DIFFERENTIAL EQUATIONS ON HYPERPLANE COMPLEMENTS II

NOTES BY VALERIO TOLEDANO LAREDO AND YAPING YANG

Contents

1. The simply connected torus and the adjoint torus

1.1. **Motivation.** We will introduce trigonometric connections associated to root systems. Since quantum differential equations are trigonometric. We will also focus on a special example of AKZ-connection, which corresponds to quantum differential equations on cotangent bundles of flag varieties.

1.2. Let E be a Euclidean vector space, $\Phi \subset E^*$ a root system. Denote $Q \subset \mathfrak{h}^*$ the root lattice, and $P \subset \mathfrak{h}^*$ the weight lattice.

Let $Q^\vee \subset E$ be the lattice generated by the coroots $\alpha^\vee, \alpha \in \Phi$, the coroot lattice is dual to the weight lattice $P \subset \mathfrak{h}^*$, and $P^\vee \subset E$ the dual weight lattice, which is dual to the root lattice Q.

Let $H = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{C}^*) = Q^\vee \otimes_{\mathbb{Z}} \mathbb{C}^*$ be the complex algebraic torus with Lie algebra $\mathfrak{h} = Q^\vee \otimes_{\mathbb{Z}} \mathbb{C}$. We call H the torus of simply connected type. For any root $\alpha \in \Phi$, we have the following diagram

$$
\begin{array}{ccc}
\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q^\vee & \xrightarrow{\alpha} & \mathbb{C} \\
\downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\
H = \mathbb{C}^* \otimes_{\mathbb{Z}} Q^\vee & \xrightarrow{e^\alpha} & \mathbb{C}^*
\end{array}
\tag{1}
$$

set

$$H_{\mathrm{reg}} = H \setminus \bigcup_{\alpha \in \Phi} \{e^\alpha = 1\}.$$

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The Weyl group W acts on H_{reg} freely. Since the subtori $\{e^\alpha = 1\}$ we are removing are the fixed points of the Weyl group action. Indeed, think $H = Q^\vee \otimes \mathbb{C}/\mathbb{Z}$, we have for $s_\beta \in W$, $z \in Q^\vee \otimes \mathbb{C}$, then, $s_\beta(z) = z - (\beta, z)\beta^\vee$. Thus,

$$\begin{aligned} & s_\beta(z) = z \\ \Longleftrightarrow & -(\beta, z)\beta^\vee \in Q^\vee \otimes \mathbb{Z} \\ \Longleftrightarrow & -(\beta, z) \in \mathbb{Z} \\ \Longleftrightarrow & e^\beta(z) = 1 \end{aligned}$$

Let $T = \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{C}^*) = P^\vee \otimes_{\mathbb{Z}} \mathbb{C}^*$ be the complex algebraic torus with Lie algebra $\mathfrak{h} = P^\vee \otimes_{\mathbb{Z}} \mathbb{C}$. We call T the adjoint torus. For any root $\alpha \in \Phi$, we have the following diagram

$$\begin{array}{ccc} \mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee & \xrightarrow{\alpha} & \mathbb{C} \\ \downarrow \exp & & \downarrow \exp \\ T = \mathbb{C}^* \otimes_{\mathbb{Z}} P^\vee & \xrightarrow{e^\alpha} & \mathbb{C}^* \end{array} \tag{2}$$

set

$$T_{\mathrm{reg}} = H \setminus \bigcup_{\alpha \in \Phi} \{e^\alpha = 1\}.$$

The action of Weyl group W on T_{reg} is not free. See the following example

Example 1.1. In the case of $\mathfrak{sl}_2$, we have only one positive root α. The root lattice Q is generated by α. The weight lattice is generated by λ, with $2\lambda = \alpha$. The coroot lattice $Q^\vee$ is generated by $\alpha^\vee$ and the coweight lattice is generated by $\lambda^\vee$, with $\alpha^\vee = 2\lambda^\vee$, and $(\alpha, \lambda^\vee) = 1$.

In this case, $H_{\mathrm{reg}} = \mathbb{C}^* \setminus \{\pm 1\}$, while $T_{\mathrm{reg}} = \mathbb{C}^* \setminus \{1\}$. The nontrivial element σ in the Weyl group $\mathbb{Z}_2$ acts by $z \mapsto z^{-1}$ in both case. It's obvious that W action on T_{reg} is not free, since it fixes the element -1.

The fundamental group of H_{reg}/W is called affine Braid group, the following proposition gives a presentation of the affine Braid group, which can be found in [3], Proposition 1.3.

Proposition 1.2. *The affine Braid group $\widehat{B_{\mathfrak{g}}}$ is generated by the finite braid group $B_{\mathfrak{g}}$ and the coroot lattice $Q^\vee$, such that the following relations are satisfied for all $1 \leq j \leq n$.*

- $S_j X_\mu = X_\mu S_j$, *if* $(\mu, \alpha_j) = 0$;
- $S_j X_\mu S_j = X_{s_j(\mu)}$, *if* $(\mu, \alpha_j) = 1$;

The orbifold fundamental groups of the space T_{reg}/W is called extended affine Braid group, the following proposition gives a presentation of the extended affine Braid group.

The presentation of $\widehat{B_{\mathfrak{g}}}^{\mathrm{ex}}$ is described in the following theorem, see [2], page 61, Theorem 1.2.5.

Theorem 1.3. *$\widehat{B_{\mathfrak{g}}}^{\mathrm{ex}}$ is generated by the finite braid group $B_{\mathfrak{g}}$ and the coweight lattice $P^\vee$, with the following relations*

(1) *S_i satisfies the braid relations, that is,*

$$\underbrace{S_i S_j S_i \cdots}_{m_{ij}} = \underbrace{S_j S_i S_j \ldots}_{m_{ij}},$$

where m_{ij} is the order of $s_i s_j$ is the Weyl group W.

(2) $[X_i, X_j] = 0$.
(3) $[S_i, X_j] = 0$, *for* $i \neq j$.
(4) $S_i X_i S_i = X_{s_i(\lambda_i^\vee)}$.

Let u^0 be a base point in T_{reg}/W, The generators X_i in $\widehat{B_{\mathfrak{g}}}^{\text{ex}}$ correspond to the path $u^0 + 2\pi\sqrt{-1}\lambda_i^\vee t$, for $0 \leq t \leq 1$. That is, the path $(u_1^0, \ldots, u_j^0 + 2\pi\sqrt{-1}t, \ldots, u_n^0)$. While the generator S_i corresponds to a path from u^0 to $s_i(u^0)$. More precisely, it's the path $u^0 + \frac{e^{\pi\sqrt{-1}t}-1}{2}(u^0, \alpha_i^\vee)\alpha_i = u^0 + \frac{\cos(\pi t)-1+\sqrt{-1}\sin(\pi t)}{2}u_i^0\alpha_i^\vee$.

Remark 1.4. In the following diagram:

$$\begin{array}{ccc} \widehat{B}_{\mathfrak{sl}_2} & \longrightarrow & \widehat{B}^{\text{ex}}_{\mathfrak{sl}_2} \\ \downarrow & & \downarrow \\ \widehat{B}_{\mathfrak{g}} & \longrightarrow & \widehat{B}^{\text{ex}}_{\mathfrak{g}}, \end{array} \tag{3}$$

where the left vertical map is given by: $S \mapsto S_i$, and $X_{\alpha^\vee} \mapsto X_{\alpha_i^\vee}$.

