SINGULAR SYMPLECTIC MODULI SPACES

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Abstract. These are notes of a talk given at the NEU-MIT graduate student seminar. It is based on the paper by Kaledin-Lehn-Sorger, showing examples of singular symplectic moduli spaces of sheaves that do not admit a symplectic resolution.

1. Introduction

Let X be a projective K3 surface and H be an ample divisor. Let $v \in H^{\text{even}}(X, \mathbb{Z})$ be the Mukai vector of a sheaf. Let M_v be the moduli space of Gieseker semistable sheaves with respect to the polarization H. Suppose

$$v = mv_0$$

for a primitive v_0, i.e. not an integral multiple of another Mukai vector, and $m \in \mathbb{N}$.

When v is primitive, that is $m = 1$, and H is generic, we know that M_v is an irreducible symplectic manifold. This reflects the geometry of the surface. Barbara Bolognese [Bol16] has demonstrated an example that the moduli space is actually a K3 surface. When the moduli space has higher dimension, Isabel Vogt [Vog16] has explained that it is deformation equivalent to Hilbert scheme of points.

When v is not primitive, the moduli space M_v is singular. However, the stable locus M_v^s still admits a non-degenerate 2-form. We are interested in the question whether the 2-form can be extended to resolutions of singularities of M_v. (Actually, if it extends to one, it extends to all.) Bolognese [Bol16] has shown us O'Grady's example [O'G99] where the answer is positive. This article is primarily interested in the cases where the 2-form does not extend to a resolution of singularities.

These are summarized in Table 2.[1] In this article, we will concentrate on the case where $v_0 = (r_0, c_0, a_0)$ and m satisfy the following conditions.

(1) Either $r_0 > 0$ and $c_0 \in \text{NS}(X)$, or $r = 0$, $c_0 \in \text{NS}(X)$ is effective, and $a_0 \neq 0$.
(2) $m \geq 3$ and $\langle v_0, v_0 \rangle \geq 2$, or $m = 2$ and $\langle v_0, v_0 \rangle \geq 4$.

The first condition makes sure that v_0 is the Mukai vector of a coherent sheaf. In the rest of this article, we will assume that v_0 and m satisfy these conditions.

We aim to demonstrate the following result.

Theorem. *If either $m \geq 2$ and $\langle v_0, v_0 \rangle > 2$ or $m > 2$ and $\langle v_0, v_0 \rangle \geq 2$, then M_{mv_0} is a locally factorial singular symplectic variety, which does not admit a proper symplectic resolution.*

[1] Similar statements also hold for abelian surfaces.

We have summarized the beautiful argument by Kaledin-Lehn-Sorger in Table 1. For the reader's convenience, we recall the Serre's condition (S_k) and regularity (R_k) in codimension k.

(S_k): A ring A satisfies condition S_k if for every prime ideal $\mathfrak{p} \subset A$, $\operatorname{depth} A_{\mathfrak{p}} \geq \min\{k, \operatorname{ht}(\mathfrak{p})\}$.
(R_k): A ring A satisfies condition S_k if for every prime ideal $\mathfrak{p} \subset A$ such that $\operatorname{ht}(\mathfrak{p}) \leq k$, $A_{\mathfrak{p}}$ is regular.

2. Preliminaries

2.1. **Construction of moduli spaces.** Let $v = v(E)$ be a Mukai vector and P_v be the corresponding Hilbert polynomial, i.e. $P_v(m) = \chi(E \otimes \mathscr{O}_X(mH))$. Suppose k is sufficiently large, $N = P_v(k)$, and $\mathcal{H} = \mathscr{O}_X(-kH)^{\oplus N}$. Let

$$R \subset \operatorname{Quot}_{X,H}(\mathcal{H}, P_v)$$

be the Zariski closure of the following subscheme

$$\{[q : \mathcal{H} \to E] \mid q \text{ GIT-semistable, } H^0(q(kH)) \text{ isom.}\},$$

