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1 Spherical objects & twists

$X$ -smooth proj var / field  $\mathbb{R}$

Def:  $E \in \mathcal{D}^b(X)$  is spherical if

i)  $E \otimes \omega_X \simeq E$

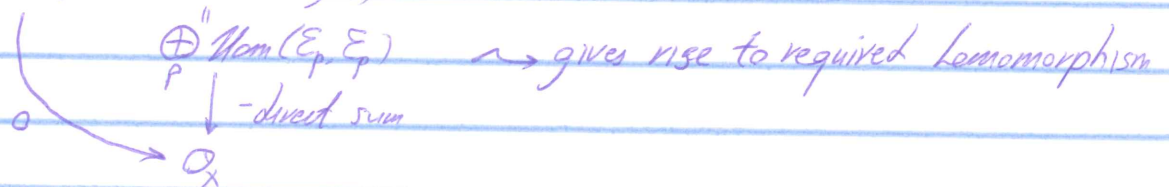
ii)  $\text{Hom}(E, E[i]) = \begin{cases} \mathbb{R}, & i=0 \text{ or } \dim X \\ 0 & \text{else} \end{cases}$

Ex:  $E$ -spherical  $\Rightarrow E^\vee, E[i], E \otimes \mathcal{L}$  are spherical  
line bundle

$R\text{Hom}(E, F)$ : can assume  $E, F$ -complexes of loc free sheaves

$\text{Tr}: R\text{Hom}(E, E) \rightarrow \mathcal{O}_X$  -morphism of complexes

$\text{Hom}^{-1}(E, E) \rightarrow \text{Hom}^0(E, E)$



$P_E \in \mathcal{D}^1(X \times X)$   $\tau: X \xrightarrow{\cong} \Delta \subset X \times X$   $X \xleftarrow{p_1} X \times X \xrightarrow{p_2} X$

$q^* E \otimes p^* E \rightarrow \tau_* \tau^*(q^* E^\vee \otimes p^* E) \rightarrow \tau_*(E^\vee \otimes E) \rightarrow \tau_*(\mathcal{O}_X) = \mathcal{O}_\Delta$

$q^* E^\vee \otimes p^* E \rightarrow \mathcal{O}_\Delta \rightarrow P_E$  -distinguished triangle

Def: Spherical twist defined by  $E$  is, by defn, FM transform

$T_E := \Phi_{P_E}: \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(X)$

lem:

(i)  $\text{cone}(\text{Hom}(E, F[*]) \otimes E \xrightarrow{\text{ev}} F) \simeq T_E(F)$

$\oplus \text{Hom}(E, F[i]) \otimes E[-i]$

(ii)  $T_E(E) \simeq E[1 - \dim X]$

$T_E(F) \simeq F$

for  $F \in \mathcal{E}^\perp$  ie  $\text{Hom}(E, F[i]) = 0 \forall i$

Prop:  $E$  is a spher. object. Then  $T_E$  is autoequivalence of  $D^b(X)$

Examples of spherical objects.

i) smooth proj. curve  $C$ ,  $x \in C - pt \rightsquigarrow$  then  $\mathcal{O}_C(x)$  (skyscraper) is spherical. &  $T_{\mathcal{O}_C(x)}: F \rightarrow F \oplus \mathcal{O}_C(x)$  - corresp line bundles  $(\mathcal{O}_C(x), L)$

Scheme of proof: • first prove for line bundles  $L$  using  $\text{Ext}^i(\mathcal{O}_C(x), L) = 0$   
• once we know this for all line bundles, the claim follows in general

follows in general

(i)  $X$  is true CY variety, i.e.  $\mathcal{O}_X \cong \omega_X$ ,  $H^i(X, \mathcal{O}_X) = 0$  for  $1 \leq i \leq \dim X - 1$   
then any line bundle is sph. object.  $T_{\mathcal{O}_X} = \mathcal{P}_{\mathbb{Z}[\dim X]}$

(ii)  $X$  is smooth proj-re surface

$C \subset X$  - smooth irred. rational curve w.  $C^2 = -2$

$\Rightarrow \mathcal{O}_C$  is a spher. object (b/c  $\omega_X|_C = \mathcal{O}_C(1)$ , &  $C^2 < 0$ ) cannot deform

(iv) let  $C$  be smooth rational curve in a true CY 3fold. Assume  $N_{C/X} \cong \mathcal{O}(-1)^{\oplus 2}$ . Then  $\mathcal{O}_C$  is spherical object.

