^{-1}(0)}]^{G_1}$ (in general, the left hand side is included into the right hand side, this inclusion might be proper because G_1 is not linearly reductive). Let us work with the deformation picture (this is convenient because individual points of $(\mu^{(1)})^{-1}(0)$ are not $\mathbb{F}^\times$-stable and hence the individual fibers of $\mathcal{D}/\mathcal{D}\Phi(\mathfrak{g})$ are not filtered): consider the Rees sheaf $\mathfrak{R} := R_\hbar(\mathcal{D}/\mathcal{D}\Phi(\mathfrak{g}))$ on $(\mu^{(1)})^{-1}(0)\times \operatorname{Spec}(\mathbb{F}[\hbar])$. The Rees sheaf of $[\mathcal{D}/\mathcal{D}\Phi(\mathfrak{g})]^{G_1}$ coincides with $\mathfrak{R}^{G_1}$ and $[\operatorname{Fr}^{\mu^{-1}(0)}_*\mathcal{O}_{\mu^{-1}(0)}]^{G_1}$ is $(\mathfrak{R}/\hbar\mathfrak{R})^{G_1}$. What we need to check is that the natural map $\mathfrak{R}^{G_1}/\hbar\mathfrak{R}^{G_1} \to (\mathfrak{R}/\hbar\mathfrak{R})^{G_1}$ is an isomorphism of sheaves on $(\mu^{(1)})^{-1}(0)$.

Step 4. The question is local so we may assume that X/G is affine. It is enough to show that $\operatorname{Ext}^1_{\mathbb{F}G_1}(\mathrm{triv}, \mathfrak{R}) = 0$. This Ext is a finitely generated $\mathbb{F}[\hbar]\otimes\mathbb{F}[(\mu^{(1)})^{-1}(0)]$-module. So it is enough to show that the multiplication by $\hbar$ on this module is an epimorphism. This amounts to checking that $\operatorname{Ext}^1_{\mathbb{F}G_1}(\mathrm{triv}, \mathfrak{R}/\hbar\mathfrak{R}) = 0$. Observe that $\mathfrak{R}/\hbar\mathfrak{R} = \mathbb{F}[\mu^{-1}(0)]$. So we are checking that $\operatorname{Ext}^1_{\mathbb{F}G_1}(\mathrm{triv}, \mathbb{F}[\mu^{-1}(0)]) = 0$. Let us note that $\mathbb{F}[\mu^{-1}(0)]$ is a $\mathbb{F}G_1\otimes\mathbb{F}[\mu^{-1}(0)]^{G_1}$-module. So $\operatorname{Ext}^1_{\mathbb{F}G_1}(\mathrm{triv}, \mathbb{F}[\mu^{-1}(0)])$ is a finitely generated $\mathbb{F}[\mu^{-1}(0)]^{G_1}$-module. Obviously, if A is a flat $\mathbb{F}[\mu^{-1}(0)]^{G_1}$-algebra, then

$$\operatorname{Ext}^1_{\mathbb{F}G_1}(\mathrm{triv}, A\otimes_{\mathbb{F}[\mu^{-1}(0)]^{G_1}}\mathbb{F}[\mu^{-1}(0)]) = A\otimes_{\mathbb{F}[\mu^{-1}(0)]^{G_1}}\operatorname{Ext}^1_{\mathbb{F}G_1}(\mathrm{triv}, \mathbb{F}[\mu^{-1}(0)]).$$

But recall that $X \to X/G$ is a principal G-bundle in étale topology. So after making a faitfully flat (and étale) base change we reduce the proof to the case when $X = G\times X/G$. In particular, $\mathbb{F}[\mu^{-1}(0)] = \mathbb{F}[G_1]\otimes\mathbb{F}[\mu^{-1}(0)]^{G_1}$. So we reduce the proof of $\operatorname{Ext}^1_{\mathbb{F}G_1}(\mathrm{triv}, \mathbb{F}[\mu^{-1}(0)]) = 0$ to $\operatorname{Ext}^1_{\mathbb{F}G_1}(\mathrm{triv}, \mathbb{F}[G_1]) = 0$. But the G_1-module $\mathbb{F}[G_1] = (\mathbb{F}G_1)^*$ is injective, so we are done.