equipped with a $\operatorname{PGL}(N)$-linearized ample line bundle. Let

$$R^s \subset R^{ss} \subset R$$

be the open subscheme of stable points and semistable points. The moduli space M_v of semistable sheaves is the GIT quotient

$$\pi : R^{ss} \to R^{ss} /\!\!/ \operatorname{PGL}(N) \cong M_v.$$

The orbit of $[q]$ is closed in R^{ss} if and only if E is polystable. In that case, the stabilizer subgroup of $[q]$ in $\operatorname{PGL}(N)$ is isomorphic to

$$\operatorname{PAut}(E) = \operatorname{Aut}(E)/\mathbb{C}^*.$$

Moreover, by Luna's slice theorem, there is a $\operatorname{PAut}(E)$-invariant subscheme $[q] \in S \hookrightarrow R^{ss}$ such that

$$(\operatorname{PGL}(N) \times S) /\!\!/ \operatorname{PAut}(E) \to R^{ss} \quad \text{and} \quad S /\!\!/ \operatorname{PAut}(E) \to M_v$$

are étale and

$$T_{[q]}S \cong \operatorname{Ext}^1(E, E).$$

2.2. **Kuranishi map and the key proposition.** Let $\mathbb{C}[\operatorname{Ext}^1(E,E)]$ be the ring of polynomial functions on $\operatorname{Ext}^1(E,E)$. Let

$$A := \mathbb{C}[\operatorname{Ext}^1(E,E)]^\wedge$$

be the completion at the maximal ideal $\mathfrak{m}$ of functions vanishing at 0. We denote the kernel of the trace map $\operatorname{Ext}^2(E,E) \to H^2(\mathscr{O}_X)$ by $\operatorname{Ext}^2(E,E)_0$. The automorphism group $\operatorname{Aut}(E)$ naturally acts on $\operatorname{Ext}^1(E,E)$ and $\operatorname{Ext}^2(E,E)_0$ by conjugation. Since scalars act trivially, this induces an action of $\operatorname{PAut}(E)$.

There is a linear map

$$\kappa : \mathrm{Ext}^2(E,E)_0^* \to \mathbb{C}[\mathrm{Ext}^1(E,E)]^\wedge,$$

called the *Kuranishi map*, with the following properties.

(1) The map κ is $\mathrm{PAut}(E)$-equivariant.
(2) Let I be the ideal generated by the image of κ. Then there are isomorphisms of complete rings

$$\hat{\mathscr{O}}_{S,[q]} \cong A/I \quad \text{and} \quad \hat{\mathscr{O}}_{M_{v},[E]} \cong (A/I)^{\mathrm{PAut}(E)}.$$

(3) For every linear form $\phi \in \mathrm{Ext}^2(E,E)_0^*$ and $e \in \mathrm{Ext}^1(E,E)$,

$$\kappa(\phi)(e) = \frac{1}{2}\phi(e \cup e) + \text{ higher order terms in e.}$$

Denote the quadratic part of the Kuranishi map by

$$\begin{aligned} \kappa_2 : \mathrm{Ext}^2(E,E)_0^* &\to S^2\mathbb{C}[\mathrm{Ext}^1(E,E)]^*, \\ \phi &\mapsto (e \mapsto \frac{1}{2}\phi(e\cup e)). \end{aligned}$$

Let $J \subset \mathbb{C}[\mathrm{Ext}^1(E,E)]$ be ideal generated by the image of κ_2. Then J is the defining ideal of $F = \mu^{-1}(0)$ where μ is the following map

$$\begin{aligned} \mu : \mathrm{Ext}^1(E,E) &\to \mathrm{Ext}^2(E,E)_0, \\ e &\mapsto \frac{1}{2}(e\cup e). \end{aligned}$$