There is no map from $\widehat{B}^{\text{ex}}_{\mathfrak{sl}_2}$ to $\widehat{B}^{\text{ex}}_{\mathfrak{g}}$. Since, the presentation of $\widehat{B}^{\text{ex}}_{\mathfrak{sl}_2}$ is giving by T, X, satisfies the relation

$$TXT = X^{-1},$$

while in general, the relation in $\widehat{B}^{\text{ex}}_{\mathfrak{g}}$ becomes

$$S_i X_i S_i = X_i X_{\alpha_i^\vee}^{-1}.$$

2. Trigonometric connections

Reference for this section, see [7]. We work over the space T_{reg}/W.

Let A_{trig} be an algebra endowed with the following data:

- a set of elements $\{t_\alpha\}_{\alpha\in\Phi} \subset A_{\text{trig}}$ such that $t_{-\alpha} = t_\alpha$.
- a linear map $X : \mathfrak{h} \to A_{\text{trig}}$.

Consider the A_{trig}-valued connection on T_{reg} given by

$$\nabla_{\text{trig}} = d - \sum_{\alpha\in\Phi_+} \frac{d\alpha}{e^\alpha - 1} t_\alpha - du_i X(u^i). \tag{4}$$

where $\Phi_+ \subset \Phi$ is a chosen system of positive roots, $\{u_i\}$ and $\{u^i\}$ are dual bases of $\mathfrak{h}^*$ and $\mathfrak{h}_*$ respectively, and the summation over i is implicit.

The tail $du_i X(u^i)$ is necessary. There are two reasons for the appearance of the tail:

- One can think the $\text{Conf}_n\,\mathbb{C}^*$ as $\mathbb{C}^n$ removing $z_i = 0$, for $i = 1, \ldots, n$, and hyperplanes $z_i = z_j$, for $i \neq j$. For the form of the rational connections discussed in [8], there is one term like $\frac{\Omega_i dz_i}{z_i}$. The appearance of the tail in trigonometric connection comes from this term $\frac{\Omega_i dz_i}{z_i}$.
- Without the tail, the connection is neither flat nor W equivariant.

Remark 2.1. Unlike its rational counterpart, the connection (4) depends upon the choice of the system of positive roots $\Phi_+ \subset \Phi$. Let however $\Phi'_+ \subset \Phi$ be another such system, then

The connection (4) may be rewritten as

$$\nabla = d - \sum_{\alpha\in\Phi'_+} \frac{d\alpha}{e^\alpha - 1}\, t_\alpha - du_i\, X'(u^i)$$

where $X' : \mathfrak{h} \to A$ is given by

$$X'(v) = X(v) - \sum_{\alpha\in\Phi_+\cap\Phi'_-} \alpha(v)\, t_\alpha \tag{5}$$

2.1. **Delta form.** The connection can also be written as

$$\nabla_{\text{trig}} = d - \sum_{\alpha\in\Phi_+} \frac{e^\alpha + 1}{e^\alpha - 1} d\alpha t_\alpha - du_i Y(u^i),$$

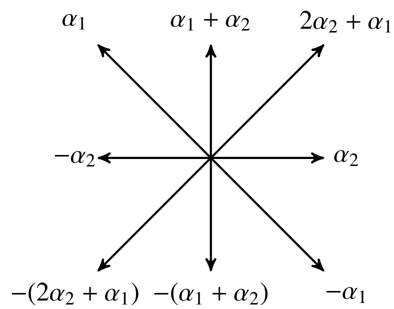
where $Y : \mathfrak{h} \to A_{\text{trig}}$ is given by:

$$Y(v) = X(v) - \frac{1}{2}\sum_{\alpha\in\Phi_+} \alpha(v) t_\alpha$$

For a subset $\Psi \subset \Phi$ and subring $R \subset \mathbb{R}$, let $\langle\Psi\rangle_R \subset E^*$ be the $R-$span of Ψ.

Definition 2.2. A root subsystem of Φ is a subset $\Psi \subset \Phi$ such that $\langle\Psi\rangle_{\mathbb{Z}} \cap \Phi = \Psi$. Ψ is *complete* if $\langle\Psi\rangle_{\mathbb{R}} \cap \Phi = \Psi$. If $\Psi \subset \Phi$ is a root subsystem, we set $\Psi_+ = \Psi \cap \Phi_+$.

Example 2.3. Let the root system be B_2. See the following picture:



$\Psi = \{\pm\alpha_2, \pm(\alpha_1 + \alpha_2)\}$ is not a root subsystem, while $\Psi = \{\alpha_1, \alpha_1 + 2\alpha_2\}$ is a root subsystem.

Theorem 2.4. *The connection (4) is flat if, and only if the following relations hold:*

(1): **(tt):** *For any rank 2 root subsystem $\Psi \subset \Phi$, and $\alpha \in \Psi$.*

$$[t_\alpha, \sum_{\beta\in\Psi_+} t_\beta] = 0.$$

(XX): *For any $u, v \in \mathfrak{h}$,*

$$[X(u), X(v)] = 0.$$

(tX): *For any $\alpha \in \Phi_+$, $w \in W$ such that $w^{-1}\alpha$ is a simple root and $u \in \mathfrak{h}$, such that $\alpha(u) = 0$,*

$$[t_\alpha, X_w(u)] = 0,$$

where $X_w(u) = X(v) - \sum_{\beta\in\Phi_+\cap w\Phi_-} \beta(v) t_\beta$.

(2): *Modulo the relations (tt), the relations (tX) are equivalent to* **(tY):**

$$[t_\alpha, Y(v)] = 0,$$

for any $\alpha \in \Phi$ and $v \in \mathfrak{h}$ such that $\alpha(v) = 0$.

Example 2.5. In the case of B_2, the (tt) relations become:

$$[t_{\alpha_1}, t_{\alpha_1+2\alpha_2}] = 0, [t_\gamma, \sum_\beta t_\beta] = 0.$$

while

$$[t_{\alpha_2}, t_{\alpha_1+\alpha_2}] \neq 0.$$

2.2. **Equivariance under W.** Assume now that the algebra A_{trig} is acted upon by the Weyl group W of Φ.

Proposition 2.6. *The connection ∇_{trig} is $W-$ equivariant if, and only if*

(1): *For any $\alpha \in \Phi$, simple reflection $s_i \in W$ and $x \in \mathfrak{h}$,*

$$s_i(t_\alpha) = t_{s_i(\alpha)} \tag{6}$$

$$s_i(X(x)) - X((s_i x)) = (\alpha_i, x) t_{\alpha_i}, \tag{7}$$

(2): *Modulo (6), the relation (7) is equivalent to $W-$ equivariance of the linear map $Y : \mathfrak{h} \to A_{\mathrm{trig}}$.*

Based on the above criterion of flatness and $W-$ Equivariance of the trigonometric connection, we make the following definition.

Definition 2.7. The holonomy Lie algebra A_{trig} is an algebra endowed generated by:

- a set of elements $\{t_\alpha\}_{\alpha\in\Phi} \subset A_{\mathrm{trig}}$ such that $t_{-\alpha} = t_\alpha$.
- a linear map $X : \mathfrak{h} \to A_{\mathrm{trig}}$.

satisfy the relations in Theorem 2.4 and Proposition 2.6.

The monodromy of the trigonometric connection (4) gives representation of the extended braid group $\widehat{B_{\mathfrak{g}}}^{\mathrm{ex}}$.

3. Basic ODE result

Proposition 3.1. *Let $\mathcal{U} \subset \mathbb{C}$ be a connected neighborhood of 0, $A \in \mathrm{End}(F)$, and $R : \mathcal{U} \to \mathrm{End}(F)$ a holomorphic function. Let $H_0 \in \mathrm{End}(F)$ be such that $[A, H_0] = 0$. Then, there exists a unique holomorphic function $H : \mathcal{U} \to \mathrm{End}(F)$ such that $H(0) = H_0$ and*

$$\frac{dH}{dz} = \frac{[A, H]}{z} + RH$$

Moreover, H is a holomorphic function of H_0.

