NOTIONS OF STABILITY OF SHEAVES

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1. Stability and Filtrations

1.1. **Semistable sheaves.** Let X be a projective scheme over a field k and E be a coherent sheaf on X. The Euler characteristic of E is denoted by $\chi(E) = \sum(-1)^i h^i(X, E)$, where $h^i(X, E) = \dim_k H^i(X, E)$. Fix $\mathcal{O}(1)$ as an ample line bundle on X.

Definition and Lemma. *The Hilbert polynomial $P(E) : m \mapsto \chi(E \otimes \mathcal{O}(m))$ is a polynomial of m and can be written as $P(E, m) = \sum_{i=0}^{\dim E} \alpha_i(E) \frac{m^i}{i!}$.*

Note. *$\alpha_{\dim X}(\mathcal{O}_X)$ is exactly the degree of X with respect to $\mathcal{O}(1)$. Furthermore, if X is reduced and irreducible, of dimension d_X, then $\alpha_{d_X}(E) = \mathrm{rank}(E) \cdot \alpha_{d_X}(\mathcal{O}_X)$.*

Definition 1.1.1. *The reduced Hilbert polynomial $p(E)$ of a coherent sheaf E of dimension d is defined by $p(E, m) = \frac{P(E,m)}{\alpha_d(E)}$.*

For two polynomials $p(m)$ and $q(m)$, we say $p(m) < q(m)$ if that holds for $m >> 0$.

Definition 1.1.2. *A coherent sheaf E purely of dimension d (i.e. every nonzero subsheaf is of support dimension d) is (semi)stable if for any proper subsheaf $F \subset E$, one has $p(F) < (\leq) p(E)$.*

Exercise 1.1.1. *E is (semi)stable if and only if for all proper quotient sheaves $E \twoheadrightarrow G$ with $\alpha_d(G) > 0$, one has $p(E) < (\leq) p(G)$.*

Exercise 1.1.2. *Suppose F, G are semistable, purely of dimension d. If $p(F) > p(G)$, then $\mathrm{Hom}(F, G) = 0$; if $p(F) = p(G)$ and $f : F \to G$ is nontrivial, then f is injective if F is stable and surjective if G is stable.*

1.2. **Slope stable.** Let X be a smooth projective curve over an algebraic closed field k and E be a locally free sheaf of rank r. Then $\chi(E) = \deg(E) + r(1 - g)$, where g is the genus of X. So $P(E, m) = (\deg(X)m + \mu(E) + (1 - g))r$, where $\mu(E) = \frac{\deg(E)}{r}$ is called the slope of E.

In this case, the stability means:

E is (semi)stable if for all subsheaves $F \subset E$ with $0 < \mathrm{rank}(F) < \mathrm{rank}(E)$, one has $\mu(F) < (\leq) \mu(E)$.

In general, this becomes the μ-stability. Denote $d = \dim X$.

Definition 1.2.1. *Suppose that E is a coherent sheaf of dimension $d = \dim X$. The degree of E is defined to be*

$$\deg(E) = \alpha_{d-1}(E) - \mathrm{rank}(E) \cdot \alpha_{d-1}(\mathcal{O}_X).$$

And its slope is

$$\mu(E) = \frac{\deg(E)}{\mathrm{rank}(E)}.$$

Definition 1.2.2. *A coherent sheaf E of dimension $d = \dim(X)$ is μ-(semi)stable if*

(i) *any torsion subsheaf of E has support of codimension at least 2;*
(ii) $\mu(F) < (\leq)\mu(E)$ *for all subsheaves* $F \subset E$ *with* $0 < \operatorname{rank}(F) < \operatorname{rank}(E)$.

