REPRESENTATIONS OF FINITE DIMENSIONAL ALGEBRAS

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1. Injectives

Let A be a finite dimensional algebra over a field. Let A-mod denote the category of finite dimensional left A-modules. Let $e_1,\ldots,e_n$ be indecomposable commuting idempotents with $e_1+\ldots+e_n=1$. Recall that $Ae_i, i=1,\ldots,n$, are the indecomposable projectives in A-mod. Prove that $(e_iA)^*$ are the indecomposable injectives (an injective object is defined dually to a projective one).

2. Categorical characterization

Here we will describe A-mod in the categorical terms. Let C be an abelian category having finitely many simples, enough projectives (=any simple – and therefore any object – has a projective cover), and all objects of finite length. Further, suppose that, for some field K, all Hom's in C are endowed with structures of finite dimensional K-vector spaces. Prove that C is equivalent to the category of modules over a finite dimensional K-algebra A constructed as follows. Take a projective object P that has nonzero Hom's to all simples (a pro-generator). Then prove that for A one can take $\operatorname{End}(P)^{opp}$ and an equivalence is given by the functor $\operatorname{Hom}_C(P,\bullet)$.

3. Right exact functors

Let C, C' be two categories as in the previous problem. Let C-proj denote the subcategory of all projective objects in C. Show that any functor C-proj$\to C'$ uniquely extends to a right-exact functor $C\to C'$.

4. Serre subcategories

a) Let C be as above. By a Serre subcategory we mean a subcategory that is closed under taking quotients, subobjects and extensions. Produce a natural bijection between the Serre subcategories in C and the subsets of the set of simples in C.

b) Let C_0 be a Serre subcategory of C. Show that the inclusion functor has both left and right adjoint functors.

c) Now let A be a finite dimensional algebra such that $C=A$-mod. Show that for any Serre subcategory C_0 of C there is an idempotent $e\in A$ such that C_0 is the category of all A-modules annihilated by $I:=AeA$. Describe the left and right adjoint functors from (b) via I.

5. Quotients

a) Let C, A, C_0, e be as above. Consider the subalgebra eAe of A with unit e. There is a functor $\pi: A$-mod$\to eA$-mod sending M to eM. Check that π is exact and has both left and right adjoints.

b) Show that π is a quotient functor in the following sense: for any other exact functor π' from C to some other category C' (of the same nature as C) such that $\pi'(C_0) = 0$ (meaning, any object of C_0 is mapped to 0) there is a unique exact functor $\iota : eA$-mod$\to C'$ such that $\pi' = \iota \circ \pi$.

6. Principal block for the category O for $\mathfrak{sl}_2$

Let P be the sum of the two indecomposable projectives in O. Describe $A = \operatorname{End}(P)^{opp}$ and also the algebras eAe for the idempotents corresponding to the two projectives. Also describe the Verma and dual Verma modules as modules over A.