ABELIAN LOCALIZATION FOR CHEREDNIK ALGEBRAS

IVAN LOSEV

1. Introduction

The goal of this talk is to prove an analog of the Beilinson-Bernstein localization theorem for Cherednik algebras. Strictly speaking we will only do this for categories $\mathcal{O}$, in fact, the localization theorem for all modules follows from here.

Let us recall the notation and some definitions. We consider the reflection representation $\mathfrak{h}$ of the symmetric group $\mathfrak{S}_n$. By X we denote the "normalized" Hilbert scheme X, a resolution of $X_0 := (\mathfrak{h} \oplus \mathfrak{h}^*)/\mathfrak{S}_n$, we write π for the Hilbert-Chow morphism $X \to X_0$. The structure sheaf $\mathcal{O}_X$ has a two-parametric deformation $\mathfrak{A}_{un}$, a sheaf of algebras over $\mathbb{C}[z, \hbar]$, where z is a parameter of a commutative deformation $\tilde{X}$. We have a two dimensional torus $T_h \times T_c$ (h for "Hamiltonian", c for "contracting") acting on X and on $\mathfrak{A}_{un}$. We can consider the specialization $\mathfrak{A}_\lambda$ of $\mathfrak{A}_{un}$ to $\hbar = 1, z = \lambda$. It still carries a T_h-action.

We also consider the algebra $S(\mathfrak{h}\oplus\mathfrak{h}^*)\#\mathfrak{S}_n$ and its two-parametric deformation H_{un} over $\mathbb{C}[c, t]$ that again comes equipped with a $T_h \times T_c$-action. We consider the specialization $H_{1,c}$ of H_{un} to numerical parameters.

Let $\mathcal{P}$ denote the Procesi bundle on X, a $T_h \times T_c$-equivariant vector bundle constructed in Gufang's talk. Let $\tilde{\mathcal{P}}_\hbar$ be its deformation to a right $\mathfrak{A}_{un}$-module (note that this is different from the previous lecture, where we used a deformation to a left module). We write H^{loc}_{un} for its endomorphism sheaf. This is a sheaf of $\mathbb{C}[z, \hbar]$-algebras or of $\mathbb{C}[c, t]$-algebras, where $c \mapsto -z, t \mapsto \hbar$ (well, there is another choice of the map, and I'm not 100% sure what one needs to take...). As we have seen in the previous lecture, $\Gamma(H^{loc}_{un}) = H_{un}$, a $T_h \times T_c$-equivariant isomorphism of $\mathbb{C}[c, t]$-algebras. We remark that $\tilde{\mathcal{P}}_\hbar \otimes_{\mathcal{A}_{un}} \bullet$ is an equivalence of $\operatorname{Coh}(\mathfrak{A}_{un})$ and $\operatorname{Coh}(H^{loc}_{un})$. We also have functors $\Gamma : \operatorname{Coh}(H^{loc}_{un}) \to H_{un}\text{-mod}$ of taking global sections and $\operatorname{Loc} : H_{un}\text{-mod} \to \operatorname{Coh}(H^{loc}_{un})$, the localization functor given by $N \mapsto H^{loc}_{un} \otimes_{H_{un}} N$. We also consider the specializations of these functors to numerical parameters. We have similar functors between $\operatorname{Coh}(H^{loc}_{1,c}), H_{1,c}\text{-mod}$. Let us denote those by $\Gamma_{1,c}, \operatorname{Loc}_{1,c}$, they are obtained from $\Gamma, \operatorname{Loc}$ by specialization to numerical values of parameters. We also remark that $R\Gamma$ and $L\operatorname{Loc}$ are mutually inverse derived equivalences. This was basically established in the previous lecture (with slightly different functors).