Step 5. So we have proved the claims in the second paragraph of Step 1. Since $(\mu^{(1)})^{-1}(0) \to (\mu^{(1)})^{-1}(0)/G^{(1)}$ is a principal bundle in étale topology, the Azumaya property for $\mathcal{D} /\!/\!/_0 G$ follows from that of $\mathcal{D} /\!/\!/_0 G_1$ (because the fibers of the two algebras are the same). Also the claim that $\operatorname{gr} \mathcal{D} /\!/\!/_0 G = \operatorname{Fr}_*^{X /\!/\!/ G} \mathcal{O}_{X /\!/\!/ G}$ follows from $\operatorname{gr} \mathcal{D} /\!/\!/_0 G_1 = [\operatorname{Fr}_*^{\mu^{-1}(0)} \mathcal{O}_{\mu^{-1}(0)}]^{G_1}$ (it is more convenient to see this in the deformed setting; one needs to use the fact that the category of coherent sheaves on $(\mu^{(1)})^{-1}(0)/G^{(1)} \times \operatorname{Spec}(\mathbb{F}[\hbar])$ is naturally equivalent to the category of $G^{(1)}$-equivariant coherent sheaves on $(\mu^{(1)})^{-1}(0) \times \operatorname{Spec}(\mathbb{F}[\hbar])$ – via the functor $\pi_*^{(1)}(\bullet)^{G^{(1)}}$). $\square$

4. Quantization of $\operatorname{Hilb}_n(\mathbb{F}^2)$

As in the characteristic 0 story, take the group $G = \operatorname{GL}_n(\mathbb{F})$ and a vector space $V_{\mathbb{F}} = \mathfrak{g}_{\mathbb{F}} \oplus \mathbb{F}^n$. Set $X := (T^*V_{\mathbb{F}})^{\det^{-1}-ss}$ and let $\mathcal{D}$ be the restriction of $\mathcal{D}_{V_{\mathbb{F}}}$ to $X^{(1)}$ (we remark that $G^{(1)}$ is still GL_n and $X^{(1)}$ is the set of semistable points for the $G^{(1)}$-action on $T^*V_{\mathbb{F}}^{(1)}$). Then the conditions of Theorem 3.1 are met and we get a Frobenius constant quantization $\mathfrak{A}$ of $X /\!/\!/^{\det^{-1}} G = \operatorname{Hilb}_n(\mathbb{F}^2)$.

Theorem 4.1. *Let $p \gg 0$. Then the following holds:*

1. *The Hilbert-Chow morphism $\pi : \operatorname{Hilb}_n(\mathbb{F}^2) \to \operatorname{Sym}_n(\mathbb{F}^2)$ is a resolution of singularities. The global sections of the structure sheaf on $\operatorname{Hilb}_n(\mathbb{F}^2)$ coincide with $\mathbb{F}[\mathbb{F}^{2n}]^{\mathfrak{S}_n}$ and the higher cohomology groups vanish.*
2. *We have isomorphisms $\mathbb{F}[T^*V_{\mathbb{F}} /\!/\!/ G] \cong \mathbb{F}[\mathbb{F}^{2n}]^{\mathfrak{S}_n}$ of graded algebras and $D(V_{\mathbb{F}}) /\!/\!/_0 G \cong D(\mathbb{F}^n)^{\mathfrak{S}_n}$ of filtered algebras.*
3. *A natural homomorphism $D(V_{\mathbb{F}}) /\!/\!/_0 G \to \Gamma(\operatorname{Hilb}_n(\mathbb{F}^2)^{(1)}, \mathfrak{A})$ is an isomorphism and $H^i(\operatorname{Hilb}_n(\mathbb{F}^2)^{(1)}, \mathfrak{A}) = 0$ for $i > 0$.*