4. Large volume limit solutions

4.1. Let F be a finite–dimensional complex vector space, and consider a flat connection on the trivial vector bundle over H_{reg} with fiber F of the form

$$\nabla = d - \sum_{\alpha\in\Phi_+} \frac{d\alpha}{e^\alpha - 1} t_\alpha - dX \tag{8}$$

where $\Phi_+ \subset \Phi$ is a chosen system of positive roots, $X : \mathfrak{h} \to \mathrm{End}(F)$ is a linear map, and dX is regarded as a translation–invariant 1–form on H. Note that if $\{u_i\}$ and $\{u^i\}$ are dual bases of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively, then $dX = du_i\, X(u^i)$, where the summation over i is implicit.

4.2. The connection ∇ descends to the trivial vector bundle over T_{reg}, where $T \cong \mathfrak{h}/P^\vee$ is the adjoint torus corresponding to the root system Φ. Let $\overline{T} \cong \mathbb{C}^n$ be the partial compactification determined by the embedding $T \hookrightarrow (\mathbb{C}^*)^n$ given by sending $p \in T$ to the point with coordinates $z_i = e^{-\alpha_i}(p)$, and let us rewrite ∇ with respect to the coordinates z_i.

Choosing $u_i = \alpha_i$ as basis of $\mathfrak{h}^*$, so that the dual basis $\{u^i\}$ of $\mathfrak{h}$ is given by the fundamental coweights $\{\lambda_i^\vee\}$ yields $du_i = -dz_i/z_i$ and

$$dX = du_i\, X(u^i) = -\frac{dz_i}{z_i}\, X(\lambda_i^\vee)$$

Further, if $\alpha = \sum_i m_\alpha^i \alpha_i$ is a positive root, then $e^\alpha = \prod_i z_i^{-m_\alpha^i}$ so that

$$\frac{d\alpha}{e^\alpha - 1} = \frac{e^{-\alpha}}{1 - e^{-\alpha}}\, d\alpha = -\sum_{i:m_\alpha^i \geq 1} m_\alpha^i \frac{z_i^{m_\alpha^i - 1} \prod_{j \neq i} z_j^{m_\alpha^j}}{1 - \prod_j z_j^{m_\alpha^j}}\, dz_i$$

which is a regular on the neighborhood of 0 in $\mathbb{C}^n$. Thus, in the coordinates z_i, ∇ takes the form

$$\nabla = d - \sum_{i=1}^n \frac{dz_i}{z_i} A_i - R(z) \tag{9}$$

where $A_i = X(\lambda_i^\vee)$, and R is a holomorphic 1–form on $\mathcal{U}$ with values in $\mathrm{End}(F)$.

4.3. **Existence.** Let $\mathcal{U} \subset \mathbb{C}^n$ be a polydisc centered at the origin, and ∇ a connection on $\mathcal{U} \times F$ of the form (9), where $A_i \in \mathrm{End}(F)$ and $R = \sum_i R_i dz_i$ is a holomorphic 1–form on $\mathcal{U}$ with values in $\mathrm{End}(F)$. The following is straighforward.

Lemma 4.1. *The connection ∇ is integrable iff the following holds for any $1 \leq i \neq j \leq n$*

$$[A_i, A_j] = 0 \qquad\qquad [A_i, R_j] = 0$$
$$\partial_i R_j - \partial_j R_i = [R_i, R_j]$$

Assume that ∇ is integrable and *non–resonant*, that is such that the eigenvalues of each A_i do not differ by non–zero integers.

Proposition 4.2. *Let $H_0 \in GL(F)$ be such that $[A_i, H_0] = 0$ for any i. Then, there exists a unique holomorphic function $H : \mathcal{U} \to GL(F)$ such that $H(0) = H_0$ and, for any determination of the logarithm, the function*

$$\Psi(z) = H(z) \cdot \prod_{i=1}^n z_i^{A_i}$$

is a fundamental solution of ∇.

Proof. H is required to satisfy the system of PDEs

$$\partial_i H = \frac{[A_i, H]}{z_i} + R_i H \tag{10}$$

together with the initial condition $H(0) = H_0$. The case $n = 1$ is covered by Proposition 3.1. Assume now that $n \geq 2$, set $\mathcal{U}^{(j)} = \mathcal{U} \cap \{z_j = \cdots = z_n = 0\}$ and assume by induction on $j = 1, \ldots, n-1$ the existence and uniqueness of a holomorphic function $H^{(j)} : \mathcal{U}^{(j)} \to GL(F)$ which satisfies (10) for all $i \leq j$, together with $H^{(j)}(0) = H_0$. Since A_{j+1} commutes with A_i and R_i for $i \leq j$, $[A_{j+1}, H^{(j)}]$ is a solution of (10) for any $i \leq j$ with initial condition 0 so that, by uniqueness, A_{j+1} commutes with $H^{(j)}$. For each $(z_1, \ldots, z_j) \in \mathcal{U}^{(j)}$, we may apply Proposition 3.1 to find a unique $H^{(j+1)} = H^{(j+1)}(z_{j+1}; z_1, \ldots, z_j)$ which satisfies (10) for $i = j+1$ together with the initial condition $H^{(j+1)}(0; z_1, \dots, z_j) = H^{(j)}(z_1, \dots, z_j)$. Since $H^{(j+1)}$ varies holomorphically in $z_1, \dots, z_j$, there remains to show that it satisfies (10) for $i = 1, \dots, j$. Denote by ∇_i the covariant derivative $\partial_i - z_i^{-1}\,\mathrm{ad}(A_i) - R_i$. Since $[\nabla_k, \nabla_{j+1}] = 0$ for $k = 1, \dots, j$, $\nabla_k H^{(j+1)}$ solves (10) for $i = j+1$, with initial condition $\nabla_k H^{(j)} = 0$ whence, by uniqueness $\nabla_k H^{(j+1)} = 0$. □

4.4. Fix henceforth a given determination of the logarithm.

Corollary 4.3. *If the eigenvalues of each $X(\lambda_i^\vee)$ do not differ by non–zero integers, there is a unique fundamental solution of the trigonometric connection (8) of the form*

$$\Psi = H \cdot \prod_i z_i^{-X(\lambda_i^\vee)}$$

where H is holomorphic on a neighborhood of the point 0 and such that $H(0) = 1$.

The fundamental solution Ψ will be called the *large volume limit* solution of ∇ (another name: asymptotically free).

Corollary 4.4. *Assume the eigenvalues of $X(\lambda_i^\vee)$ do not differ by non-zero integers, then the monodromy of the generators $X_{\lambda_j^\vee}$ is giving by:*

$$\mu_\Psi(X_{\lambda_j^\vee}) = \exp(2\pi\sqrt{-1}X(\lambda_j^\vee)).$$

5. Rank 1 reduction

The reference for this part is [2]. We would like to compute the monodromy of the generators S_j in the large volume limit solution Ψ. The idea is to reduce the calculations into rank 1 case.