Ideals $I \subset \mathbb{C}[\mathrm{Ext}^1(E,E)]^\wedge$ and $J \subset \mathbb{C}[\mathrm{Ext}^1(E,E)]$ are related as follows. First, notice the graded ring $\mathrm{gr}\, A$ associated to the $\mathfrak{m}$-adic filtration of $\mathbb{C}[\mathrm{Ext}^1(E,E)]^\wedge$ is canonically isomorphic to $\mathbb{C}[\mathrm{Ext}^1(E,E)]$. For any ideal $\mathfrak{a} \subset A$, let $\mathrm{in}(\mathfrak{a}) \subset \mathrm{gr}\, A$ denote the ideal generated by the leading terms (lowest degree terms) $\mathrm{in}(f)$, for all $f \in \mathfrak{a}$. Then,

$$J \subset \mathrm{in}(I).$$

and we have the following inequalities

$$\begin{aligned} \dim F &= \dim \mathrm{gr} A/J \\ &\geq \dim \mathrm{gr}\, A/\mathrm{in}(I) = \dim \mathrm{gr}(A/I) \\ &= \dim(A/I) \geq \mathrm{ext}^1(E,E) - \mathrm{ext}^2(E,E)_0. \end{aligned} \tag{1}$$

Suppose $v = mv_0$ where v_0 and m satisfy the conditions in the introduction. Then, the inequalities above are all equalities:

Proposition 1. *The null-fiber F is an irreducible normal complete intersection of dimension $\mathrm{ext}^1(E,E) - \mathrm{ext}^2(E,E)_0$. Moreover, it satisfies R_3.*

This statement actually holds more generally for a class of symplectic moment map. This is the key proposition in the paper [KLS06].

3. Normality, regularity and factoriality

In this section, we will show various regularity results.

Proposition 2. *Let H be an arbitrary ample divisor Let $E = \oplus_{i=1}^{s} E_i^{\oplus n_i}$ be a polystable sheaf such that $v(E_i) \in N v_0$. Consider $[q : \mathcal{H} \to E] \in R^{ss}$ and a slice $S \subset R^{ss}$ to the orbit of $[q]$. Then, $\mathscr{O}_{S,[q]}$ is a normal complete intersection domain of dimension*

$$\operatorname{ext}^1(E,E) - \operatorname{ext}^2(E,E)_0 \tag{2}$$

that has property R_3.

Proof. By Proposition 1, $F = \mu^{-1}(0) = \operatorname{Spec}(\operatorname{gr} A/J)$ is a normal complete intersection variety of dimension (2). Thus, we have equalities at all places in (1). Therefore, $J = \operatorname{in}(I)$. It follows that

$$\operatorname{gr}\hat{\mathscr{O}}_{S,[q]} = \operatorname{gr}(A/I) = \operatorname{gr} A/\operatorname{in}(I) = \Gamma(F, \mathscr{O}_F) \tag{3}$$

is a normal complete intersection. In particular, $\operatorname{gr}(\hat{\mathscr{O}}_{S,[q]})$ is Cohen-Macaulay, hence satisfies S_k for all $k \in \mathbb{N}$.

Moreover, $\operatorname{gr}(\mathscr{O}_{S,[q]}) = \operatorname{gr}(\hat{\mathscr{O}}_{S,[q]})$ is smooth in codimension 3. Then by Proposition 3, $\mathscr{O}_{S,[q]}$ itself is a normal complete intersection which satisfies R_3. □

Equalities (3) is crucial to the argument, relating the slice to the key proposition, Proposition 1.

The following statement in commutative algebra allows us to recover regularity properties of a local ring from those of its associated graded ring.

Proposition 3. *Let $(B, \mathfrak{m})$ be a noetherian local ring with residue field $B/\mathfrak{m} \cong \mathbb{C}$. Let $\operatorname{gr} B$ be the graded ring associated to the $\mathfrak{m}$-adic filtration. Then, $\dim B = \dim \operatorname{gr} B$ and if $\operatorname{gr} B$ is an integral domain, normal or a complete intersection, then the same is true for B. Moreover, if $\operatorname{gr} B$ satisfies R_k and S_{k+1}, for some $k \in \mathbb{N}$, then B satisfies R_k.*

The following result of R^{ss} being local factorial will be the basis to apply Drezet's result to prove the M_v is local factorial.