Proof. The open subset $V_{\mathbb{C}}^{ss} \subset V^{ss}$ and the morphism $\mu_{\mathbb{C}} : V_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ are defined over some finite localization R of $\mathbb{Z}$ and so is the G-torsor $\mu_{\mathbb{C}}^{-1}(0)^{ss} \to \mu_{\mathbb{C}}^{-1}(0)/G_{\mathbb{C}}$. We can do the base change to any R-algebra, in particular to $\mathbb{F}$ provided $p \gg 0$, and still get a torsor. So we have an R-scheme $\operatorname{Hilb}_n(R^2)$ such that $\operatorname{Hilb}_n(\mathbb{F}^2), \operatorname{Hilb}_n(\mathbb{C}^2)$ are obtained from that scheme by base change. After some additional localization, we can assume that the claims of (1) hold over R (because they hold over $\mathbb{C}$). (1) follows.

Of course, $H^i(\operatorname{Hilb}_n(\mathbb{F}^2), \mathcal{F}) = H^i(\operatorname{Hilb}_n(\mathbb{F}^2)^{(1)}, \operatorname{Fr}_* \mathcal{F})$ for any coherent sheaf $\mathcal{F}$. It follows that $H^i(\operatorname{Hilb}_n(\mathbb{F}^2)^{(1)}, \operatorname{Fr}_* \operatorname{Hilb}_n(\mathbb{F}^2)) = 0$ for $i > 0$. Since $\mathfrak{A}$ is a deformation of $\operatorname{Fr}_* \mathcal{O}$, we see that $H^i(\operatorname{Hilb}_n(\mathbb{F}^2)^{(1)}, \mathfrak{A}) = 0$ for $i > 0$, while $\operatorname{gr} \Gamma(\operatorname{Hilb}_n(\mathbb{F}^2)^{(1)}, \mathfrak{A}) = \mathbb{F}[\operatorname{Hilb}_n(\mathbb{F}^2)] = \mathbb{F}[\mathbb{F}^{2n}]^{\mathfrak{S}_n}$ (an isomorphism of graded algebras).

Also let us point out that we have a natural homomorphism $\iota : D(V_{\mathbb{F}}) /\!/\!/_0 G \to \Gamma(\operatorname{Hilb}_n(\mathbb{F}^2)^{(1)}, \mathfrak{A})$. Its associated graded composed with a natural homomorphism $\kappa : \mathbb{F}[T^*V] /\!/\!/ G \to \operatorname{gr} D(V_{\mathbb{F}}) /\!/\!/_0 G$ becomes a natural homomorphism $\upsilon : \mathbb{F}[T^*V] /\!/\!/ G \to \mathbb{F}[\operatorname{Hilb}_n(\mathbb{F}^2)]$ (we suppress the subscript $\mathbb{F}$ when the ground field is clear). Finally let us note that the inclusion $\mathbb{F}^{2n} \hookrightarrow T^*V_{\mathbb{F}}$ (as pairs of diagonal matrices) induces a homomorphism $\mathbb{F}[T^*V] /\!/\!/ G \to \mathbb{F}[\mathbb{F}^{2n}]^{\mathfrak{S}_n}$. Under the identification $\mathbb{F}[\operatorname{Hilb}_n(\mathbb{F}^2)] \cong \mathbb{F}[\mathbb{F}^{2n}]^{\mathfrak{S}_n}$, this homomorphism is identified with υ.

To prove (2) it is enough to show that κ and υ are isomorphisms and that $D(V_{\mathbb{F}}) /\!/\!/_0 G \cong D(\mathbb{F}^n)^{\mathfrak{S}_n}$. (3) will follow.

