CATEGORICAL AFFINE BRAID GROUP ACTION ON SPRINGER RESOLUTIONS

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Contents

1. Splitting on Springer fibers

We have an equivalence of triangulated categories $D^b(\mathrm{Coh}_\lambda(\tilde{\mathscr{D}})) \to D^b(\mathrm{Mod}_\lambda U)$. Now we link them to the category of coherent sheaves on X.

We state the main theorem of this section. Recall $\mathscr{B}_{\lambda,\chi} = \mathscr{B}^{(1)}_\chi \cap T^*_\lambda \mathscr{B}^{(1)} \subseteq \tilde{T}^*\mathscr{B}^{(1)} \times_{\mathfrak{h}^{*(1)}} \{\lambda\}$.

Theorem 1.1 ([BMR1])**.** *For all integral $\lambda \in \mathfrak{h}^*$, the Azumaya algebra $\tilde{\mathscr{D}}$ splits on the formal neighborhood of $\mathscr{B}_{\lambda,\chi}$ in $\tilde{T}^*\mathscr{B}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}$.*

As a consequence of Theorem 1.1, Morita theory gives equivalence of categories.

Theorem 1.2. *We have equivalence of abelian categories*

$$\mathrm{Coh}_{\mathscr{B}^{(1)}_{\lambda,\chi}}(\tilde{T}^*\mathscr{B} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*) \cong \mathrm{Mod}_{\chi,\lambda} \tilde{\mathscr{D}};$$

$$\mathrm{Coh}_{\mathscr{B}^{(1)}_{\lambda,\chi}}(T^*_\nu \mathscr{B}^{(1)}) \cong \mathrm{Mod}_\chi \mathscr{D}^\lambda.$$

The rest of this section will be devoted to the proof of Theorem 1.1.

Proposition 1.3 ([BG] § 3)**.** *Let $\chi = 0$, and $\zeta = (0, -\rho) \in \mathfrak{g}^{*(1)} \times_{\mathfrak{h}^{*(1)}//W} \mathfrak{h}^*_{unr}$, we have $U_0^{-\rho} \cong \mathrm{End}_k(\delta^\zeta)$.*

Corollary 1.4. *Let $\mu^{(1)} : T^*_{-\rho}\mathscr{B}^{(1)} \cong T^*\mathscr{B}^{(1)} \to \mathcal{N}^{(1)}$ be the moment map, then the natural map $\phi : \mu^{(1)*}U^{-\rho} \to \mathscr{D}^{-\rho}$ is an isomorphism.*

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Proof. The restriction of ϕ to the zero section $\mathscr{B}_{-\rho,0} \subseteq T^*\mathscr{B}^{(1)}$ is an isomorphism, since up to a faithfully flat base change, every fiber of this map is the isomorphism $U^{-\rho} \to \mathscr{E}\mathrm{nd}(\delta^\zeta)$. Let $\mathcal{K}$ and $\mathcal{C}$ be respectively the kernel an cokernel of ϕ. Then $\mathcal{C}$ restricted to $\mathscr{B}_{-\rho,0}$ is trivial, by the right exactness of restriction. Note that ϕ is G-equivariant, hence so are $\mathcal{C}$ and $\mathcal{K}$. Then by upper-semi-continuity, $\mathcal{C}$ is trivial, since every G-equivariant neighborhood of $\mathscr{B}_{-\rho,0}$ is the entire $T^*\mathscr{B}^{(1)}$. Now we have a short exact sequence $0 \to \mathcal{K} \to \mu^{(1)*}U^{-\rho} \to \mathscr{D}^{-\rho} \to 0$, with $\mathscr{D}^{-\rho}$ locally free, restriction of this sequence to $\mathscr{B}_{-\rho,0}$ is exact. □

Lemma 1.5. *Let $U^{\widehat{-\rho}}$ be the completion of U at the Harish-Chandra central character $-\rho$. It is an Azumaya algebra over $\mathfrak{g}^{*(1)}_{\widehat{\mathcal{N}^{(1)}}}$, the formal neighborhood of $\mathcal{N}^{(1)}$ in $\mathfrak{g}^{*(1)}$.*

Proof. Note that $U^{\widehat{-\rho}}|_{\mathcal{N}^{(1)}} \cong U^{-\rho}$ is a matrix algebra. Only need to show that $U^{\widehat{-\rho}}$ is locally free, which in turn amounts to show it is flat.