Fix i, and let D_i be the trigonometric connection for the rank 1 root system corresponding to α_i, that is,

$$D_i = d - \frac{t_{\alpha_i}}{e^{\alpha_i} - 1} + X(\lambda_i^\vee)d\alpha_i$$

Let also Ψ_i be the large volume limit solution of D_i (corresponding to the neighborhood of the point $z_i := \exp(-\alpha_i) = 0$, then,

Theorem 5.1. *The monodromy of S_i in Ψ is equal to the monodromy of S_i in Ψ_i,*

$$\mu_\Psi(S_i) = \mu_{\Psi_i}(S_i).$$

Proof. By the existence of the large volume limit solution, we know that

$$\Psi(z) = H(z)\prod_{i=1}^n z_i^{X(\lambda_i^\vee)}.$$

Consider

$$\widetilde{\Psi}_i := \left(\lim_{(z_j \to 0, j \neq i)} H(z)\right)\prod_{i=1}^n z_i^{X(\lambda_i^\vee)},$$

Then, the AKZ system satisfied by $\widetilde{\Psi}_i$ is

$$\frac{\partial \widetilde{\Psi}_i}{\partial \alpha_i} = (k\frac{t_{\alpha_i}}{e^{\alpha_i} - 1} + X(\lambda_i^\vee))\widetilde{\Psi}_i,$$

and

$$\frac{\partial \widetilde{\Psi}_i}{\partial \alpha_j} = X(\lambda_j^\vee)\widetilde{\Psi}_i,$$

Since the monodromy $\mu(S_i)$ does not depend on the base point z^0, and the path connecting z^0 and $s_i(z^0)$, the path may be replaced by any deformation in T_{reg}/W. We can also degenerate this system by sending the parameters of such a deformation to the limits if the resulting system is well defined. Then the resulting monodromy will remain unchanged. Using this flexibility, we conclude that

$$\mu_\Psi(S_i) = \mu_{\widetilde{\Psi}_i}(S_i).$$

the latter is the "limiting monodromy" for a path with z_j, $(j \neq i)$ approaching zero.

Note that the following elements is equivariant under the action of s_i:

$$X(\lambda_j^\vee),\, j \neq i, X(\lambda_i^\vee) - \frac{1}{2}X(\alpha_i^\vee),$$

The reason is that:

$$s_i(X(x)) - X((s_i x)) = (\alpha_i, x)t_{\alpha_i}$$

Thus, $s_i(X(\lambda_j^\vee)) = X(s_i(\lambda_j^\vee))$, for $j \neq i$. Note the element

$$\lambda_i^\vee - \frac{1}{2}X(\alpha_i^\vee) = -\sum_{k\neq i} \frac{(\alpha_i, \alpha_k)}{(\alpha_i, \alpha_i)}\lambda_k^\vee.$$

Define $E^{(i)}(z)$ by

$$E^{(i)}(z) = \prod_{j=1}^{n} z_j^{X(\lambda_j^\vee)} \cdot z_i^{-X(\frac{\alpha_i^\vee}{2})}$$

it enjoys the following properties:

- $E^{(i)}(z)$ commutes with s_i, that is, $s_i \cdot E^{(i)}(z) = E^{(i)}(z) \cdot s_i$, where $\cdot$ means the composition as elements of $\mathrm{End}(F)$.
- $E^{(i)}(s_i(z)) = E^{(i)}(z)$.

The fact that $E^{(i)}(z)$ commutes with s_i follows from the observation that the following elements commute with s_i:

$$X(\lambda_j^\vee),\, j \neq i, X(\lambda_i^\vee) - \frac{1}{2}X_{\alpha_i^\vee}.$$

Setting

$$\Psi_i(z) = \widetilde{\Psi}_i(z)E^{(i)}(z) = \left(\lim_{z_j \to 0, j\neq i} H(z)\right) z_i^{X(\frac{\alpha_i^\vee}{2})}$$

Then, the system of equations satisfied by Ψ_i becomes precisely the AKZ equation in the rank A_1 case:

$$\frac{\partial \Psi_i}{\partial \alpha_i} = \left(\frac{t_{\alpha_i}}{e^{\alpha_i} - 1} + \frac{X_{\alpha_i^\vee}}{2}\right)\Psi_i,$$

and for $j \neq i$,

$$\frac{\partial \Psi_i}{\partial \alpha_j} = 0,$$

Let's check the monodromy of Ψ_i coincides with the monodromy of $\widetilde{\Psi}_i$ by using the properties of $E^{(i)}(z)$:

$$\mu_{\Psi_i}(S_i) = s_i(\Psi_i)^{-1}\Psi_i = E^{(i)}(z)s_i(\widetilde{\Psi}_i)^{-1}\widetilde{\Psi}_i E^{(i)}(z)^{-1} = E^{(i)}(z)\mu_{\widetilde{\Psi}_i}(S_i)E^{(i)}(z)^{-1} = \mu_{\widetilde{\Psi}_i}(S_i)$$

where the last equality follows from the relations in the extended affine braid group.

□

6. Affine KZ-connections

Even in the case of rank 1, the calculations of $\mu_\Psi(T)$ is not easy.

Definition 6.1. The degenerate affine Hecke algebra $\mathcal{H}'$ is the associative algebra generated by $\mathbb{C}W$ and the symmetric algebra $S\mathfrak{h}$, subject to the relations,

$$s_i x_u - x_{s_i(u)} s_i = k_i(u, \alpha_i),$$

for for any simple reflection $s_i \in W$ and linear generator x_u, for $u \in \mathfrak{h}$, and k_α a complex number.

Consider the following $\mathcal{H}'$ connection on X

$$\nabla_{AKZ} = d - \sum_{\alpha \in R_+} \frac{k_\alpha s_\alpha d\alpha}{e^\alpha - 1} - du_i x_{u^i}$$

By Proposition 2.6, the defining relations of $\mathcal{H}'$ are equivalent to integrability and equivariance of the AKZ connection. Thus, the connection is flat and $W-$equivariant if and only if s_α, x_j satisfy the relations from the definition of degenerate affine Hecke algebra $\mathcal{H}'$.

Definition 6.2. The affine Hecke algebra $\mathbb{H}_\mathfrak{g}$ associated with root system R is the quotient of the group algebra $\mathbb{C}\widehat{B_\mathfrak{g}}$ modulo the following quadratic relations

$$(S_i - q_i)(S_i + q_i^{-1}) = 0.$$

Proposition 6.3. *The monodromy of the flat connection ∇_{AKZ} factors through the affine Hecke algebra.*

Proof. Let $z_i := e^{-\alpha_i}$, choose a point $u \in \mathbb{C}^{*n}$, such that, $u_i = 1$, but $u_j > 1$, for $j \neq i$. Restrict the AKZ connection to the 1-dimensional subtorus $u\mathbb{C}^* = \{u \cdot t\}$, for $t \in \mathbb{C}^*$. We get

$$\nabla_{AKZ}|_{u\mathbb{C}^*} = d + \frac{k_{\alpha_i} s_{\alpha_i}}{1-t} dt + R,$$

where R is a regular form around the neighborhood of $t = 1$. Since assume $\alpha \neq \alpha_j$, then, there exists some j, such that $m_\alpha^j \neq 0$, and $u_j > 1$. In this case, since $1 - \prod_k u_k^{m_\alpha^k} < 1$, then the term $\sum_{\alpha \neq \alpha_i} \sum_k m_\alpha^k \frac{t^{-1} \prod_k u_k^{m_\alpha^k} t^{m_\alpha^k} s_\alpha}{1 - \prod_k u_k^{m_\alpha^k} t^{m_\alpha^k}}$ is regular around the neighborhood of $t = 1$.

THen, $\nabla_{AKZ}|_{u\mathbb{C}^*}$ around $e_i^\alpha = 1$ has a unique solution $\Phi = H(z_i)(1 - z_i)^{s_{\alpha_i}}$, if ks_{α_i} is non-resonant and $H(z_i)$ is regular around $z_i = 1$.

Since eigenvalues of s_{α_i} are ± 1, thus, eigenvalues of $\mu_\Phi(S_i)$ are $\pm e^{\pm\pi\sqrt{-1}k_{\alpha_i}}$, which gives the relation:

$$(\mu(S_i) - q_i)(\mu(S_i) + q_i^{-1}) = 0,$$

with $q_i = e^{\pi\sqrt{-1}k_{\alpha_i}}$.

