Xiaolei Zhao

1 Spherical objects & twists

X-smooth proj var / field k

Def: $\mathcal{E} \in D^b(X)$ is <u>spherical</u> if

i) $\mathcal{E} \otimes \omega_X \simeq \mathcal{E}$

ii) $\mathrm{Hom}(\mathcal{E}, \mathcal{E}[i]) = \begin{cases} k, & i = 0 \text{ or } \dim X \\ 0 & \text{else} \end{cases}$

Ex: $\mathcal{E}$-spherical $\Rightarrow$ $\mathcal{E}^\vee$, $\mathcal{E}[i]$, $\mathcal{E} \otimes \mathcal{L}$ are spherical

$\mathcal{L}$ – line bundle

$R\mathcal{H}om(\mathcal{E}, \mathcal{F})$: can assume $\mathcal{E}, \mathcal{F}$ - complexes of loc free sheaves

$\mathrm{Tr}: R\mathcal{H}om(\mathcal{E}, \mathcal{E}) \to \mathcal{O}_X$ - morphism of complexes

$$\begin{tikzcd}
\mathcal{H}om^{-1}(\mathcal{E}, \mathcal{E}) \arrow[r] \arrow[dr, bend right, "0"'] & \mathcal{H}om^{0}(\mathcal{E}, \mathcal{E}) = \bigoplus_p \mathcal{H}om(\mathcal{E}_p, \mathcal{E}_p) \arrow[d, "\text{-direct sum}"] \\
& \mathcal{O}_X
\end{tikzcd}$$

$\rightsquigarrow$ gives rise to required homomorphism

$P_{\mathcal{E}} \in D^b(X \times X)$ $\quad \tau: X \xrightarrow{\sim} \Delta \hookrightarrow X \times X$ $\quad X \xleftarrow{p} X \times X \xrightarrow{q} X$

$$q^*\mathcal{E} \otimes p^*\mathcal{E} \to \tau_*\tau^*(q^*\mathcal{E}^\vee \otimes p^*\mathcal{E}) \to \tau_*(\mathcal{E}^\vee \otimes \mathcal{E}) \to \tau_*(\mathcal{O}_X) = \mathcal{O}_\Delta$$

$$q^*\mathcal{E}^\vee \otimes p^*\mathcal{E} \to \mathcal{O}_\Delta \to P_{\mathcal{E}}$$ - distinguished triangle

Def: Spherical twist defined by $\mathcal{E}$ is, by def-n, FM transform

$T_{\mathcal{E}} := \Phi_{P_{\mathcal{E}}} : D^b(X) \to D^b(X)$

Lem:

(i) $\mathrm{cone}\big(\underbrace{\mathrm{Hom}(\mathcal{E}, F[*]) \otimes \mathcal{E}}_{= \bigoplus \mathrm{Hom}(\mathcal{E}, F[i]) \otimes \mathcal{E}[-i]} \xrightarrow{ev} F\big) \simeq T_{\mathcal{E}}(F)$

(ii) $T_{\mathcal{E}}(\mathcal{E}) \simeq \mathcal{E}[1 - \dim X]$

$T_{\mathcal{E}}(F) \simeq F$

for $F \in \mathcal{E}^\perp$ i.e. $\mathrm{Hom}(\mathcal{E}, F[i]) = 0 \ \forall i$

Prop1: $\mathcal{E}$ is a spher. object. Then $T_{\mathcal{E}}$ is autoequivalence of $D^b(X)$

Examples of spherical objects.

i) smooth proj curve $\mathcal{C}$, $x \in \mathcal{C}$ - pt $\rightsquigarrow$ then $k(x)$ (skyscraper) is spherical. & $T_{k(x)}: F \to F \otimes \mathcal{O}(x)$ - corresp line bundles

scheme of proof: • first prove for line bundles L using $\mathrm{Ext}^1_{(k(x), L)} = k$

• once we know this for all line bundles, the claim follows in general

(i) X is true CY variety, i.e. $\mathcal{O}_X \simeq \omega_X$, $H^i(X, \mathcal{O}_X) = 0$ for $1 \le i \le \dim X - 1$ then any line bundle is sph. object. $T_{\mathcal{O}_X} = \Phi_{I_\Delta[1]}$

(ii) X is smooth proj-ve surface

$C < X$ - smooth irred. rational curve w. $C^2 = -2$

$\Rightarrow \mathcal{O}_C$ is a spher. object (b/c $\omega_X|_C = \mathcal{O}_C$, & $C^2 < 0$) cannot deform

iv) Let C be smooth rational curve in a true CY 3fold. Assume $N_{C/X} \simeq \mathcal{O}(-1)^{\oplus 2}$. Then $\mathcal{O}_C$ is spherical object.