There are two facts: $\mathfrak{g}^{*(1)}$ is flat over $\mathfrak{h}^{*(1)}/W$; and $U(\mathfrak{g})$ is flat over $\mathfrak{h}^{*(1)}/W$ for p large enough. Therefore, $U^{\widehat{0}}$ is flat over $\mathfrak{g}^{*(1)}_{\widehat{\mathcal{N}^{(1)}}}$. So is $U^{\widehat{-\rho}}$ which is a translation of $U^{\widehat{0}}$. □

Corollary 1.6. *For any closed point $\chi \in \mathcal{N}^{(1)}$, $U^{\widehat{-\rho}}$ is an Azumaya algebra on $\mathfrak{g}^{*(1)}_{\hat{\chi}}$, the formal neighborhood of χ in $\mathfrak{g}^{*(1)}$. Moreover, it splits on $\mathfrak{g}^{*(1)}_{\hat{\chi}}$.*

To summarize, $\tilde{\mathscr{D}}$ splits on the formal neighborhood of $\mathscr{B}_{-\rho,\chi}$ in $\tilde{T}^*\mathscr{B} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$.

Now we look at the effect of twisting by a group character on twisted differential operators. Let $\pi : \tilde{X} \to X$ the the torus torsor. We look at $(\pi_*\mathscr{D}_{\tilde{X}} \otimes_k k_\eta)^H$. This sheaf clearly has an action by $\tilde{\mathscr{D}}_X$. But this sheaf can also be interpreted as the isotypical component in $\pi_*\mathscr{D}_{\tilde{X}}$ transforms under H by the character η. On the other hand, let τ_η be the translation automorphism on $\tilde{T}^*X^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$ shifting the second factor by η. Then $\tau_\eta^*\tilde{\mathscr{D}}_X$ also acts on $(\pi_*\mathscr{D}_{\tilde{X}} \otimes_k k_\eta)^H$. One can check this bimodule induces Morita equivalence between $\tau_\eta^*\tilde{\mathscr{D}}_X$ and $\tilde{\mathscr{D}}_X$.

If $\tilde{\mathscr{D}}_{\mathscr{B}}$ splits on the formal neighborhood of $\mathscr{B}_{-\rho,\chi}$, it also splits on the formal neighborhood of $\mathscr{B}_{\lambda,\chi}$ for integral η. This completes the proof of Theorem 1.1.

2. Affine braid group action

2.1. **Review of affine braid group.** For α a coroot and $n \in \mathbb{Z}$, let the hyperplanes $H_{\check{\alpha},n}$ given by $\{\lambda \in \Lambda \mid \langle \check{\alpha}, \lambda + \rho \rangle = np\}$. Open facets are called alcoves and codimension one facets are called faces. There is a special alcove, called the fundamental alcove, denoted by A_0, i.e., the alcove containing $(\epsilon + 1)\rho$ for small $\epsilon > 0$. It consists of those weights λ such that $0 < \langle \lambda + \rho, \check{\alpha} \rangle < p$ for all $\alpha \in \Phi^+$. The set of faces of A_0 will be denoted by I_{aff}.

Let $W_{\mathrm{aff}} := W \ltimes Q$ be the affine Weyl group. It acts naturally on Λ via the dot-action as follows. Elements in W acts via the usual dot-action. Element ν in the lattice acts by $\lambda \mapsto \lambda + p\nu$. The group W_{aff} is generated by reflections in affine hyperplanes $H_{\check{\alpha},n}$. The $(W_{\mathrm{aff}},\bullet)$-orbits in the set of faces are canonically identified with I_{aff}, the faces in the closure of the fundamental alcove A_0. The (Coxeter) generators of the group W_{aff} can be chosen to be the reflections in the faces of the alcove A_0.

For $\alpha \in I_{\mathrm{aff}}$, let $s_\alpha \in W_{\mathrm{aff}}$ be the reflection. Associated to α a standard generator $\widetilde{s_\alpha} \in B_{\mathrm{aff}}$. Then we define a set theoretical lifting $C : W_{\mathrm{aff}} \to B_{\mathrm{aff}}$, sending a minimal length decomposition $w = s_{\alpha_1}\cdots s_{\alpha_{l(w)}}$ to $\tilde{w} = \widetilde{s_{\alpha_1}}\cdots\widetilde{s_{\alpha_{l(w)}}}$. Then B_{aff} can be presented as follows. The generators are taken to be the image of C, and relations are given by $\widetilde{wu} = \tilde{w}\tilde{u}$ when $l(wu) = l(w) + l(u)$.