Proposition 4. *(1) Let H be a v-general ample divisor. Then R^{ss} is normal and locally a complete intersection of dimension $\langle v, v\rangle + 1 + N^2$. It satisfies R_3 and hence is locally factorial.*

(2) Suppose that $E = E_0^{\oplus m}$ for some stable sheaf E_0 with $v(E_0) = v_0$. Let H be an arbitrary ample divisor. There is an open neighborhood U of $[E] \in M_v$ such that $\pi^{-1}(U) \subset R^{ss}$ is locally factorial of dimension $\langle v, v\rangle + 1 + N^2$.

Proof. (1) Let $[q : \mathcal{H} \to E] \in R^{ss}$ be a point with closed orbit, and let $S \subset R^{ss}$ be a $\operatorname{PAut}(E)$-invariant slice through $[q]$. By Proposition 2, the local ring $\mathscr{O}_{S,[q]}$ is a normal complete intersection satisfying R_3. Being normal, locally a complete intersection, or having property R_3 are open properties [Gro61, 19.3.3, 6.12.9]. Hence there is a neighborhood U of $[q]$ in S that is normal, locally a complete intersection and has property R_3.

The natural morphism $\operatorname{PGL}(N) \times S \to R^{ss}$ is smooth. Therefore, every closed orbit in R^{ss} has a neighborhood that has the same properties.

Finally, every $\mathrm{PGL}(N)$ orbit of R^{ss} meets such an open neighborhood. Then, R^{ss} has the same properties. Hence, R^{ss} is locally factorial due to the following theorem of Grothendieck [Gro62, XI Corollary 3.14].

(2) The second assertion follows analogously.

□

Theorem 1 (Grothendieck)**.** *Let B be a noetherian local ring. If B is a complete intersection and regular in codimension ≤ 3, then B is factorial.*

Then, a result of Drezet [Dre91, Theorem A] implies that

Theorem 2. *Let H be a v-general ample divisor. The moduli space M_v is locally factorial.*

Remark. This is the property that distinguishes the examples studied here from O'Grady's examples. The examples studied here do not admit symplectic resolution.

4. Irreducibility

Before showing the irreducibility, let us first state the following preparatory result: if the moduli space has a "nice" connected component, then the component will be all of the moduli space.

Theorem 3. *Let X be a projective K3 or abelian surface. Suppose $Y \subset M_v$ be a connected component parametrizing only stable sheaves. Then $M_v = Y$.*

The idea of the proof of this theorem is as follows. Fix a point $[F] \in Y$ and suppose that there is a point $[G] \in M_v \backslash Y$. We can assume that there is a universal family $\mathbb{E} \in \mathrm{Coh}(Y \times X)$. Let $p : Y \times X \to Y$ and $q : Y \times X \to X$ be the projections. Since F and G are numerically equal, the same is true for the relative Ext-sheaves $\mathrm{Ext}^{\bullet}_p(q^*F, \mathbb{E})$ and $\mathrm{Ext}^{\bullet}_p(q^*G, \mathbb{E})$, by Grothendieck-Riemann-Roch. This will lead to a contradiction. For details of the argument, see [KLS06].

This theorem has the following important consequence.

Theorem 4. *Let $v = mv_0$ and H be a v-general ample divisor. Then, M_v is a normal irreducible variety of dimension $2 + \langle v, v \rangle$.*

Proof. By Proposition 4, R^{ss} is normal, therefore M_v is normal.

If $m = 1$, $M_v = M_{v_0}$ parametrizes stable sheave and hence M_v is smooth. Theorem 3 implies that M_v is irreducible.

By induction, assume now $m \geq 2$ and assertions have been proved for $1 \leq m' < m$. For every partition $m = m' + m''$, such that $1 \leq m' \leq m''$, consider

$$\begin{aligned} \phi(m', m'') : M_{m'v_0} \times M_{m''v_0} &\to M_{mv_0}, \\ ([E'], [E'']) &\mapsto [E' \oplus E''], \end{aligned} \tag{4}$$

and let $Y(m', m'') \subset M_v$ denote its image. Then, $Y(m', m'')$ are irreducible components of strictly semistable locus of M_v. Since all $Y(m', m'')$ are irreducible (by induction) and intersect in the points of the form $[E_0^{\oplus m}]$, $[E_0] \in M_{v_0}$, the strictly semistable locus is connected. Since M_v is normal, connected components are irreducible. In particular, there is a unique irreducible component that meets the strictly semistable locus. Theorem 3 excludes the possibility of other components. Therefore, M_v is irreducible. □

5. Proof of the main theorem

We will first show that the moduli space is indeed singular, and the singular locus has high comdimension.