Theorem 1.1. *Suppose c is not of the form $\frac{r}{m}$ with $1 < m \leqslant n$ and $r < 0$. Then $\Gamma_{1,c}, \operatorname{Loc}_{1,c}$ are mutually inverse equivalences.*

We will only prove this theorem for categories $\mathcal{O}$ (this is the most interesting case anyway, and the general one follows from this). Let us recall that by the category $\mathcal{O}$ for $H^{loc}_{1,c}$ we mean the full subcategory of $\operatorname{Coh}(H^{loc}_{1,c})$ consisting of all objects supported on $\pi^{-1}(\mathfrak{h}/\mathfrak{S}_n)$ that have a T_h-equivariant structure (under the equivalence $\operatorname{Coh} H^{loc}_{1,c} \to \operatorname{Coh} \mathfrak{A}_\lambda$, this becomes the category $\mathcal{O}$ for $\mathfrak{A}_\lambda$ introduced in the previous lecture). For technical reasons we will only consider $c > 0$ (if c satisfies neither this nor the conditions of the theorem, then the category $\mathcal{O}$ is semisimple (=boring)).

Let us provide some references for the previous theorem. It was first proved by Gordon and Stafford, [GS], in the context of so called $\mathbb{Z}$-algebras (noncommutative analogs of homogeneous coordinate rings). Then the theorem was proved by Kashiwara and Rouquier in the setting close to ours, [KR]. Both approaches had a restriction on parameters: $c \notin \frac{1}{2} + \mathbb{Z}$ that was later removed in [BE]. I am going to explain my own proof, [L], that has an advantage to generalize to so called cyclotomic Rational Cherednik algebras.

2. Correspondence between Verma modules

We will need two facts mentioned in Kostya's lecture:

- (i) $\mathcal{P}^*$ is flat over $S(\mathfrak{h})$, in other words, a basis $y_1, \ldots, y_{n-1}$ of $\mathfrak{h}$ forms a regular sequence in $\mathcal{P}^*$.
- (ii) Let e_λ stand for a primitive idempotent corresponding to an irreducible representation λ of $\mathfrak{S}_n$. Then $[\mathcal{P}^*/\mathcal{P}^*\mathfrak{h}]e_\lambda$ is supported on the union of the contracting components (for the T_h-action) Y_μ with $\mu \leqslant \lambda$ (well, Kostya had something about $\mathcal{P}$ and $\mathfrak{h}^*$ and we are dealing with $\mathcal{P}^*$ and $\mathfrak{h}$, but those are similar things).

Recall that Jose has defined Verma modules over $H_{1,c}$. We can use the same construction for H_{un}: we get $\Delta_{un}(\lambda) := H_{un} \otimes_{S(\mathfrak{h})\#\mathfrak{S}_n} \lambda = [H_{un}/H_{un}\mathfrak{h}]e_\lambda$ (let us note that $H_{un}/H_{un}\mathfrak{h} = H_{un} \otimes_{S(\mathfrak{h})\#\mathfrak{S}_n} \mathbb{C}\mathfrak{S}_n$). This module is flat over $\mathbb{C}[c,t]$, its specialization to $1, c$ gives $\Delta_{t,c}(\lambda)$. Similarly, we can define the Verma modules for H_{un}^{loc}: $\Delta_{un}^{loc}(\lambda) = [H_{un}^{loc}/H_{un}^{loc}\mathfrak{h}]e_\lambda$. Similarly, we can define their specializations $\Delta_{t,c}^{loc}(\lambda)$.

The following proposition establishes some properties of $\Delta_{un}^{loc}(\lambda)$ based on (i).

Proposition 2.1. *The following is true:*

1. H_{un}^{loc} *is flat as a right module over* $S(\mathfrak{h})[c,t]$.
2. $\Delta_{un}^{loc}(\lambda)$ *is flat as a* $\mathbb{C}[c,t]$*-module.*
3. $\Delta_{0,0}^{loc}(\lambda) = \mathcal{P} \otimes [\mathcal{P}^*/\mathcal{P}^*\mathfrak{h}]e_\lambda$ *and so has the support as specified in (ii).*
4. *We have* $R\Gamma(\Delta_{un}^{loc}(\lambda)) = \Delta_{un}(\lambda)$ *(meaning that the usual global sections are as specified and the higher derived sections vanish).*
5. *We have* $L\,\mathrm{Loc}(\Delta_{un}(\lambda)) = \Delta_{un}^{loc}(\lambda)$.