Similarly, the extended affine Weyl group $W'_{\mathrm{aff}} := W \ltimes \Lambda$ has the length function extending that on W_{aff}. We write W'_{aff} as $W_{\mathrm{aff}} \rtimes \mathrm{Stab}_{W'_{\mathrm{aff}}}(A_0)$. Then the length function on W'_{aff} is given by $l(w\omega) = l(w)$ for $\omega \in \mathrm{Stab}_{W'_{\mathrm{aff}}}(A_0)$. The extended affine Braid group B'_{aff} can be presented in a fashion similar to the non-extended one. The generators are $\tilde{w}$ for $w \in W'_{\mathrm{aff}}$, and relations are given by $\widetilde{wu} = \tilde{w}\tilde{u}$ when $l(wu) = l(w) + l(u)$. As $\mathrm{Stab}_{W'_{\mathrm{aff}}}(A_0)$ permutes I_{aff}, we have naturally $B'_{\mathrm{aff}} = B_{\mathrm{aff}} \rtimes \mathrm{Stab}_{W'_{\mathrm{aff}}}(A_0)$. A smaller set of generators of B'_{aff} can be chosen to be I_{aff} and $\mathrm{Stab}_{W'_{\mathrm{aff}}}(A_0)$.

2.2. **Review of intertwining functors.** Note that $\mathrm{Mod}_\lambda U = \mathrm{Mod}_\mu U$ for any λ and μ in the same $W'_{\mathrm{aff}}\bullet$-orbit. For any $\lambda,\ \mu \in \Lambda$ we define $I_{\mu\lambda} : D^b(\mathrm{Mod}_\lambda U) \to D^b(\mathrm{Mod}_\mu U)$ as the composition $R\Gamma_{\tilde{\mathscr{D}},\mu} \circ (\mathscr{O}_{\mu-\lambda} \otimes_{\mathscr{O}_{\mathscr{B}}} -) \circ \mathscr{L}^\lambda$. In the case when λ and μ are in the same $W'_{\mathrm{aff}}\bullet$-orbit and are both regular, this functor become an auto-equivalence.

The main goal of this section is to explain how these functors fit together to an affine braid group action. In characteristic zero, we have a braid group action on $D^b(\mathrm{Mod}_\lambda U)$ for regular λ. (See e.g., [B] and [T].) The action of generators are built up using translation functors.

For $\lambda,\ \mu \in \Lambda$, we define $T^\mu_\lambda : \mathrm{Mod}_\lambda U \to \mathrm{Mod}_\mu U$ sending M to $[V_{\mu-\lambda} \otimes M]_\mu$ here $V_{\mu-\lambda}$ is a finite dimensional representation with extremal weight $\mu - \lambda$, and $[-]_\mu$ means taking the component supported on the point μ in $\mathfrak{h}^*/\!/W$. As this functor is exact, it has clear counterpart on the level of D-modules. On $\mathscr{B}$ we take $\mathscr{V}_\eta$ as the vector bundle corresponding to the G-module V_η. We have

$$T^\mu_\lambda(R\Gamma_{\tilde{\mathscr{D}},\lambda}M) = [V_{\mu-\lambda}\otimes R\Gamma_{\tilde{\mathscr{D}},\lambda}M]_\mu = [R\Gamma_{\tilde{\mathscr{D}}}(V_{\mu-\lambda}\otimes M)]_\mu \cong R\Gamma_{\tilde{\mathscr{D}},\mu}([V_{\mu-\lambda}\otimes M)]_\mu).$$

The bundle $\mathscr{V}_\eta$ has a filtration by line bundles, or better by $V_\eta[\nu] \otimes \mathscr{O}_\nu$ and the smaller ν appears earlier in the filtration.

Proposition 2.1. (1) *If μ is in the closure of the facet of λ ($\lambda \to \mu$ for short), then $T^\mu_\lambda(R\Gamma_{\tilde{\mathscr{D}},\lambda}M) \cong R\Gamma_{\tilde{\mathscr{D}},\mu}(\mathscr{O}_{\mu-\lambda} \otimes M)$.*

(2) *If μ is regular and λ lies in a codimension 1 wall H, and $s_H(\mu) < \mu$, then there is an exact triangle*

$$R\Gamma_{\tilde{\mathscr{D}},s_H\mu}(\mathscr{O}_{\lambda-\mu} \otimes M) \to T^\mu_\lambda(R\Gamma_{\tilde{\mathscr{D}},\lambda}M) \to R\Gamma_{\tilde{\mathscr{D}},\mu}(\mathscr{O}_{\mu-\lambda} \otimes M) \to [1].$$

To prove this Propositon, we only need to count the weights occur in $(\lambda + \text{weights in } V_{\mu-\lambda}) \cap W\mu$. In case (1) there is only one which is μ. In case (2) there are two of them μ and $s_H\mu$, and μ occurs later.

Proposition 2.2. *If $\nu \to \mu \to \lambda$, then $T_\mu^\nu \circ T_\lambda^\mu \cong T_\lambda^\nu$ and $T_\mu^\lambda \circ T_\nu^\mu \cong T_\nu^\lambda$. In particular, if $\mu \to \nu \to \mu$ then $T_\mu^\nu \cong T_\nu^{\mu -1}$.*

Proof. By adjointness, we only need to prove one of them.