First of all, let us notice that enlarging R (by finite localization), we can assume that $\mu_R^{-1}(0)$ is flat over $\operatorname{Spec}(R)$ and is still a reduced complete intersection, hence

$$\operatorname{gr} D(V_R)/D(V_R)\Phi(\mathfrak{g}_R) = R[\mu^{-1}(0)].$$

It follows that $R[\mu^{-1}(0)]^{\mathrm{GL}_n(R)}$ is flat (=torsion-free) over R and is an R-form of

$$T^*V_{\mathbb{C}} /\!/\!/ \mathrm{GL}_n(\mathbb{C}) = \mathbb{C}^{2n}/\mathfrak{S}_n.$$

Similarly, $D(V_R) /\!/\!/ \mathrm{GL}_n(R)$ is flat over R and is an R-form of $D(\mathbb{C}^n)^{\mathfrak{S}_n}$. One corollary of this is that for any i the graded component $R[\mu^{-1}(0)]_i^{\mathrm{GL}_n(R)}$ is a free R-module of rank $\dim_{\mathbb{C}} \mathbb{C}[\mathbb{C}^{2n}]_i^{\mathfrak{S}_n}$. Similarly, the filtered component $D(V_R) /\!/\!/ \mathrm{GL}_n(R)_{\leqslant i}$ is a free R-module of rank equal to $\dim_{\mathbb{C}} D(\mathbb{C}^n)_i^{\mathfrak{S}_n}$.

Also it is easy to see that, say, $R[\mu^{-1}(0)]/R[\mu^{-1}(0)]^{\mathrm{GL}_n(R)}$ is flat over R. It follows that $\mathbb{F}\otimes_R R[\mu^{-1}(0)]^{\mathrm{GL}_n(R)} \hookrightarrow \mathbb{F}\otimes_R R[\mu^{-1}(0)] = \mathbb{F}[\mu^{-1}(0)]$. Clearly, the image is contained in $\mathbb{F}[\mu_{\mathbb{F}}^{-1}(0)]^{\mathrm{GL}_n(\mathbb{F})}$. Since $\mu_{\mathbb{F}}^{-1}(0)$ is a reduced complete intersection and every closed $\mathrm{GL}_n(\mathbb{F})$-orbit in $\mu_{\mathbb{F}}^{-1}(0)$ intersects the space of pairs of diagonal matrices, we deduce that $\upsilon : \mathbb{F}[\mu_{\mathbb{F}}^{-1}(0)]^{\mathrm{GL}_n(\mathbb{F})} \to \mathbb{F}[\mathbb{F}^{2n}]^{\mathfrak{S}_n}$ is injective. But $\dim_{\mathbb{C}} \mathbb{C}[\mathbb{C}^{2n}]_i^{\mathfrak{S}_n} = \dim_{\mathbb{F}} \mathbb{F}[\mathbb{F}^{2n}]_i^{\mathfrak{S}_n}$ for all i provided $p > n$. We have graded embeddings $\mathbb{F}\otimes_R R[\mu^{-1}(0)]^{\mathrm{GL}_n(R)} \hookrightarrow \mathbb{F}[\mu^{-1}(0)]^{\mathrm{GL}_n(\mathbb{F})} \hookrightarrow \mathbb{F}[\mathbb{F}^{2n}]^{\mathfrak{S}_n}$. Since the graded dimensions in the first term and in the third term coincide, we see that all three embeddings are isomorphisms. Similarly, we see that $\mathbb{F}\otimes_R D(V_R) /\!/\!/_0 \mathrm{GL}_n(R) = D(V_{\mathbb{F}}) /\!/\!/_0 \mathrm{GL}_n(\mathbb{F})$ and $\operatorname{gr} D(V_{\mathbb{F}}) /\!/\!/_0 \mathrm{GL}_n(\mathbb{F}) = \mathbb{F}[T^*V] /\!/\!/ \mathrm{GL}_n(\mathbb{F})$, this is an exercise.

To check that $D(V_{\mathbb{F}}) /\!/\!/_0 \mathrm{GL}_n(\mathbb{F}) \cong D(\mathbb{F}^n)^{\mathfrak{S}_n}$ let us notice that we still have a homomorphism of filtered algebras from the left hand side to the right hand side, just as in characteristic 0. The main ingredient there was an identification $D(\mathfrak{g}_{reg}) /\!/\!/_0 G = D(\mathfrak{h}_{reg})^W$ which followed from $\mathfrak{g}_{reg} = G\times_{N_G(\mathfrak{h})} \mathfrak{h}_{reg} = (G\times \mathfrak{h}^{reg})/N_G(\mathfrak{h}_{reg})$. Of course the latter equality holds in characteristic p as well and we claim that it gives rise to $D(\mathfrak{g}_{reg}) /\!/\!/_0 G = D(\mathfrak{h}_{reg})^W$. This is so called reduction in stages. In order to establish this, we need the following standard lemma.