Exercise 1.2.1.
- *If E is purely of dimension $d = \dim X$, then μ-stable $\Longrightarrow$ stable $\Longrightarrow$ semistable $\Longrightarrow$ μ-semistable.*
- *Given X being integral, if the coherent sheaf E of dimension $d = \dim X$ is μ-semistable, and $\operatorname{rank}(E)$ is coprime to $\deg(E)$, then E is μ-stable.*

1.3. Harder-Narasimhan Filtration.

Definition 1.3.1. *Suppose a coherent sheaf E over X is purely of dimension d. A Harder-Narasimhan filtration for E is an increasing filtration*

$$0 = HN_0(E) \subset HN_1(E) \subset \cdots \subset HN_\ell(E) = E,$$

such that $\operatorname{gr}_i^{HN} := HN_i(E)/HN_{i-1}(E)$ for $i = 1, \cdots, \ell$ are semistable sheaves of dimension d with reduced Hilbert polynomials p_i satisfying

$$p_{\max}(E) := p_1 > \cdots > p_\ell =: p_{\min}(E).$$

Theorem 1.3.1. *Every pure sheaf E has a unique HN filtration.*

Proof. We first need the following lemma.

Lemma 1.3.1. *Suppose E is pure of dimension d. Then there exists $F \subset E$ such that for all $G \subset E$, one has $p(F) \geq p(G)$, and in case of equality $F \supset G$. Moreover F is unique and semistable. We call F the maximal destabilizing sheaf of E.*

Proof of Lemma. We define an order '$\leq$' on the nontrivial subsheaves of E: $F_1 \leq F_2$ if $F_1 \subset F_2$ and $p(F_1) \leq p(F_2)$. We say a sheaf is $\leq$-maximal if it is maximal with respect to this order. By ascending property, for each $F \subset E$, there exists a subsheaf F' such that $F \subset F' \subset E$ and F' is $\leq$-maximal. Let $F \subset E$ be the $\leq$-maximal subsheaf with minimal $\alpha_d(F)$. We claim that F has the asserted properties.

Suppose there exists $G \subset E$ with $p(G) \geq p(F)$. First we show that we can assume $G \subset F$ by replacing G by $G \cap F$. Indeed, if $G \not\subset F$, F is a proper subsheaf of $F + G$, so $p(F) > p(F + G)$. Consider

$$0 \to F \cap G \to F \oplus G \to F + G \to 0.$$

We have

$$P(F) + P(G) = P(F \cap G) + P(F + G),$$
$$\alpha_d(F) + \alpha_d(G) = \alpha_d(F \cap G) + \alpha_d(F + G).$$

Hence

$$\alpha_d(F \cap G)(p(G) - p(F \cap G)) = \alpha_d(F + G)(p(F + G) - p(F)) + (\alpha_d(G) - \alpha_d(F \cap G))(p(F) - p(G)).$$

Therefore $p(F) \leq p(G) < p(F \cap G)$.

Next, fix $G \subset F$ with $p(G) > p(F)$ which is $\leq$-maximal in F. Let G' be the $\leq$-maximal sheaf in E containing G. In particular, $p(F) < p(G) \leq p(G')$. By definition, $G' \not\subset F$ (otherwise $\alpha_d(G') < \alpha_d(F)$), hence F is a proper subsheaf of $F + G'$. Therefore $p(F) > p(F + G')$. As before, we have $p(F \cap G') > p(G') \geq p(G)$. Since $G \subset F \cap G' \subset F$, this is a contradiction to the assumption on G.

The other two properties follow from the first property. □

Existence of HN-filtration: Let E_1 be the maximal destabilizing subsheaf. By induction, we can assume E/E_1 has an HN-filtration

$$0 = G_0 \subset G_1 \subset \cdots \subset G_{\ell-1} = E/E_1.$$

Let $E_{i+1} \subset E$ be the preimage of G_i. We just need to show $p(E_1) \geq p(E_2/E_1)$. This follows from the maximal property of E_1.