On the D-module level, twisting by line bundles composes as they should. This means $T_\mu^\nu \circ T_\lambda^\mu R\Gamma_{\tilde{\mathscr{D}},\lambda} \cong T_\lambda^\nu R\Gamma_{\tilde{\mathscr{D}},\lambda}$. Composing $\mathscr{L}$, and using the commutative diagram

$$\begin{array}{ccc} D^b(\mathrm{Coh}_\lambda \tilde{\mathscr{D}}) & \xleftarrow{\mathscr{L}^{\hat{\lambda}}} & D^b(\mathrm{Mod}_\lambda U) \\ {\scriptstyle R\Gamma_{\tilde{\mathscr{D}},\lambda}} \big\downarrow & \searrow^{R\tilde{\Gamma}} & \big\downarrow {\scriptstyle Ind_{U^{\hat{\lambda}}}^{\tilde{U}^{\hat{\lambda}}}} \\ D^b(\mathrm{Mod}_\lambda U) & \xleftarrow[Res_{U^{\hat{\lambda}}}^{\tilde{U}^{\hat{\lambda}}}]{} & D^b(\mathrm{Mod}_\lambda \tilde{U}) \end{array}$$

we have $T_\mu^\nu \circ T_\lambda^\mu Res_{U^{\hat{\lambda}}}^{\tilde{U}^{\hat{\lambda}}} Ind_{U^{\hat{\lambda}}}^{\tilde{U}^{\hat{\lambda}}} \cong T_\lambda^\nu Res_{U^{\hat{\lambda}}}^{\tilde{U}^{\hat{\lambda}}} Ind_{U^{\hat{\lambda}}}^{\tilde{U}^{\hat{\lambda}}}$.

Then $T_\mu^\nu \circ T_\lambda^\mu$ sits in the left hand side as a direct summand and T_λ^ν is a factor of the right hand side. We get $T_\mu^\nu \circ T_\lambda^\mu \to T_\lambda^\nu$. Applying them to the generator of the category U^λ to see that they are isomorphic as functors. □

Now we have the translation functors we can use them to build the reflection functors and intertwining functors as in [B]. Assume ν lies in a codimension 1 wall of the facet of μ. Define

$$R_{\mu|\nu} := T_\nu^\mu T_\mu^\nu : \mathrm{Mod}_\mu U \to \mathrm{Mod}_\mu U.$$

Corollary 2.3. *If μ is regular and ν, ν' lie in the same codimension 1 wall, then $R_{\mu|\nu} \cong R_{\mu|\nu'}$.*

This means $R_{\mu|\nu}$ depends only on the wall, not the character itself.

As $R_{\mu|\nu}$ is self-adjoint, we have two adjunctions. We define

$$\Theta_{\mu|\nu} := cone(\mathrm{id} \to R_{\mu|\nu}) \text{ and } \Theta'_{\mu|\nu} := cone(R_{\mu|\nu} \to \mathrm{id}).$$

Corollary 2.4. *If μ is regular and ν, ν' lies in the same codimension 1 wall, then $\Theta_{\mu|\nu} \cong \Theta_{\mu|\nu'}$.*

In the case when μ is regular, these two functors can be expressed as the intertwining functors defined as follows.

Lemma 2.5. *When μ is regular and ν lies in a codimension 1 wall H in the facet of μ, and $s_H\mu < \mu$, we have $\Theta'_{\mu|\nu} \cong I_{(s_H\mu)\mu}$ and $\Theta_{\mu|\nu} \cong I_{\mu(s_H\mu)}$.*

Note that $\mathrm{Mod}_{s_H\mu} U = \mathrm{Mod}_\mu U$.

Proof. For any module M we take the Γ-acyclic resolution of $\mathscr{L}M$. For acyclic C, have have from Proposition 2.1 (1) that $C \otimes \mathscr{O}(\nu - \mu)$ is acyclic. So is $[C \otimes \mathscr{O}(\mu - \nu) \otimes V_{\mu-\nu}]_\mu$. Using Proposition 2.1 (2) we know

$$C \otimes \mathscr{O}(s_H\mu - \mu) \to [C \otimes \mathscr{O}(\mu - \nu) \otimes V_{\mu-\nu}]_\mu \to C \to [1],$$

hence applying Γ we are done. □

2.3. **The affine braid group action on representation categories.** Now we describe how the functors Θ fit together to give an affine braid group action. As noted above, for any regular μ, and an arbitrary ν lying in a face of the alcove containing μ, the functor $\Theta_{\mu|\nu}$ depends only on the wall containing ν. For a regular λ, the faces of the alcove containing λ are naturally labeled by I_{aff}.