Ideas of proof:

Claim: $F: \mathcal{D} \to \mathcal{D}'$ exact functor with adjoints $G \vdash F \dashv H$

Suppose Ω is a spanning class of $\mathcal{D}$, i.e:

i) $F \in \mathcal{D}$ w. $\mathrm{Hom}(A, F[i]) = 0$ $\forall A \in \Omega \Rightarrow F = 0$

ii) — · — · — · — $\mathrm{Hom}(F, A[i]) = 0$ — · — · — — · — · —

1) $\mathrm{Hom}(A, B^{[i]}) \xrightarrow{\sim} \mathrm{Hom}(FA, FB[i])$ $\forall A, B \in \Omega \Rightarrow F$ is fully faithful

If moreover 2) $\mathcal{D}, \mathcal{D}'$ admit Serre functors $S_{\mathcal{D}}, S_{\mathcal{D}'}$ w.

$F(S_{\mathcal{D}}(A)) \simeq S_{\mathcal{D}'}(F(A))$ (in our case $S_{\mathcal{D}} = \otimes \omega_X[\dim X]$) $\forall A \in \Omega$

3) If $\mathcal{D}'$ is indecomposable & $\mathcal{D}$ is non-trivial then F is equivalence (provided it's fully faithful)

Proof: skipped

Proof of Prop 1: Take $\Omega = \{\mathcal{E}\} \cup \mathcal{E}^{\perp}$. This is a spanning class

2) Braid group action

B_{m+1} - type A_m braid group

Def: A_m-configuration of spherical objects in $D^b(X)$ consists of spherical objects $\mathcal{E}_1, \dots, \mathcal{E}_m$ s.t.

$$\bigoplus_{\ell} Hom(\mathcal{E}_i, \mathcal{E}_j[\ell]) = \begin{cases} k & |i-j|=1 \\ 0, & |i-j|>1 \end{cases}$$

Prop 2 Suppose $\mathcal{E}_1, \dots \mathcal{E}_m \in D_b(X)$ be an A_m-configuration. Let $T_i = T_{\mathcal{E}_i}$ then T_i's satisfy braid rel-ns

Thm (Seidel-Thomas) The resulting homomorphism $B_{m+1} \hookrightarrow Aut(D^b(X))$ is injective if $\dim X \geq 2$

Ex: 1) surface w. rational curves of self-intersection -2 forming type A_m Dynkin diagram

2) C smooth elliptic $x_1, x_2 \in C$ w. $x_1 - x_2$ is 2-torsion

Consider $T_{k(x_1)} \circ T^{-1}_{k(x_2)} = \otimes O(x_1 - x_2)$ - is of order 2

Rem $k(x_1), O_C, k(x_2)$ - A_3-configuration

Note that $\beta_1 \circ \beta_3^{-1}$ is of infinite order so the Seidel-Thomas thm fails when $\dim X = 1$

Ideas of proof of Prop 2

Lem: Let $\mathcal{E} \in D^b(X)$ be spher. object

$\varphi: D^b(X) \to D^b(X)$ - autoequiv. Then $\varphi \circ T_{\mathcal{E}} \simeq T_{\varphi(\mathcal{E})} \circ \varphi$

Proof of Prop 2: $T_{\mathcal{E}_i} T_{\mathcal{E}_j} = T_{T_{\mathcal{E}_i}(\mathcal{E}_j)} T_{\mathcal{E}_i} = T_{\mathcal{E}_j} T_{\mathcal{E}_i}$

$$T_{\mathcal{E}_i} T_{\mathcal{E}_{i+1}} T_{\mathcal{E}_i} \simeq T_{\mathcal{E}_{i+1}} T_{\mathcal{E}_i} T_{\mathcal{E}_{i+1}}$$

$$\parallel$$

$$T_i T_{T_{i+1}(\mathcal{E}_i)} T_{i+1} \simeq T_{T_i T_{i+1}(\mathcal{E}_i)} T_i T_{i+1}$$

So need to prove $T_i T_{i+1}(\mathcal{E}_i) \simeq \mathcal{E}_{i+1}[1]$

Assume: $Hom(\mathcal{E}_{i+1}, \mathcal{E}_i) = k \Rightarrow$ this

$$\mathcal{E}_{i+1} \xrightarrow{\text{only Hom}} \mathcal{E}_i \longrightarrow T_{i+1}(\mathcal{E}_i) \text{ - distinguished}$$

$$\Rightarrow \quad T_i(\mathcal{E}_{i+1}) \longrightarrow \underset{\mathcal{E}_i[1-\dim X]}{\overset{\|}{T_i\mathcal{E}_i}} \longrightarrow T_i T_{i+1}(\mathcal{E}_i) \qquad (*)$$