Lemma 4.2. *Let G be a reductive algebraic group acting freely on a smooth affine variety X_0. Then $D(X_0) /\!/\!/ G = D(X_0/G)$ (an isomorphism of filtered algebras).*

Proof of the lemma. We are going to produce a natural homomorphism $D(X_0/G) \to D(X_0) /\!/\!/_0 G$. Namely, $\mathbb{F}[X_0/G] = \mathbb{F}[X_0]^G$ naturally embeds into $D(X_0)^G$ and hence maps to $D(X_0) /\!/\!/_0 G$. Also $\operatorname{Vect}(X_0/G)$ is naturally identified with $[\operatorname{Vect}(X_0)/\operatorname{Vect}_{vert}(X_0)]^G$. Here we write $\operatorname{Vect}_{vert}(X_0)$ for the space of vector fields tangent to the orbits. This space coincides with $\mathbb{F}[X_0]\Phi(\mathfrak{g})$, which is a consequence of the claim that X_0 is a principal G-bundle over X_0/G. So $\operatorname{Vect}(X_0/G) = [\operatorname{Vect}(X_0)/\mathbb{F}[X_0]\Phi(\mathfrak{g})]^G$ naturally maps to $D(X_0) /\!/\!/_0 G$. It is not so hard to see that the maps $\mathbb{F}[X_0/G], \operatorname{Vect}(X_0/G) \to D(X_0) /\!/\!/_0 G$ extends to an algebra homomorphism $\iota_{X_0} : D(X_0/G) \to D(X_0) /\!/\!/_0 G$.

To see that this is an isomorphism we note that ι_{X_0} is functorial with respect to X_0/G (where we only consider etale morphisms of affine varieties). When $X_0 = G\times X_0/G$, the claim that ι_{X_0} is an isomorphism is straightforward. Now we are done thanks to the claim that $X_0 \to X_0/G$ is a principal G-bundle in etale topology. $\square$

From $\mathfrak{g}_{reg} = (G\times\mathfrak{h}^{reg})/N_G(\mathfrak{h}_{reg})$ we see that $D(\mathfrak{g}_{reg}) = D(G\times \mathfrak{h}^{reg}) /\!/\!/_0 N_G(\mathfrak{h})$. Also $D(\mathfrak{h}_{reg}) = D(G\times\mathfrak{h}^{reg}) /\!/\!/_0 G$ (for the left action of G). Since the $N_G(\mathfrak{h})$-action on $\mathfrak{h}$ factors through W, we see that the reduction of $D(\mathfrak{h}_{reg})$ with respect to $N_G(\mathfrak{h})$ amounts to taking W-invariants. So

$$D(\mathfrak{g}_{reg}) ///_0 G = D(G \times \mathfrak{h}^{reg}) ///_0 (G \times N_G(\mathfrak{h})) = D(\mathfrak{h}_{red})^W$$

(more precisely, there are homomorphisms from from the first and the third terms to the second one and they can be shown to be isomorphisms from the commutative level).

The associated graded of the homomorphism $D(V_{\mathbb{F}}) ///_0 \mathrm{GL}_n(\mathbb{F}) \to D(\mathbb{F}^n)^{\mathfrak{S}_n}$ is

$$\mathbb{F}[T^*V] /// \mathrm{GL}_n(\mathbb{F}) \xrightarrow{\sim} \mathbb{F}[\mathbb{F}^{2n}]^{\mathfrak{S}_n}.$$

As we have already checked, it is an isomorphism, so $D(V_{\mathbb{F}}) ///_0 \mathrm{GL}_n(\mathbb{F}) \cong D(\mathbb{F}^n)^{\mathfrak{S}_n}$. $\square$

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