For regular λ, the orbit $W'_{\text{aff}} \bullet \lambda$ is a free orbit. We define a right action of W'_{aff} on this orbit by $(u \bullet \lambda)w = uw \bullet \lambda$ for u and $w \in W'_{\text{aff}}$.

For $w \in W'_{\text{aff}}$ and $\mu \in W'_{\text{aff}} \bullet \lambda$, we say w increases μ if $\mu s_1 \cdots s_i < \mu s_1 \cdots s_{i+1}$ for all i, where $w = s_1 \cdots s_{l(w)}\omega$ is a reduced decomposition with $l(\omega) = 0$.

Lemma 2.6. *Assume $\alpha \in I_{\text{aff}}$ and $\mu \in W'_{\text{aff}} \bullet \lambda$ is such that $\mu s_\alpha > \mu$. Let $\mu w = \nu$ then*

$$\begin{array}{ccc} D^b(\mathrm{Coh}_\mu \tilde{\mathscr{D}}_{\mathscr{B}}) & \xrightarrow{\mathscr{O}_{\nu-\mu} \otimes_{\mathscr{O}_{\mathscr{B}}} -} & D^b(\mathrm{Coh}_\nu \tilde{\mathscr{D}}_{\mathscr{B}}) \\ \uparrow{\scriptstyle \mathscr{L}^{\hat{\mu}}} & & \uparrow{\scriptstyle \mathscr{L}^{\hat{\nu}}} \\ D^b(\mathrm{Mod}_\mu U) & \xrightarrow{\Theta_{\mu|\nu}} & D^b(\mathrm{Mod}_\nu U), \end{array}$$

where ν is in the face of the alcove containing μ labeled by α.

Theorem 2.7 ([BMR2]). *Let $\lambda \in \Lambda$ be regular. The assignment*

$$\alpha \in I_{\text{aff}} \mapsto \Theta_{\lambda|\nu} =: \Theta_\alpha$$

for an arbitrary ν in the face of the alcove containing λ labeled by $\alpha \in I_{\text{aff}}$, and

$$\omega \in \mathrm{Stab}_{W'_{\text{aff}}}(A_0) \mapsto T_\lambda^{\omega\bullet\lambda} =: T^\omega$$

defines a (weak) right action of B'_{aff} on $D^b(\mathrm{Mod}_\lambda U)$.

The proof is the same as in [T].

Proof of Theorem 2.7. For $w \in^\lambda W'_{\text{aff}}$, let $w = \omega s_{\alpha_1} \cdots s_{\alpha_{l(w)}}$ be a decomposition with $l(\omega) = 0$ and $\alpha_i \in^\lambda I_{\text{aff}}$. We have

$$(\mathscr{O}(\lambda\omega\alpha_1 \cdots \alpha_{l(w)} - \lambda) \otimes_{\mathscr{O}_{\mathscr{B}}} -) \circ \mathscr{L}^{\lambda\omega\widehat{\alpha_1 \cdots \alpha}_{l(w)}} \cong \mathscr{L}^{\hat{\lambda}} T_\omega \Theta_{\alpha_1} \circ \cdots \circ \Theta_{\alpha_{l(w)}}.$$

□

Proposition 2.8. *Assume $w \in W'_{\mathrm{aff}}$, and $\mu \in W'_{\mathrm{aff}} \bullet \lambda$ is such that w increases μ. Let $\mu w = \nu$ then*

$$\begin{CD}
D^b(\mathrm{Coh}_\mu \tilde{\mathscr{D}}_{\mathscr{B}}) @>{\mathscr{O}_{\nu-\mu}\otimes_{\mathscr{O}_{\mathscr{B}}} -}>> D^b(\mathrm{Coh}_\nu \tilde{\mathscr{D}}_{\mathscr{B}}) \\
@A{\mathscr{L}^{\hat{\mu}}}AA @A{\mathscr{L}^{\hat{\nu}}}AA \\
D^b(\mathrm{Mod}_\mu U) @>{\Theta_{\tilde{w}}}>> D^b(\mathrm{Mod}_\nu U).
\end{CD}$$

2.4. **Action on the level of D-modules.** The followings are straightforward consequences of the construction of the affine braid group action.