Proposition 5. *The singular locus $M_{v,\mathrm{sing}}$ of M_v is nonempty and equals to the locus of strictly semistable sheaves. The irreducible components of $M_{v,\mathrm{sing}}$ correspond to integers m', $1 \le m' \le m/2$, and have codimension $2m'(m-m')\langle v_0, v_0\rangle - 2$, respectively. In particular,* $\operatorname{codim} M_{v,\mathrm{sing}} \ge 4$.

Proof. Recall that the strictly semistable locus is the union of $Y(m', m'')$, (4). Also notice that

$$\phi(m', m''): M_{m'v_0} \times M_{m''v_0} \to Y(m', m'')$$

is finite and surjective. A simple dimension calculation shows that they have the desired codimension.

Since M_v is smooth at stable points, it suffices to show that strictly semistable points are singular. It is enough to show that M_v is singular at a generic

$$[E' \oplus E''] \in Y(m', m''),$$

where E' and E'' are stable. In this case, $\mathrm{PAut}(E) \cong \mathbb{C}^*$, $\mathrm{Ext}^2(E, E) \cong \mathbb{C}$, and the Kuranishi map is completely determine by an invariant $f \in \mathbb{C}[\mathrm{Ext}^1(E, E)]^\wedge$. Moreover, according to properties of Kuranishi map,

$$\hat{\mathscr{O}}_{M_v,[E]} \cong (\mathbb{C}[\mathrm{Ext}^1(E,E)]^\wedge)^{\mathbb{C}^*}/(f).$$

The group $\mathbb{C}^*$ acts on

$$\mathrm{Ext}^1(E,E) \cong \mathrm{Ext}^1(E',E') \oplus \mathrm{Ext}^1(E',E'') \oplus \mathrm{Ext}^1(E'',E') \oplus \mathrm{Ext}^1(E'',E'')$$

with weights 0, 1, −1, and 0. Then

$$\mathrm{Ext}^1(E,E) /\!\!/ \mathbb{C}^* = \mathrm{Ext}^1(E',E') \times C \times \mathrm{Ext}^1(E'',E'')$$

where $C \subset M(d, \mathbb{C})$ is the cone of matrices of rank ≤ 1 and $d = \mathrm{ext}^1(E', E'') = m'm''\langle v_0, v_0\rangle \ge 2$. In particular, C is singular. The quotient of a singular local ring by a non-zero divisor cannot become regular. Therefore, $\hat{\mathscr{O}}_{M_v,[E]}$ is singular. □

A more precise statement of the main theorem is as follows

Theorem 5. *The moduli space M_v is a locally factorial symplectic variety of dimension $2 + \langle v, v\rangle$. The singular locus is non-empty and has codimension ≥ 4. All singularities are symplectic, but there is no open neighborhood of a singular point in M_v that admits a projective symplectic resolution.*

Symplectic singularities are in the sense of Beauville [Bea00]. A normal variety V has *symplectic singularities* if the nonsingular locus V_{reg} carries a closed symplectic 2-form whose pull-back in any resolution $Y \to V$ extends to a holomorphic 2-form on Y. In particular, this last condition is automatic if the singular locus V_{sing} has codimension ≥ 4, by Flenner [Fle88].

Proof. We have seen that M_v is locally factorial.

Mukai constructed a closed non-degenerate 2-form on M_v^s. We also know that the singular locus has codimension ≥ 4. Therefore, singularities are symplectic.