□

7. Monodromy of Affine KZ-connections

7.1. Let $\mathrm{Rep}_{f.d}^{n.r.}(\mathcal{H}')$ be a category consisting of finite dimensional representation of $\mathcal{H}'$, such that the eigenvalues of $x(\lambda_i^\vee)$ don't differ by $\mathbb{Z}^*$, for any $i = 1, \ldots, n$. (where n.r. =non-resonant=representations where the eigenvalues of $x(\lambda_i^\vee)$ do not differ by non-zero integers). Under the non-resonant condition, the large volume limit solution Ψ exists.

Proposition 7.1. *The monodromy functor μ_Ψ induces an exact, faithful functor:*

$$\mu_\Psi : \mathrm{Rep}^{n.r.}_{f.d}(\mathcal{H}') \to \mathrm{Rep}_{f.d}(\mathbb{H}_{\mathfrak{g}}).$$

This functor has a right-sided inverse.

Remark 7.2. The functor μ_Ψ is consistent with the inclusions of rank 1 subalgebras of degenerate affine Hecke algebra

$$\mathcal{H}'_{sl_2^i} = \langle S_i, \frac{X_{\alpha_i^\vee}}{2} \rangle \hookrightarrow \mathcal{H}'_{\mathfrak{g}},$$

and affine Hecke algebra $\mathbb{H}_{sl_2^i} \hookrightarrow \mathbb{H}_{\mathfrak{g}}$, that is, we have the following commuting diagram:

$$\begin{array}{ccc} \mathrm{Rep}^{n.r.}_{f.d}(\mathcal{H}'_{sl_2^i}) & \xrightarrow{\mu_{\Psi}{}^i} & \mathrm{Rep}_{f.d}(\mathbb{H}_{sl_2^i}) \\ \uparrow & & \uparrow \\ \mathrm{Rep}^{n.r.}_{f.d}(\mathcal{H}'_{\mathfrak{g}}) & \xrightarrow{\mu_\Psi} & \mathrm{Rep}_{f.d}(\mathbb{H}_{\mathfrak{g}}) \end{array} \tag{11}$$

where the vertical maps are restrictions induced by the inclusions of algebras in diagram (3).

7.2. We wish to compute the monodromy of the AKZ connection on any finite dimensional non-resonant representation M of $\mathcal{H}'$. By the rank 1 reduction, it suffices to do this when $W = \mathbb{Z}_2$.

Let $\mathcal{H}'$ be the rank 1 degenerate affine Hecke algebra. Then, $\mathcal{H}'$ is generated by $s, x = x_{\lambda^\vee}$, with the relations

$$s^2 = 1, sx + xs = k.$$

Since $M = \mathcal{H}' \otimes_{\mathcal{H}'} M$, it suffices to do this when M is $\mathcal{H}'$ with the left regular action. But, this is an infinite-dimensional representation, so monodromy is not defined a priori.

However, as a $(\mathcal{H}', \mathbb{C}[x])$-bimodule, $\mathcal{H}'$ is the space of (algebraic) sections of the vector bundle $\mathcal{I}$ over $\mathbb{C}$ with fibre at $m \in \mathbb{C}$ over a given by the (finite-dimensional) induced module

$$I_m := \mathcal{H}' \otimes_{\mathbb{C}[x]} \mathbb{C}_m,$$

where $\mathbb{C}_m$ is endowed with the $\mathbb{C}[x]$ module structure given by evaluation at m. By the PBW theorem for $\mathcal{H}'$. I_m is isomorphic to $\mathbb{C}\mathbb{Z}_2$ as a left $\mathbb{Z}_2-$ module.

Since $\mathcal{H}'$ acts fibrewise on $\mathcal{I}$, the monodromy is well defined on those fibres which are non-resonant. Let's determine the point m for which I_m is non-resonant.

Choose a basis of I_m to be $\frac{s+e}{2} \otimes 1$, and $\frac{-s+e}{2} \otimes 1$. Then, the action of s under this basis is giving by the matrix:

$$s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Use the relation $sx + xs = k$, we get $x(s \otimes 1) = m(-s \otimes 1) + k(e \otimes 1)$, $x(-s \otimes 1) = m(s \otimes 1) - k(e \otimes 1)$, and $x(e \otimes 1) = m(e \otimes 1)$.

Then, the action of x under this basis $\frac{s+e}{2} \otimes 1$, and $\frac{-s+e}{2} \otimes 1$ is giving by the matrix:

$$x = \begin{pmatrix} \frac{k}{2} & \frac{k}{2} + m \\ -\frac{k}{2} + m & -\frac{k}{2} \end{pmatrix}$$

Then, $\det(x - \lambda) = \lambda^2 - m^2$, which implies that the eigenvalues of x are $\pm m$.

Thus, the induced representation I_m is non-resonant, if and only if $m \notin \frac{\mathbb{Z}}{2}$.

Since $x_{\alpha^\vee} = 2x$, the matrix for $x_{\alpha^\vee}$ is giving by:

$$x_{\alpha^\vee} = \begin{pmatrix} k & k+2m \\ -k+2m & -k \end{pmatrix} = k\left(s - \begin{pmatrix} 0 & -1-\frac{2m}{k} \\ 1-\frac{2m}{k} & 0 \end{pmatrix}\right)$$

7.3. **Rank 1 calculations.** The reference for this part is [1]. The goal of this subsection is to show the following explicit formula of $\mu(S)$ acting on the induced representation I_m.

Theorem 7.3. *Assume $m \notin \frac{\mathbb{Z}}{2}$, then $\mu(S)$ action on I_m is giving by:*

$$\mu(S) + \frac{q-q^{-1}}{\mu(X_{\alpha^\vee})^{-1}-1} = g(x_{\alpha^\vee})(s - kx_{\alpha^\vee}^{-1}), \tag{12}$$

where $g(v) = \frac{\Gamma^2(1+v)}{\Gamma(1+k+v)\Gamma(1-k+v)}$, and Γ is the gamma function.

Remark 7.4. Under the assumption $m \notin \frac{\mathbb{Z}}{2}$, the two operators $\mu(X_{\alpha^\vee})^{-1} - 1$ and $x_{\alpha^\vee}$ are invertible.

Let $v = \exp(-\alpha_i)$, then, the A_1- AKZ system becomes

$$\frac{\partial \Phi}{\partial v} + (\frac{ks}{1-v} + \frac{x_{\alpha^\vee}}{2v})\Phi = 0 \tag{13}$$

where $s, x_{\alpha^\vee} \in \mathcal{H}'$, such that: $s^2 = 1$, and $sx_{\alpha^\vee} + x_{\alpha^\vee}s = 2k$.

One may assume that

$$s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, x_{\alpha^\vee} = kS - k\begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}$$

acting on the induced representation I_m, where

$$\lambda = -1 - \frac{2m}{k}, \mu = 1 - \frac{2m}{k}.$$

Consider the following vector version of (13) with

$$\varphi = \begin{pmatrix} v^{-k/2}(v-1)^k f_1 \\ v^{k/2}(v-1)^{-k} f_2 \end{pmatrix}$$

Plug the above vector version φ in (13), we get

$$\frac{\partial f_1}{\partial v} = k\lambda v^k (v-1)^{-2k} f_2$$

and

$$\frac{\partial f_2}{\partial v} = k\mu v^{-k}(v-1)^{2k} f_1$$

Take the second derivation of both of them, we get

$$\frac{\partial^2 f_1}{\partial v^2} + (\frac{2k}{v-1} + \frac{1-k}{v})\frac{\partial f_1}{\partial v} - \frac{k^2\lambda\mu}{4v^2} f_1 = 0.$$

The classical hypergeometric equation is

$$z(1-z)\frac{\partial^2 u}{\partial z^2} + (c - (a+b+1)v)\frac{\partial u}{\partial z} - abu = 0.$$

It has two solutions $F(a,b,c;z)$ and $z^{1-c}F(a-c+1, b-c+1, 2-c; z)$. Here $F(a,b,c;z)$ is the hypergeometric function defined by

$$F(a,b,c;z) = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{2c(c+1)}z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{2\cdot 3c(c+1)(c+2)}z^3 + \cdots$$

From the above definition, it's obvious that, $F(a,b,c;0) = 1$ and

$$F(a,b,c;z) = F(b,a,c;z)$$

There are two properties of $F(a,b,c;z)$ we are going to use:

(1) $\frac{\partial F(a,b,c;z)}{\partial z} = F(a+1,b+1,c+1;z)$,

(2) There is a formula, see Page 289 of [10],

(14)

$$F(a,b,c;z) = \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}\frac{F(a,1-c+a,1-b+a,z^{-1})}{(-z)^a}+\frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}\frac{F(b,1-c+b,1-a+b,z^{-1})}{(-z)^b},$$

which computes the parallel transport of $F(a,b,c;z)$ from $z=0$ to $z=\infty$ in terms of the basis if solutions of the hypergeometric equation at $z=\infty$.