Corollary 2.9. *Fix a regular $\lambda \in \Lambda$. For $\nu \in \Lambda^+ \subseteq W'_{\mathrm{aff}}$, let $\mu = \lambda + p\nu$. Then we have*

$$\begin{CD}
D^b(\mathrm{Coh}_\lambda \tilde{\mathscr{D}}_{\mathscr{B}}) @>{\mathscr{O}(\nu)\otimes_{\mathscr{O}_{\mathscr{B}^{(1)}}} -}>> D^b(\mathrm{Coh}_\mu \tilde{\mathscr{D}}) \\
@A{\mathscr{L}^{\hat{\lambda}}}AA @A{\mathscr{L}^{\hat{\mu}}}AA \\
D^b(\mathrm{Mod}_\lambda U) @>{\Theta_\nu}>> D^b(\mathrm{Mod}_\mu U).
\end{CD}$$

Corollary 2.10. *Fix a regular $\lambda \in \Lambda$. For $\nu \in \Lambda^+ \subseteq W'_{\mathrm{aff}}$, we have*

$$\begin{CD}
D^b(\mathrm{Coh}_{\mathscr{B}^{(1)}_\chi} \tilde{\mathfrak{g}}^{(1)}) @>{\mathscr{O}(\nu)\otimes_{\mathscr{O}_{\tilde{\mathfrak{g}}^{(1)}}} -}>> D^b(\mathrm{Coh}_{\mathscr{B}^{(1)}_\chi} \tilde{\mathfrak{g}}^{(1)}) \\
@V{\gamma_{\chi,\lambda}}VV @V{\gamma_{\chi,\lambda}}VV \\
D^b(\mathrm{Mod}_\lambda U) @>{\Theta_\nu}>> D^b(\mathrm{Mod}_\lambda U).
\end{CD}$$

3. Translation functors on the level of coherent sheaves

As we need to consider the singular character λ, hence in order to do this we need to consider a singular version of the localization theorem. This is essentially the same as the regular case, except that $D^b(\mathrm{Mod}_{\lambda,\chi} U)$ is localized to twisted D-modules on a partial flag variety.

Let $P \subseteq G$ be a parabolic subgroup, with unipotent radical J and Levi $\bar{P} = P/J$. Let $\mathscr{P} = G/P$ and $\tilde{\mathscr{P}} = G/J$ which is a $\bar{P}$-torsor over $\mathscr{P}$. Let $\tilde{\mathscr{T}}_{\mathscr{P}}$ be $(\pi_* \mathscr{T}_{\tilde{\mathscr{P}}})^{\bar{P}}$. The sheaf of enveloping algebras is $\tilde{\mathscr{D}}_{\mathscr{P}}$. The total space of $\tilde{\mathscr{T}}^*_{\mathscr{P}}$ is denoted by $\tilde{T}^*\mathscr{P}$. Let $\tilde{\mathfrak{g}}^*_{\mathscr{P}}$ be the subset of $\mathscr{P} \times \mathfrak{g}^*$ consisting of pairs $(\mathfrak{p}, \chi)$ with $\mathfrak{p} \in \mathscr{P}$ and $\chi \in \mathfrak{g}^*$ such that $\chi|_{\mathrm{nilp}(\mathfrak{p})} = 0$. It is endowed with two projections $p_{\mathfrak{g}} : \tilde{\mathfrak{g}}^*_{\mathscr{P}} \to \mathfrak{g}^*$ and $p_{\bar{\mathfrak{p}}^*} : \tilde{\mathfrak{g}}^*_{\mathscr{P}} \to \bar{\mathfrak{p}}^* =: \mathrm{Lie}(\bar{P})^*$. The center of $\tilde{\mathscr{D}}_{\mathscr{P}} =: \mathfrak{Z}(\tilde{\mathscr{D}}_{\mathscr{P}}) \cong \mathscr{O}_{\tilde{\mathfrak{g}}^{*(1)}_{\mathscr{P}} \times_{\bar{\mathfrak{p}}^{(1)}} \bar{\mathfrak{p}}}$.

Note that there is a natural map $\tilde{\pi}^{\mathscr{Q}}_{\mathscr{P}} : \tilde{\mathfrak{g}}^* = \tilde{\mathfrak{g}}^*_{\mathscr{B}} \to \tilde{\mathfrak{g}}^*_{\mathscr{P}}$ such that $\mathrm{pr}_1 : \tilde{\mathfrak{g}}^*_{\mathscr{B}} \to \mathfrak{g}^*$ factors through $\tilde{\pi}^{\mathscr{Q}}_{\mathscr{P}}$. As pr_1 is a proper morphism, so are $\tilde{\pi}^{\mathscr{Q}}_{\mathscr{P}}$ and $p_{\mathfrak{g}}$.

Let $\mathscr{P} = G/P$ be a partial flag variety. We say λ is $\mathscr{P}$-regular if it has singularity exactly $\mathscr{P}$.