Proof. Because of the T_c-equivariance and the claim that H_{un}^{loc} is a flat over $\mathbb{C}[c,t]$ deformation of $\mathcal{E}nd(\mathcal{P})$, it is enough to show that $\mathcal{E}nd(\mathcal{P})$ is flat as a right module over $S(\mathfrak{h})$, but this follows from (i). So (1) is proved. (2) is a corollary of (1) (an exercise). (3) is a direct consequence of the definition of $\Delta_{0,0}^{loc}(\lambda)$.

Let us prove (4), (5) is given as an exercise. Because of (i), as an object of $D^b(\mathrm{Coh}^{T_c} H_{un}^{loc})$, $\Delta_{un}^{loc}(\lambda)$ is $K(H_{un}^{loc}, \mathfrak{h})e_\lambda$, where $K(H_{un}^{loc}, \mathfrak{h})$ is the Koszul complex:

$$\to H_{un}^{loc} \otimes \Lambda^2\mathfrak{h} \to H_{un}^{loc} \otimes \mathfrak{h} \to H_{un}^{loc}$$

All terms in $K(H_{un}^{loc}, \mathfrak{h})$ are acyclic for $R\Gamma$. Indeed, $R^i\Gamma(H_{un}^{loc}) = H^i(H_{un}^{loc}) = 0$ for $i > 0$ (because H_{un}^{loc} is a flat T_c-equivariant deformation of $\mathcal{E}nd(\mathcal{P})$, where the cohomology vanishing holds). So $R\Gamma(\Delta_{un}^{loc}(\lambda))$ is given by the complex $\Gamma(K(H_{un}^{loc}, \mathfrak{h}))e_\lambda$. But $\Gamma(H_{un}^{loc} \otimes \Lambda^i\mathfrak{h}) = \Gamma(H_{un}^{loc}) \otimes \Lambda^i\mathfrak{h} = H_{un} \otimes \Lambda^i\mathfrak{h}$. So $\Gamma(K(H_{un}^{loc}, \mathfrak{h})) = K(H_{un}, \mathfrak{h})$. We deduce that $R\Gamma(\Delta_{un}^{loc}(\lambda)) = K(H_{un}, \mathfrak{h})e_\lambda$ – which is a resolution of $\Delta_{un}(\lambda)$. (4) is proved. □

3. Localization theorem

Now we are ready to prove Theorem 1.1 for categories $\mathcal{O}$. Recall that $\mathcal{C} := \mathcal{O}(H_{1,c})$ is a highest weight category. There is an additional structure: the labeling set of its simples

– the set of partitions of n to be denoted by $P(n)$ is equipped with a partial order given by $\lambda < \mu$ if $c(\mathrm{cont}(\mu) - \mathrm{cont}(\lambda)) \in \mathbb{Z}_{>0}$. Then there are axioms. First, $\mathcal{C}$ is a length category, with finitely many simples, finite dimensional Hom's and enough projectives. Second, there are standard objects (that happen to coincide with Verma modules) $\Delta(\lambda)$ satisfying upper triangularity conditions explained in Jose's lecture. One consequence of those conditions is that standard objects are uniquely determined from the order: namely, consider the Serre subcategory $\mathcal{C}_{\leqslant\lambda}$ spanned by all simples $L(\mu)$ with $\mu \leqslant \lambda$. Then $\Delta(\lambda)$ is the projective cover of $L(\lambda)$ in $\mathcal{C}_{\leqslant\lambda}$. This is an exercise.

Now it turns out that $\mathcal{C}' := \mathcal{O}(\mathfrak{A}_\lambda) \cong \mathcal{O}(H^{loc}_{1,c})$ is also a highest weight category, this is established in a yet unpublished paper [BLPW]. The labeling set is $X^{T_h} \cong P(n)$ (the T_h-fixed points are the same as $T_c \times T_h$-fixed points – this is a VERY nice exercise). The order on $P(n)$ is the geometric order on the fixed points: we first define a preorder $\lambda \leqslant \mu$ if $Y_\lambda \cap \overline{Y}_\mu \neq \emptyset$, and then take its transitive closure. This order was described in Sasha's lecture – this is a standard dominance order on the set of partitions. The category $\mathcal{C}'_{\leqslant\lambda}$ consists of all objects supported on $\bigsqcup_{\mu\leqslant\lambda} Y_\mu$.