Now the equation satisfied by f_1 is a variant of hypergeometric equation. Making the change of variable that $f_1 = v^{\frac{k}{2}(\sqrt{1+\lambda\mu}+1)}F$, then the function F satisfies the classical hypergeometric equation with

$$a = k+\zeta, b = k, c = 1+\zeta,$$

where $\zeta = k\sqrt{1+\lambda\mu}$.

Thus, we get two solutions of f_1, that is:

$f_1 = v^{\frac{k}{2}(\sqrt{1+\lambda\mu}+1)}F(a,b,c,v)$, and $f_1 = v^{\frac{k}{2}(\sqrt{1+\lambda\mu}+1)}v^{1-c}F(a-c+1,b-c+1,2-c;z)$, which gives a fundamental solution of Φ:

$$\Phi = \begin{pmatrix} \varphi_1 & \varphi_1^* \\ \varphi_2 & \varphi_2^* \end{pmatrix}$$

where $\varphi_1 = v^{\zeta/2}(v-1)^k F(a,b,c,v)$, and $\varphi_2 = v^{\zeta/2}(v-1)^k(aF+2vabc^{-1}F(a+1,b+1,c+1,v))$. The notation $f^* := f(-\zeta)$ for any function f depending on ζ.

Using the fact that $F(a,b,c;v)\to 1$ as $v\to 0$, we have: as $v\to 0$,

$$\begin{pmatrix} \varphi_1 & \varphi_1^* \\ \varphi_2 & \varphi_2^* \end{pmatrix} \to \begin{pmatrix} v^{\zeta/2}(v-1)^k & v^{-\zeta/2}(v-1)^k \\ a(k\lambda)^{-1}v^{\zeta/2}(v-1)^k & a^*(k\lambda)^{-1}v^{-\zeta/2}(v-1)^k \end{pmatrix}$$

Denote

$$G := \begin{pmatrix} 1 & 1 \\ a(k\lambda)^{-1} & a^*(k\lambda)^{-1} \end{pmatrix},$$

and let $\Phi_0(v) := \Phi(v)G^{-1}$, we have, $\Phi_0(v)$ is a fundamental solution of A_1-AKZ system.

Lemma 7.5. *The solution $\Phi_0(v)$ is the large limit volume solution.*

Proof. First, let's diagonalize the matrix $x_{\alpha^\vee}$, we have

$$G^{-1}x_{\alpha^\vee}G = \begin{pmatrix} -\zeta & 0 \\ 0 & \zeta \end{pmatrix}.$$

To check $\Phi_0(v)$ is a large limit volume solution, we need to show that $\Phi_0(v)v^{\frac{x_{\alpha^\vee}}{2}} \sim 1$, as $v\to 0$.

Since

$$v^{\frac{x_{\alpha^\vee}}{2}} = G\begin{pmatrix} v^{\frac{-\zeta}{2}} & 0 \\ 0 & v^{\frac{\zeta}{2}} \end{pmatrix}G^{-1},$$

then,

$$\Phi_0(v)v^{\frac{x_{\alpha^\vee}}{2}} \sim \begin{pmatrix} v^{\zeta/2} & v^{-\zeta/2} \\ a(k\lambda)^{-1}v^{\zeta/2} & a^*(k\lambda)^{-1}v^{-\zeta/2} \end{pmatrix}\begin{pmatrix} v^{\frac{-\zeta}{2}} & 0 \\ 0 & v^{\frac{\zeta}{2}} \end{pmatrix}G^{-1} = 1,$$

as $v\to 0$. ∎

We are going to use the large volume limit solution $\Phi_0(v)$ to calculate the function g. By the formula (14), we have as $w = v^{-1} \to 0$,

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \to w^{\zeta/2}(w-1)^k sG \begin{pmatrix} \exp(-\pi i\zeta)\frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)} \\ \\ \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)} \end{pmatrix}$$

Hence the monodromy

$$\begin{aligned}\mu_\Phi(T) &= (s(\Phi(v)))^{-1}\Phi \\ &= (\Phi(w))^{-1}s\Phi \\ &= \begin{pmatrix} t_1 & t_1^* \\ t_2 & t_2^* \end{pmatrix},\end{aligned}$$

where

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} \exp(-\pi i\zeta)\frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)} \\ \\ \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)} \end{pmatrix}$$

So we get $\mu_{\Phi_0}(T) = G\mu_\Phi(T)G^{-1} = G\begin{pmatrix} t_1 & t_1^* \\ t_2 & t_2^* \end{pmatrix}G^{-1}$.

To finish the proof, we need the following formulas

$$x_{\alpha^\vee}G = G\begin{pmatrix} -\zeta & 0 \\ 0 & \zeta \end{pmatrix}$$

and

$$G^{-1}(S - kx_{\alpha^\vee}^{-1})G = \zeta^{-1}\begin{pmatrix} 0 & \zeta - k \\ \zeta + k & 0 \end{pmatrix}$$

Now rewrite both sides of the equality:

$$\mu(T) + \frac{q - q^{-1}}{\mu(X)^{-1} - 1} = g(x_{\alpha^\vee})(s - \frac{k}{x_{\alpha^\vee}}),$$

We have:

$$\begin{pmatrix} t_1 & t_1^* \\ t_2 & t_2^* \end{pmatrix} + \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} = \zeta^{-1}\begin{pmatrix} 0 & g(-\zeta)(\zeta - k) \\ g(\zeta)(\zeta + k) & 0 \end{pmatrix}$$

Compare both sides on the left lower corner of the matrices, we get $g(\zeta) = t_2\zeta(\zeta + k)^{-1}$. Use the property of Gamma function that $\Gamma(1 + z) = z\Gamma(z)$, we have:

$$g(\zeta) = t_2\zeta(\zeta+k)^{-1} = \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}\zeta(\zeta+k)^{-1} = \frac{\Gamma(\zeta)\Gamma(1+\zeta)\zeta}{\Gamma(k+\zeta)\Gamma(1+\zeta-k)(\zeta+k)} = \frac{\Gamma^2(1+\zeta)}{\Gamma(1+k+\zeta)\Gamma(1+\zeta-k)}.$$

By the rank 1 reduction, we get the following theorem:

Theorem 7.6. *Assume eigenvalues of x_{λ_i} do not differ by integers, for any $i = 1, \ldots, n$, then,*

$$\mu(S_i) + \frac{q_i - q_i^{-1}}{\mu(X_{\alpha_i^\vee})^{-1} - 1} = g(x_{\alpha_i^\vee})(s_i - kx_{\alpha_i^\vee}^{-1}),$$

where $g(x) = \frac{\Gamma^2(1+x)}{\Gamma(1+k+x)\Gamma(1-k+x)}$, and Γ is the gamma function.