Theorem 3.1 ([BMR2])**.** *Under the assumption that λ is $\mathscr{P}$-regular, we have an equivalence of categories*

$$R\Gamma_{\tilde{\mathscr{D}}_{\mathscr{B}},\lambda} : D^b(\mathrm{Coh}_{\lambda,\chi} \tilde{\mathscr{D}}_{\mathscr{P}}) \to D^b(\mathrm{Mod}_{\lambda,\chi} U).$$

Similarly we have the notion of generalized Springer fibers and $\tilde{\mathscr{D}}_{\mathscr{P}}$ splits on their formal neighborhoods. We summarize the equivalences of categories as follows

$$\begin{array}{ccc} D^b \operatorname{Coh}_{\mathfrak{Z}(\tilde{\mathscr{D}}_{\mathscr{P}})\times_{\mathfrak{Z}(U)}(\chi,W\lambda)}(\mathfrak{Z}(\tilde{\mathscr{D}}_{\mathscr{P}})) & \xrightarrow{\gamma_M} & D^b(\mathrm{Mod}_{\lambda,\chi} U) \\ & \searrow^{\otimes_{\mathfrak{Z}(\tilde{\mathscr{D}}_{\mathscr{P}})} M} \quad \nearrow^{R\Gamma_{\tilde{\mathscr{D}}_{\mathscr{B}},\lambda}} & \\ & D^b(\mathrm{Coh}_{\lambda,\chi} \tilde{\mathscr{D}}_{\mathscr{P}}). & \end{array}$$

We say an integral weight $\lambda \in \Lambda$ is $\mathscr{P}$-unramified for a parabolic subgroup P, if the map $\mathfrak{h}^*/W_P \to \mathfrak{h}^*/W$ is unramified at $W_P \bullet \lambda$.

For two parabolic subgroups $P \subseteq Q \subseteq G$ and $\pi : \mathscr{P} \to \mathscr{Q}$. The natural map $\tilde{\pi}^{\mathscr{P}}_{\mathscr{Q}} : \tilde{\mathfrak{g}}^*_{\mathscr{P}} \to \tilde{\mathfrak{g}}^*_{\mathscr{Q}}$ is also a proper morphism.

Proposition 3.2. [BMR2] *For $P \subseteq Q \subseteq G$ be two parabolic subgroups, and μ, $\nu \in \Lambda$ which are respectively $\mathscr{P}$ and $\mathscr{Q}$-regular unramified, for any $\chi \in \mathfrak{g}^{*(1)}$ we have*

$$T^\nu_\mu \circ \gamma^{\mathscr{P}}_{\chi,\mu} \cong \gamma^{\mathscr{Q}}_{\chi,\nu} \circ R\tilde{\pi}^{\mathscr{P}(1)}_{\mathscr{Q}*} \text{ and } T^\mu_\nu \circ \gamma^{\mathscr{Q}}_{\chi,\nu} \cong \gamma^{\mathscr{P}}_{\chi,\mu} \circ L\tilde{\pi}^{\mathscr{P}(1)*}_{\mathscr{Q}}$$

Proof. Again by ajointness, we only need to prove one.

$$\begin{aligned} & T^\nu_\mu[R\Gamma(M^{\mathscr{P}}_{\chi,\mu} \otimes_{\mathscr{O}_{Z(\tilde{\mathscr{D}}_{\mathscr{P}})}} \mathscr{F})] \\ \cong\ & T^\nu_\mu[R\Gamma(\pi^{\mathscr{B}}_{\mathscr{P}})^*(M^{\mathscr{P}}_{\chi,\mu} \otimes_{\mathscr{O}_{Z(\tilde{\mathscr{D}}_{\mathscr{P}})}} \mathscr{F})] \\ \cong\ & R\Gamma[\mathscr{O}_{\mathscr{B}}(\nu-\mu) \otimes_{\mathscr{O}_{\mathscr{B}}} (\pi^{\mathscr{B}}_{\mathscr{P}})^*(M^{\mathscr{P}}_{\chi,\mu} \otimes_{\mathscr{O}_{Z(\tilde{\mathscr{D}}_{\mathscr{P}})}} \mathscr{F})] \\ \cong\ & R\Gamma[(\pi^{\mathscr{B}}_{\mathscr{P}})^*(\mathscr{O}_{\mathscr{P}}(\nu-\mu) \otimes_{\mathscr{O}_{\mathscr{P}}} (M^{\mathscr{P}}_{\chi,\mu} \otimes_{\mathscr{O}_{Z(\tilde{\mathscr{D}}_{\mathscr{P}})}} \mathscr{F}))] \\ \cong\ & R\Gamma[(\pi^{\mathscr{B}}_{\mathscr{P}})^*(M^{\mathscr{P}}_{\chi,\nu} \otimes_{\mathscr{O}_{Z(\tilde{\mathscr{D}}_{\mathscr{P}})}} \mathscr{F})] \\ \cong\ & R\Gamma(M^{\mathscr{P}}_{\chi,\nu} \otimes_{\mathscr{O}_{Z(\tilde{\mathscr{D}}_{\mathscr{P}})}} \mathscr{F}) \\ \cong\ & R\Gamma[(\tilde{\pi}^{(1)})^*(M^{\mathscr{Q}}_{\chi,\nu}) \otimes_{\mathscr{O}_{Z(\tilde{\mathscr{D}}_{\mathscr{P}})}} \mathscr{F}] \\ \cong\ & R\Gamma[M^{\mathscr{Q}}_{\chi,\nu} \otimes_{\mathscr{O}_{Z(\tilde{\mathscr{D}}_{\mathscr{Q}})}} \tilde{\pi}^{(1)}_* \mathscr{F}]. \end{aligned}$$