Now the proof of Theorem 1.1 is based on two statements.

Proposition 3.1. *Suppose that $c > 0$. Then the object $\Delta^{loc}_{1,c}(\lambda)$ is a standard object in $\mathcal{C}'$ corresponding to λ.*

Proof. The support of $\Delta^{loc}_{1,c}(\lambda)$ is the same thing as that of $\Delta^{loc}_{0,0}(\lambda)$. So $\Delta^{loc}_{1,c}(\lambda)$ lies in $\mathcal{C}'_{\leqslant\lambda}$ by (ii) and (3) of the lemma above. Now we can argue by the ascending induction on λ. Suppose we know our claim for all $\mu < \lambda$, i.e., $\Delta^{loc}_{1,c}(\mu) = \Delta'(\mu)$. Recall that $R\Gamma_{1,c}(\Delta^{loc}_{1,c}(\lambda)) = \Delta_{1,c}(\lambda)$ and that $R\Gamma$ is a derived equivalence. We have $\mathrm{RHom}_{\mathcal{C}}(\Delta(\lambda), \Delta(\mu)) = 0$ (exercise: why?) and hence $\mathrm{RHom}_{\mathcal{C}'}(\Delta^{loc}_{1,c}(\lambda), \Delta^{loc}_{1,c}(\mu)) = 0$. For similar reason, $\mathrm{RHom}_{\mathcal{C}'}(\Delta^{loc}_{1,c}(\lambda), \Delta^{loc}_{1,c}(\lambda)) = \mathbb{C}$ (in homological degree 0). It follows from the inductive assumption that $\Delta^{loc}_{1,c}(\lambda)$ is an indecomposable projective not lying in $\mathcal{C}'_{<\lambda}$, and so we are done. □

The second part of the proof is a purest abstract nonsense. Let us observe that if we refine a highest weight order, it is still a highest weight order (upper triangularity properties become just less restrictive). Under the conditions of the theorem (this is where we use the assumption on c) the orders for our categories are refined by a single order: $\lambda < \mu$ if $\mathrm{cont}(\lambda) < \mathrm{cont}(\mu)$. And now the abstract nonsense.

Proposition 3.2. *Let $\mathcal{C}, \mathcal{C}'$ be two highest weight categories whose simples are indexed by the same poset. Let $\mathcal{F} : \mathcal{C} \leftrightarrows \mathcal{C}' : \mathcal{G}$ be a pair of adjoint functors ($\mathcal{G}$ is right adjoint to $\mathcal{F}$). Suppose that $R\mathcal{G}(\Delta'(\lambda)) = \Delta(\lambda)$, $L\mathcal{F}(\Delta(\lambda)) = \Delta'(\lambda)$. Then $\mathcal{F}, \mathcal{G}$ are mutually inverse equivalences.*

I don't want to provide the proof. People who's done exercises on highest weight categories (=Pasha) may want to prove this is a problem (well, not quite an exercise).

This completes the proof of Theorem 1.1.

References

[BE] R. Bezrukavnikov, P. Etingof, *Parabolic induction and restriction functors for rational Cherednik algebras.* Selecta Math., 14(2009), 397-425.

[BLPW] T. Braden, A. Licata, N. Proudfoot, B. Webster, *Quantizations of conical symplectic resolutions II: category O and symplectic duality.* Unpublished manuscript.

[GS] I. Gordon, J.T. Stafford. *Rational Cherednik algebras and Hilbert schemes*. Adv. Math. 198 (2005), no. 1, 222-274.

[KR] M. Kashiwara, R. Rouquier. *Microlocalization of rational Cherednik algebras*. Duke Math. J. 144 (2008), no. 3, 525-573.

[L] I. Losev. *Abelian localization for cyclotomic Cherednik algebras*. arXiv:1402.0224.