7.4. Define an adapted completion of $\mathcal{H}'$. Note first that the defining relations of $\mathcal{H}'$ can be written as

$$sf(x) = f(-x)s + k/2\frac{(f(x) - f(-x))}{x} \tag{15}$$

where $f \in \mathbb{C}[x]$. Let O be the algebra of meromorphic functions on $\mathbb{C}$ with poles contained in $\frac{\mathbb{Z}}{2}$. We can define an algebra $\widehat{\mathcal{H}'}$ as the quotient of $\mathbb{C}W \otimes O$ by the relations (15). Then

- finite dimensional representations of $\widehat{\mathcal{H}'}$ are the same as finite dimensional representations of $\mathcal{H}'$ supported (as $\mathbb{C}[x]$-modules) away from $\frac{\mathbb{Z}}{2}$.
- the same holds if we restrict to non-resonant representations on both sides.

Since the monodromy can be regarded as a map

$$\mu : \mathbb{H} \to \widehat{\mathcal{H}'},$$

it follows that the formula (12) compute the monodromy on finite dimensional non-resonant representations of $\mathcal{H}'$ supported away from $\frac{\mathbb{Z}}{2}$.

Appendix A. Yangians and Trigonometric Casimir connection

Recall the definition of Yangian:

Definition A.1. The Yangian $Y(\mathfrak{g})$ is the associative algebra over $\mathbb{C}[\hbar]$ generated by elements $x, J(x), x \in \mathfrak{g}$ subject to the relations

in terms of generators $z, J(z)$, for $z \in \mathfrak{g}$, with the property that

- $\lambda x + \mu y$ (in $Y(\mathfrak{g})$) $= \lambda x + \mu y$ (in $\mathfrak{g}$).
- $xy - yx = [x, y]$
- $J(\lambda x + \mu y) = \lambda J(x) + \mu J(y)$
- $[x, J(y)] = J([x, y])$
- $[J(x), J([y, z])]+[J(z), J([x, y])]+[J(y), J([z, x])] = \hbar^2([x, x_a], [[y, x_b], [z, x_c]])\{x^a, x^b, x^c\}$
- $[[J(x), J(y)], [z, J(w)]]+[[J(z), J(w)], [x, J(y)]] = \hbar^2([x, x_a], [[y, x_b], [[z, w], x_c]])\{x^a, x^b, J(x^c)\}$,

for any $x, y, z, w \in \mathfrak{g}$ and $\lambda, \mu \in \mathbb{C}$, where $\{x_a\}, \{x^a\}$are dual bases of $\mathfrak{g}$ with respect to $(,)$ and

$$\{z_1, z_2, z_3\} = \frac{1}{24}\sum_{\sigma \in S_3} z_{\sigma(1)}z_{\sigma(2)}z_{\sigma(3)}$$

The following is Drinfeld's new realization of $Y(\mathfrak{g})$. Let $a_{ij} := \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ be the entries of the Cartan matrix A of $\mathfrak{g}$. Set $d_i := \frac{(\alpha_i,\alpha_i)}{2}$, so that $d_i a_{ij} = d_j a_{ji}$ for any $i, j \in I$.

Definition A.2. The Yangian $Y_\hbar(\mathfrak{g})$ is the associative algebra, free over $\mathbb{C}[\hbar]$, generated by $X^\pm_{i,r}$, and $H_{i,r}(i \in I, r \in \mathbb{N})$, with the following defining relations

$$[H_{i_1,r_1}, H_{i_2,r_2}] = 0, [H_{i_1,0}, X^\pm_{i_2,s}] = \pm d_{i_1} a_{i_1,i_2} X^\pm_{i_2,s}, [X^+_{i_1,r_1}, X^-_{i_2,r_2}] = \delta_{i_1 i_2} H_{i_1,r+s} \tag{16}$$

$$[H_{i_1,r_1+1}, X^\pm_{i_2,r_2}] - [H_{i_1,r_1}, X^\pm_{i_2,r_2+1}] = \pm\hbar\frac{d_{i_1}a_{i_1,i_2}}{2}S(H_{i_1,r_1}, X^\pm_{i_2,r_2}) \tag{17}$$

$$[X^\pm_{i_1,r_1+1}, X^\pm_{i_2,r_2}] - [X^\pm_{i_1,r_1}, X^\pm_{i_2,r_2+1}] = \pm\hbar\frac{d_{i_1}a_{i_1,i_2}}{2}S(X^\pm_{i_1,r_1}, X^\pm_{i_2,r_2}) \tag{18}$$

$$\sum_{\pi \in S_j}[X^\pm_{i_1,r_{\pi(1)}}, [X^\pm_{i_1,r_{\pi(2)}}, \dots, [X^\pm_{i_1,r_{\pi(j)}}, X^\pm_{i_2,s}] \dots]] = 0, \tag{19}$$

where $j = 1 - a_{i_1,i_2}$, $r_1, \dots, r_j, s \in \mathbb{N}$.

Choose root vectors $X_\alpha \in \mathfrak{g}_\alpha$ for any $\alpha \in \Phi$ such that $(X_\alpha, X_{-\alpha}) = 1$ and let

$$\kappa_\alpha = X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha$$

be the truncated Casimir operator.

Then, the relation between the two presentations is giving by the following formula:

$$X^{\pm}_{i,1} = J(X^{\pm}_i) - \lambda\omega^{\pm}_i, H^{\pm}_{i,1} = J(H^{\pm}_i) - \lambda\nu^{\pm}_i,$$

where

$$\omega^{\pm}_i = \pm\frac{1}{4}\sum_{\alpha\in\Phi^+} S([X^{\pm}_i, X^{\pm}_\alpha], X^{\mp}_\alpha) - \frac{1}{4}S(X^{\pm}_i, H_i),$$

and

$$\nu_i = \frac{1}{4}\sum_{\alpha\in\Phi^+}(\alpha_i, \alpha)\kappa_\alpha - \frac{H_i^2}{2}.$$

Lemma A.3. *There is a Lie algebra homomorphism* $A_{\text{trig}} \to Y(\mathfrak{g})$*, given by:*

$$t_\alpha \mapsto \kappa_\alpha$$

and

$$Y(t) \mapsto -2J(t)$$

In terms of the generator $X(t)$*, we have:*

$$X(t) \mapsto \frac{\hbar}{2}\sum_{\alpha\in\Phi_+}(t, \alpha)\kappa_\alpha - 2J(t)$$

Definition A.4. The trigonometric Casimir connection of $\mathfrak{g}$ is the connection $\nabla_{\text{trig},C}$ given by

$$\nabla_{\text{trig},C} = d - \sum_{\alpha\in\Phi_+}\frac{e^\alpha + 1}{e^\alpha - 1}d\alpha\kappa_\alpha + 2du_i J(u^i)$$

Theorem A.5. *The trigonometric Casimir connection is flat and* W*-equivariant.*

Remark A.6. Let V be a finite-dimensional $Y(\mathfrak{g})$-module and $\mathbb{V}$ the holomorphically trivial vector bundle over H_{reg} with fibre V . The connection $\nabla_{\text{trig},C}$ induces a flat connection on $\mathbb{V}$. To push it down to the quotient by W we use the "up and down" trick to circumvent the fact that W does not in general act on V.