Let $[E] \in M_v$ be a singular point and $U \subset M_v$ a neighborhood of $[E]$. Suppose there is a projective symplectic resolution $\sigma : U' \to U$. A result of Kaledin [Kal06] implies that σ is semismall. Let E be the exceptional locus and $d = \dim E - \dim U_{\mathrm{sing}}$. Then $\dim U_{\mathrm{sing}} + 2d \leq \dim U'$. This, combined with $\operatorname{codim} U_{\mathrm{sing}} = 4$, implies

$$\operatorname{codim} E \geq 2.$$

On the other hand, since $\mathscr{O}_{M_v,[E]}$ is factorial, the exceptional locus has codimension 1 (see [Deb01]), contradiction. □

References

[Bea00] A. Beauville, *Symplectic singularities*, Invent. Math. 139(3), 541–549, 2000.

[Bol16] B. Bolognese, *Examples of hyperkähler manifolds as moduli spaces of sheaves on K3 surfaces*, preprint.

[Deb01] O. Debarre, *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, New York, xiv+233 pp., 2001.

[Dre91] J.-M. Drezet, *Points non factoriels de variétés de modules de faisceaux semi-stables sur une surface rationnelle*, J. Reine. Angew.Math. 413, 99–126, 1991.

[Fle88] *Extendability of differential forms on non-isolated singularities*, Invent. Math. 94(2), 317–326, 1988.

[Gro61] A. Grothendieck, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I.*, Inst. Hautes Études Sci. Publ. Math. No. 11, 1961, 167 pp.

[Gro62] A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2).* Sminaire de Géométrie Algébrique du Bois-Marie, 1962. North-Holland Publishing Co., Amsterdam, 1968.

[Kal06] D. Kaledin, *Symplectic singularities from the Poisson point of view*, J. Reine Angew. Math. 600, 135–156, 2006.

[KLS06] D. Kaledin, M. Lehn, and Ch. Sorger, *Singular symplectic moduli spaces*, Invent. Math. 164 (3), 591–614, 2004.

[O'G99] K. O'Grady, *Desingularized moduli spaces of sheaves on a K3*, J. Reine Angew. Math. 512, 49–117, 1999.

[Vog16] I. Vogt, *Deformation types of moduli spaces of stable sheaves on a K3 surface*, preprint.

Table 1. Road map

A key estimate (Prop. 1)	
↓	Prop. 2
S (étale slice) normal	
↓	Prop. 4
R^{ss} normal, loc. factorial	
↓	R^{ss} loc. factorial and Drezet's result
M_v loc. factorial	
↓	
$M_v = R^{ss} /\!\!/ \mathrm{PGL}$ normal	
↓	M_v conn. (Thm. 3)
M_v irreducible	
↓	
M_v singular, does not admit a symplectic resolution	

TABLE 2. M_{mv_0}

	$m = 1$	$m \geq 2$
$\langle v_0, v_0 \rangle = -2$	(Mukai) $M_{v_0} = \{[E_0]\}$	$M_v = \{[E_0^{\oplus m}]\}$
$\langle v_0, v_0 \rangle = 0$	(Mukai) X = K3 or abelian surface $\Rightarrow$ M_{v_0} = K3 or abelian surface	$M_v = S^m(M_{v_0})$ • sing. in codim. 2 • admits symp. resolution $M_{v_0}^{[n]} \to M_v$
$\langle v_0, v_0 \rangle \geq 2$	(Mukai, Huybrechts, O'Grady, Yoshioka) • X = K3 $\Rightarrow$ M_{v_0} def. equ. to $X^{[1+\frac{1}{2}\langle v_0, v_0 \rangle]}$ • X = ab. surf. $\Rightarrow$ M_{v_0} def. equ. to $\mathrm{Pic}_0(X) \times X^{[\frac{1}{2}\langle v_0, v_0 \rangle]}$	$m = 2$ & $\langle v_0, v_0 \rangle = 2$ (O'Grady, Rapagnetta, Lehn-Sorger) M_v admits symp. desing. by blowing up reduced singular locus ——— else (Kaledin-Lehn-Sorger) • M_{mv_0} loc. fact. sing. symp. var. • does not admit proper symp. resolution