Ideas of proof:

Claim:  $F: \mathcal{D} \rightarrow \mathcal{D}'$  exact functor with adjoints  $G \dashv F \dashv H$

Suppose  $\Omega$  is a spanning class of  $\mathcal{D}$ , i.e.:

i)  $F(A) \in \mathcal{D}$  w.  $\text{Hom}(A, F[i]) = 0 \forall A \in \Omega \Rightarrow F = 0$

ii)  $\text{Hom}(F, A[i]) = 0$

1)  $\text{Hom}(A, B) \xrightarrow{\sim} \text{Hom}(FA, FB[i]) \forall A, B \in \Omega \Rightarrow F$  is fully faithful

2)  $\mathcal{D}, \mathcal{D}'$  admit Serre functors  $S_{\mathcal{D}}, S_{\mathcal{D}'}$  w.

$F(S_{\mathcal{D}}(A)) \cong S_{\mathcal{D}'}(F(A))$  (in our case  $S_{\mathcal{D}} = \otimes \omega_X[\dim X]$   $\forall A \in \Omega$ )

3) If  $\mathcal{D}'$  is indecomposable &  $\mathcal{D}$  is non-trivial

then  $F$  is equivalence (provided it's fully faithful)

Proof: skipped



Proof of Prop 1: Take  $\Omega = \{E \in \mathcal{E} \cup E^\perp\}$ . This is a spanning class

2) Braid group action

$B_{m+1}$ -type  $A_m$  braid group

Def:  $A_m$ -configuration of spherical objects in  $\mathcal{D}^b(X)$

consists of spherical objects,  $E_1, \dots, E_m$  st

$$\bigoplus_{i \neq j} \text{Hom}(E_i, E_j[1]) = \begin{cases} \mathcal{K} & |i-j|=1 \\ 0 & |i-j| > 1 \end{cases}$$

Prop 2 Suppose  $E_1, \dots, E_m \in \mathcal{D}^b(X)$  be an  $A_m$ -configuration. Let  $T_i = T_{E_i}$  then  $T_i$ 's satisfy braid rel-ns

Thm (Seidel-Thomas) The resulting homomorphism  $B_{m+1} \hookrightarrow \text{Aut}(\mathcal{D}^b(X))$  is injective if  $\dim X \geq 2$

Ex: 1) surface w. rational curves of self-intersection  $-2$  forming type  $A_m$  Dynkin diagram

2)  $C$  smooth elliptic  $x_1, x_2 \in C$  w.  $x_1 - x_2$  is 2-torsion

Consider  $T_{\mathcal{K}(x_1)} \circ T_{\mathcal{K}(x_2)}^{-1} = \otimes \mathcal{O}(x_1 - x_2)$  - is of order 2

Rem  $\mathcal{K}(x_1), \mathcal{O}_C, \mathcal{K}(x_2)$  -  $A_3$ -configuration

Note that  $\beta_1 \circ \beta_3^{-1}$  is of infinite order so the Seidel-Thomas thm fails when  $\dim X = 1$

Ideas of proof of Prop 2

Lem: let  $E \in \mathcal{D}^b(X)$  be spler. object

$\mathcal{P}: \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(X)$  - autoequiv. Then  $\mathcal{P} \circ T_E \simeq T_{\mathcal{P}(E)} \circ \mathcal{P}$

Proof of Prop 2:  $T_{E_i} T_{E_j} = T_{T_{E_i}(E_j)}$   $T_{E_i} = T_{E_j} T_{E_i}$

$$\# - T_{E_i} T_{E_{i+1}} T_{E_i} \simeq T_{E_{i+1}} T_{E_i} T_{E_{i+1}}$$

$$\parallel$$

$$T_i T_{T_{i+1}(E_i)} T_{i+1} \simeq T_{T_i T_{i+1}(E_i)} T_i T_{i+1}$$

So need to prove  $T_i T_{i+1}(E_i) \simeq E_{i+1}[1]$

Assume:  $\text{Hom}(E_{i+1}, E_i) = \mathcal{K} \Rightarrow$  this

only Hom  
↓

$$\begin{aligned} \mathcal{E}_{i+1} &\rightarrow \mathcal{E}_i \rightarrow T_{\mathcal{E}_i}(\mathcal{E}_i) \text{ - distinguished} \\ \Rightarrow T_i(\mathcal{E}_{i+1}) &\rightarrow T_i \mathcal{E}_i \rightarrow T_i \bullet T_{i+1}(\mathcal{E}_i) \quad (*) \\ &\mathcal{E}_i''[1 - \dim X] \end{aligned}$$