Specifically, since V is an integrable $\mathfrak{g}$-module, the triple exponentials

$$\exp(e_{\alpha_i})\exp(-f_{\alpha_i})\exp(e_{\alpha_i}) \in \mathrm{GL}(V)$$

defined by a choice of simple root vectors $e_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$, $f_{\alpha_i} \in \mathfrak{g}_{-\alpha_i}$ are well-defined elements of $\mathrm{GL}(V)$. They give rise to an action on V of an extension $\tilde{W}$ of W by the sign group $\mathbb{Z}_2^{\dim\mathfrak{h}}$ called the Tits extension $\tilde{W}$ of W. It's a fact that $\tilde{W}$ is a quotient of the affine braid group $\hat{E}_\mathfrak{g}$ which may therefore be made to act on V . It is then easy to check that the pull-back of the flat vector bundle $(\mathbb{V}, \nabla)$ to the universal cover of H_{reg} is equivariant under $\hat{B}_\mathfrak{g}$ acting by deck transformations on the base and through the $\tilde{W}$-action on the fibres.

A.1. **The affine KZ connection.** The degenerate affine Hecke algebra $\mathcal{H}'$ of W is, very roughly speaking, the Weyl group of the Yangian $Y(\mathfrak{g})$. Let K be the vector space of W-invariant fuctions $\Phi \to \mathbb{C}$ and denote the natural linear coordinates on K by $k_\alpha, \alpha \in \Phi/W$.

Definition A.7. The degenerate affine Hecke algebra $\mathcal{H}'$ associated to Weyl group W is the algebra over $\mathbb{C}[K]$ generated the by group algebra $\mathbb{C}W$ and the symmetric algebra $S\mathfrak{h}$ subject to the relations

$$s_i x_u - x_{s_i(u)} s_i = k_{\alpha_i} \alpha_i(u),$$

for any simple reflection $s_i \in W$ and linear generator $x_u, u \in \mathfrak{h}$, of $S\mathfrak{h}$.

The AKZ connection is the trigonometric, $\mathcal{H}'$-valued connection given by

$$\nabla_{\text{aff},KZ} = d - \sum_{\alpha \in \Phi_+} \frac{e^\alpha + 1}{e^\alpha - 1} d\alpha k_\alpha s_\alpha - du_i x(u^i)$$

Appendix B. Monodromy of trigonometric Casimir connections

Similar as the rational Casimir connection, we make a conjecture that the monodromy of the trigonometric Casimir connection $\nabla_{\text{trig},C}$ is equivalent to action of $\hat{B}_\mathfrak{g}$ coming from the quantum Weyl group operators of the quantum loop algebra $U_\hbar(L\mathfrak{g})$.

Let $L\mathfrak{g} = \mathfrak{g}[t, t^{-1}]$ be the loop algebra of $\mathfrak{g}$.

Definition B.1. The quantum loop algebra $U_\hbar(Lg)$ is generated by $E_i.F_i, H_i$, for $i = 0, 1, \dots, n$, (or $i \in I \sqcup \{0\}$), where: $H_0 = -H_\theta = -\sum_{i \in I} a_i H_i$, where $\theta \in \mathfrak{h}^*$ is the highest root and the integers a_i are given by $\theta^\vee = \sum_i a_i \alpha_i^\vee$. modulo relations.

Proposition B.2. *The quantum loop algebra $U_\hbar(Lg)$ is a Hopf algebra over the ring of formal power series $\mathbb{C}[[\hbar]]$, generated elements $E_{i,k}$, $F_{i,k}$, and $H_{i,k}$ subject to the following relations:*

(QL1): *For $i, j \in I$, and $r, s \in \mathbb{Z}$,*

$$[H_{i,r}, H_{j,s}] = 0$$

(QL2): *For any $i, j \in I$, and $k \in \mathbb{Z}$,*

$$[H_{i,0}, E_{j,k}] = a_{ij} E_{j,k}, [H_{i,0}, F_{j,k}] = -a_{ij} F_{j,k}$$

(QL3): *For any $i, j \in I$, and $k \in \mathbb{Z}^*$,*

$$[H_{i,r}, E_{j,k}] = \frac{[ra_{ij}]_{q_i}}{r} E_{j,r+k}, [H_{i,r}, F_{j,k}] = -\frac{[ra_{ij}]_{q_i}}{r} F_{j,r+k}$$

(QL4): *For any $i, j \in I$, and $k \in \mathbb{Z}$,*

$$E_{i,k+1} E_{j,l} - q_i^{a_{ij}} E_{jl} E_{i,k+1} = q_i^{a_{ij}} E_{i,k} E_{j,l+1} - E_{j,l+1} E_{i,k}$$

$$F_{i,k+1} F_{j,l} - q_i^{-a_{ij}} F_{jl} F_{i,k+1} = q_i^{-a_{ij}} F_{i,k} F_{j,l+1} - F_{j,l+1} F_{i,k}$$

(QL5): *For any $i, j \in I$, and $k, l \in \mathbb{Z}$,*

$$[E_{i,k}, F_{j,l}] = \delta_{ij} \frac{\psi_{i,k+l} - \phi_{i,k+l}}{q_i - q_i^{-1}}$$

(QL6): *Let $i \neq j$, and set $m = 1 - a_{ij}$. For every $k_1, \dots, k_m \in \mathbb{Z}$, and $l \in \mathbb{Z}$*

$$\sum_{\pi \in S_n} \sum_{s=0}^{m} (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} E_{i,k_{\pi(1)}} \cdots E_{i,k_{\pi(s)}} E_{j,l} E_{i,k_{\pi(s+1)}} \cdots E_{i,k_{\pi(m)}} = 0$$

$$\sum_{\pi\in S_n}\sum_{s=0}^{m}(-1)^s\begin{bmatrix} m \\ s \end{bmatrix}_{q_i} F_{i,k_{\pi(1)}}\cdots F_{i,k_{\pi(s)}}F_{j,l}F_{i,k_{\pi(s+1)}}\cdots F_{i,k_{\pi(m)}} = 0$$

where

$$\psi_i(z) = \exp(\frac{\hbar d_i}{2}H_{i,0})\exp\left((q_i - q_i^{-1})\sum_{s\geq 1}H_{i,s}z^{-s}\right)$$

and

$$\phi_i(z) = \exp(\frac{-\hbar d_i}{2}H_{i,0})\exp\left(-(q_i - q_i^{-1})\sum_{s\geq 1}H_{i,-s}z^{s}\right)$$

By a finite-dimensional representation of $U_\hbar(L\mathfrak{g})$, we shall mean a module $\mathcal{V}$ which is topologically free and finitely-generated over $\mathbb{C}[[\hbar]]$. Such a $\mathcal{V}$ is integrable and therefore endowed with a quantum Weyl group action of the affine braid group $\hat{B}_\mathfrak{g}$. This action is given by letting the generator corresponding to $i\in\hat{I} = I\sqcup\{0\}$ act by

$$\overline{S_i}^\hbar v = \sum_{a,b,c\in\mathbb{Z},a-b+c=-\lambda(\alpha_i^\vee)}(-1)^b q_i^{b-ac}E_i^aF_i^bE_i^c v,$$

where $v\in\mathcal{V}$ if of weight $\lambda\in\mathfrak{h}^*$ and X_i^a is the divided power $\frac{X^a}{[a]_i!}$ with

$$q = e^\hbar, q_i = q^{\frac{(\alpha_i,\alpha_i)}{2}}$$

It is known that the Yangian $Y(\mathfrak{g})$ and the quantum loop algebra $U_\hbar(L\mathfrak{g})$ have the same finite-dimensional representation theory. By analogy with the quantum Weyl group description of the monodromy of the (rational) Casimir connection of $\mathfrak{g}$, we make the following

Conjecture B.3 (V. Toledano Laredo)**.** The monodromy of the trigonometric Casimir connection is equivalent to the quantum Weyl group action of the affine braid group $\hat{B}$ on finite-dimensional $U_\hbar(L\mathfrak{g})$-modules.

What's known of the above conjecture?

Theorem B.4 (S. Gautam, V. Toledano Laredo)**.** *Let $\mathfrak{g}$ be $\mathfrak{sl}_2$ or $\mathfrak{gl}_2$, the above conjecture is true.*

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