Serre duality: $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_{i+1}[\dim X]) \simeq k$

$$\mathcal{E}_i \longrightarrow \mathcal{E}_{i+1}[\dim X] \longrightarrow T_i\,\mathcal{E}_{i+1}[\dim X] \longrightarrow \mathcal{E}_i[1]$$

$$\Rightarrow \quad T_i(\mathcal{E}_{i+1}) \longrightarrow \mathcal{E}_i[1-\dim X] \longrightarrow \mathcal{E}_{i+1}[1] \qquad (**)$$

Comparing $(*)$ and $(**)$ get required isomorphism

3 Mukai flop

X - smooth proj-ve variety of $\dim = 2n$

P - smooth subvariety, $P \simeq \mathbb{P}^n$, $n > 1$, $P = \mathbb{P}(V)$

Assume $N := N_{P/X} \simeq \Omega^1_P\ (T^*_P)$

$\widetilde{X} = Bl_P(X)$

$E = \mathbb{P}(N)$

$$\begin{tikzcd}
 & & E \simeq \mathbb{P}(N) \arrow[r, hook] \arrow[d] \arrow[dll] \arrow[drr] & P \times P^\vee \\
 & & \widetilde{X} \arrow[dl] \arrow[dr] & & \\
P \arrow[r, hook] & X & & X' & P^\vee \arrow[l, hook']
\end{tikzcd}$$

$0 \to \Omega \to V^* \otimes \mathcal{O}(-1) \to \mathcal{O} \to 0$

$\mathbb{P}(\Omega) \hookrightarrow \mathbb{P}(V^* \otimes \mathcal{O}(-1)) = \mathbb{P} \times \mathbb{P}^\vee$

$\mathbb{P}(\Omega)$ is the incidence divisor: $\{(l, H) \mid l \subset H\}$

the class of $\mathbb{P}(\Omega)$ is $(1,1)$

So $\omega_E = \mathcal{O}(-n,-n)|_E$

On the other hand $\omega_E \simeq (\omega_{\widetilde{X}} \otimes \mathcal{O}(E))|_E \simeq (\tilde{q}^*\omega_X \otimes \mathcal{O}(nE))|_E$

$= \mathcal{O}(nE)|_E$

$\Rightarrow \mathcal{O}(E)|_E \simeq \mathcal{O}(-1,-1)|_E \Rightarrow$ can contract E in both directions

Let X' be the contraction in other direction

Def: The birational map $X \dashrightarrow X'$ is Mukai flop

$$\begin{tikzcd} & \hat{X} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X \arrow[rr, dashed] & & X' \end{tikzcd}$$

$\hat{X} \leftarrow$ closure of the graph

We assume X' is ~~[illegible]~~ projective (a priori it's proper)

$\hat{X} = \tilde{X} \cup (P \times P^\vee)$ $\quad$ $\tilde{X}$ is proper transform

Fact: The normal bundle to $P^\vee$ in X' is again $\Omega_{P^\vee}$.

Thm: $\Psi = p_* \circ q^* : D^b(X) \xrightarrow{\sim} D^b(X')$

Pf: $\Omega = \{ k(x) \mid x \in X \}$ - spanning class. Things to check

1) Need to check $\mathrm{Hom}(k(x), k(x')[i]) \xrightarrow{\sim} \mathrm{Hom}(\Psi(k(x)), \Psi(k(x'))[i])$

2) $S_{X'} \circ \Psi(k(x)) \simeq \Psi \circ S_X(k(x))$ - easy

Only need to check 1) for $x, x' \in P$

$M = \mathbb{P}^n$, $X = \mathbb{P}(\mathcal{O}_M \oplus \Omega_M) \xrightarrow{f} M$

P is the section given by $\mathcal{O}_M \hookrightarrow \mathcal{O}_M \oplus \Omega_M$

Key point: formal compl-n of X along $P \simeq$ formal compl-n in special case.

Enough to prove Thm for M

In this case, $\Omega' = \{ \mathcal{O}_f(j) \otimes \pi^* \mathcal{O}_M(k) \}$ $\quad -n \leq j \leq 0$, $-n \leq k \leq 0$

spanning class

One checks 1) for this spanning class.