□

Let α be a positive root and $\mathscr{P} = \mathscr{P}_\alpha$ the maximal parabolic subgroup. Then $\tilde{\mathfrak{g}}^*_{\mathscr{P}}$ will be denoted by $\tilde{\mathfrak{g}}^*_\alpha$, and the map $\tilde{\mathfrak{g}}^* \to \tilde{\mathfrak{g}}^*_\alpha$ is denoted by $\tilde{\pi}_\alpha$.

Corollary 3.3. *Let $\mu \in \Lambda$ be regular and ν be α-regular. Then*

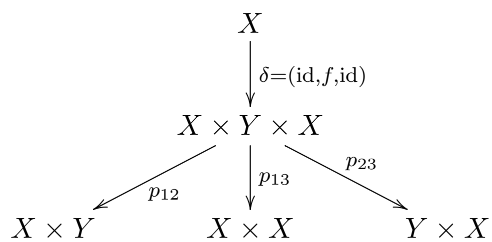
$$R_{\mu|\nu} \circ \gamma^{\mathscr{B}}_{\chi,\mu} \cong \gamma^{\mathscr{B}}_{\chi,\mu} \circ L\tilde{\pi}^{(1)*}_\alpha \circ R\tilde{\pi}^{(1)}_{\alpha*}.$$

Recall from [V] that $s_\alpha : \mathfrak{g}^*_{reg} \to \mathfrak{g}^*_{reg}$. Let $\Gamma_\alpha \subseteq \tilde{\mathfrak{g}}^* \times \tilde{\mathfrak{g}}^*$ be the closure of the graph of s_α and $\mathscr{O}_\alpha \in \mathrm{Coh}(\tilde{\mathfrak{g}}^* \times \tilde{\mathfrak{g}}^*)$ the structure sheaf of Γ_α.

Proposition 3.4. *Let $\mu \in \Lambda$ be regular and ν be α-regular. Then*

$$FM(\mathscr{O}_\alpha) \circ \gamma^{\mathscr{B}}_{\chi,\mu} \cong \gamma^{\mathscr{B}}_{\chi,\mu} \circ \Theta_{\mu|\nu}.$$

In order to prove Proposition 3.4, we need a general lemma about Fourier-Mukai transform. Let $f : X \to Y$ be a proper morphism with graph $\Gamma_f \subseteq X \times Y$ and $\Gamma^o_f \subseteq Y \times X$.



Lemma 3.5 (Lemma 1.2.2 in [R])**.** *Notations as above. We have*

(1) $Rf_* \cong FM(\mathscr{O}_{\Gamma_f})$ *and* $Lf^* \cong FM(\mathscr{O}_{\Gamma^o_f})$*;*

(2) $Rf_* \circ Lf^* \cong FM(\mathscr{O}_{\Gamma_f} * \mathscr{O}_{\Gamma^o_f})$*;*

(3) *the adjunction morphism* $Rf_* \circ Lf^* \to \mathrm{id}$ *is induced by the Fourier-Mukai of the following map*

$$\Delta_* \mathscr{O}_X \cong Rp_{13*}\mathscr{O}_{\delta X} \to Rp_{13*}\mathscr{O}_{\Gamma_f \times X \cap X \times \Gamma^o_f} \to \mathscr{O}_{\Gamma^o_f} * \mathscr{O}_{\Gamma_f}.$$

Proof of Proposition 3.4. Let $\tilde{\pi}_\alpha : \tilde{\mathfrak{g}}^* \to \mathfrak{g}^*_\alpha$. We have an isomorphism

$$p^*_{12}\mathscr{O}_{\Gamma_{\tilde{\pi}_\alpha}} \otimes^L p^*_{23}\mathscr{O}_{\Gamma^o_{\tilde{\pi}_\alpha}} \cong \mathscr{O}_{\Gamma_f \times X \cap X \times \Gamma^o_f}.$$

There is also an exact triangle

$$\mathscr{O}_\Delta \hookrightarrow \mathscr{O}_{\tilde{\mathfrak{g}}^* \times_{\tilde{\mathfrak{g}}^*_\alpha} \tilde{\mathfrak{g}}^*} \twoheadrightarrow \mathscr{O}_{\tilde{\mathfrak{g}}^*_\alpha}.$$

These facts combines to yield Proposition 3.4. □

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