Serre duality:  $\text{Hom}(\mathcal{E}_i, \mathcal{E}_{i+1}[\dim X]) \cong k$

$$\begin{aligned} \mathcal{E}_i &\rightarrow \mathcal{E}_{i+1}[\dim X] \rightarrow T_i \mathcal{E}_{i+1}[\dim X] \rightarrow \mathcal{E}_i[1] \\ \Rightarrow T_i(\mathcal{E}_{i+1}) &\rightarrow \mathcal{E}_i[1 - \dim X] \rightarrow \mathcal{E}_{i+1}[1] \quad (**) \end{aligned}$$

Comparing (\*) and (\*\*) get required isomorphism

### 3 Mukai flop

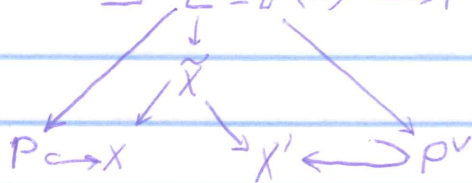
$X$  - smooth proj-ve variety of  $\dim = 2n$

$P$  - smooth subvariety,  $P \cong \mathbb{P}^n$ ,  $n > 1$ ,  $P = \mathbb{P}(V)$

Assume  $N := N_{P/X} \cong \Omega_P^1(T_P^*)$

$$\tilde{X} = \text{Bl}_P(X)$$

$$E = \mathbb{P}(N) \quad E \cong \mathbb{P}(N) \hookrightarrow P \times P^V$$



$$0 \rightarrow \Omega \rightarrow V^* \otimes \mathcal{O}(-1) \rightarrow \mathcal{O} \rightarrow 0$$

$$\mathbb{P}(\Omega) \hookrightarrow \mathbb{P}(V^* \otimes \mathcal{O}(-1)) = P \times P^V$$

$\mathbb{P}(\Omega)$  is the incidence divisor:  $\{(\ell, H) \mid \ell \subset H\}$

the class of  $\mathbb{P}(\Omega)$  is  $(1, 1)$

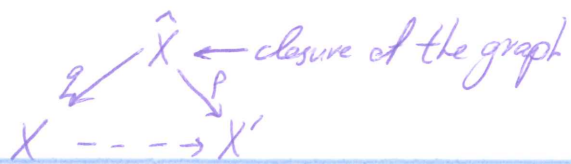
$$\text{So } \omega_E = \mathcal{O}(-n, -n)|_E$$

$$\begin{aligned} \text{On the other hand } \omega_E &\cong (\omega_{\tilde{X}} \otimes \mathcal{O}(E))|_E \cong (\tilde{g}^* \omega_X \otimes \mathcal{O}(nE))|_E \\ &= \mathcal{O}(nE)|_E \end{aligned}$$

$$\Rightarrow \mathcal{O}(E)|_E \cong \mathcal{O}(-1, -1)|_E \Rightarrow \text{can contract } E \text{ in both directions}$$

Let  $X'$  be the contraction in other direction

Def: The birational map  $X \dashrightarrow X'$  is Mukai flop



We assume  $X'$  is ~~projective~~ projective (a priori it's proper)  
 $\hat{X} = \tilde{X} \cup (P \times P^v)$   $\tilde{X}$  is proper transform

Fact: The normal bundle to  $P^v$  in  $X'$  is again  $\Omega_{P^v}$

Thm:  $\Psi = p_* \circ q^* : \mathcal{D}^b(X) \xrightarrow{\sim} \mathcal{D}^b(X')$

Pf:  $\Omega = \{R(x) \mid x \in X\}$  - spanning class. Things to check

1) Need to check  $\text{Hom}(R(x), R(x'))[i] \xrightarrow{\sim} \text{Hom}(\Psi(R(x)), \Psi(R(x')))[i]$

2)  $S_{X'} \circ \Psi(R(x)) \simeq \Psi \circ S_X(R(x))$  - easy

Only need to check 1) for  $x, x' \in P$

$$M = \mathbb{P}^n, X = \mathbb{P}(\mathcal{O}_M \oplus \Omega_M) \xrightarrow{f} M$$

$P$  is the section given by  $\mathcal{O}_M \hookrightarrow \mathcal{O}_M \oplus \Omega_M$

Key point: formal compl- $n$  of  $X$  along  $P$  = formal compl- $n$  in special case.

Enough to prove Thm for  $M$

In this case,  $\Omega' = \{ \mathcal{O}_f(j) \otimes \pi^* \mathcal{O}_M(k) \}$   $-n \leq j \leq 0$   
 $-n \leq k \leq 0$   
↑  
spanning class

One checks  $\uparrow$  for this spanning class.  
 1)