Uniqueness of HN-filtration: Assume $\mathbf{E}.$ and $\mathbf{E}.'$ are two HN-filtrations of E, with $p(E_1') \geq p(E_1)$. Let j be minimal number such that $E_1' \subset E_j$. Then

$$E_1' \to E_j \to E_j/E_{j-1}$$

in a nontrivial morphism between two semistable sheaves. Hence

$$p(E_j/E_{j-1}) \geq p(E_1') \geq p(E_1) \geq p(E_j/E_{j-1}).$$

So $j = 1$ and $E_1' \subset E_1$. Then $p(E_1') \leq p(E_1)$. Repeat the argument, we can see $E_1' = E_1$. Now by induction, E/E_1 has a unique HN-filtration. □

1.4. Jordan-Holder Filtration.

Definition 1.4.1. *Let E be a semistable coherent sheaf of dimension d on X. A Jordan-Holder filtration is a filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_\ell = E$$

such that $\mathrm{gr}_i(E) = E_i/E_{i-1}$ are stable with reduced Hilbert polynomial $p(E)$.

Proposition 1.4.1. *JH-filtration exists and $\mathrm{gr}\, E := \bigoplus_i \mathrm{gr}_i(E)$ is independent of the choice of the JH-filtration.*

Proof. The existence is straightforward: any filtration of E by semistable sheaves with reduced Hilbert polynomial $p(E)$ has a maximal refinement, whose factors are necessarily stable.

The second statement follows from the same idea as in the proof of the uniqueness of the HN-filtration. We refer to Section 1.5 of Huybrechts and Lehn's book for detail. □

Definition 1.4.2. *Two semistable sheaves E_1 and E_2 with $p(E_1) = p(E_2)$ are S-equivalent if $\mathrm{gr}(E_1) \cong \mathrm{gr}(E_2)$.*

Definition 1.4.3. *A semistable sheaf E is called polystable if E is the direct sum of stable sheaves.*

1.5. Relative case.

Theorem 1.5.1. *Let S be an integral k-scheme of finite type, $f : X \to S$ a projective morphism, $\mathcal{O}_X(1)$ an f-ample invertible sheaf on X, and F a flat family of d-dimensional coherent sheaves on the fibers of f. Then there is a projective birational morphism $g : T \to S$ of integral k-schemes and a filtration*

$$0 = HN_{0,T}(F) \subset HN_{1,T}(F) \subset \cdots \subset HN_{\ell,T}(F) = F_T,$$

such that

- (i) *$HN_{i,T}(F)/HN_{i-1,T}(F)$ are T-flat for all $i = 1, \cdots, \ell$;*
- (ii) *there is a dense open subscheme $U \subset T$ such that $\mathbf{HN}._{,\mathbf{T}}(F)_t = g_X^*(\mathbf{HN}.(F_{g(t)}))$ for all $t \in U$.*

Moreover, $(g, \mathbf{HN}._{,\mathbf{T}}(F))$ is universal, meaning that if $g' : T' \to S$ is any dominant morphism of integral schemes, and $\mathbf{F}.'$ is a filtration of $F_{T'}$ satisfying the above two properties, then there exists an S-morphism $h : T' \to T$ with $\mathbf{F}.' = h_X^*(\mathbf{HN}._{,\mathbf{T}}(F))$.

Sketch of proof. Just like the proof of the existence of the HN-filtration, the idea is to construct a family of sheaves which is generically the maximal destabilizing sheaf fiberwise. The main ingredient is the quot schemes. We refer to Section 2.3 of Huybrechts and Lehn's book for detail. □

Note. *1) In the proof, it can be shown that there exists a subscheme V of certain quot scheme* ***Quot*** *such that U is isomorphic to an open dense subscheme of S, and T is taken to be closure of V in* ***Quot****. So a priori, T is only birational to S. It is interesting to try to find an example in which this is necessarily birational.*

2) In condition ii), we can't always take $U = T$, since the graded quotients of the relative HN-filtration may degenerate to unstable sheaves on special fibers.

2. Examples of stable vector bundles

2.1. $\Omega_{\mathbb{P}^n}$.

Proposition 2.1.1. *$\Omega_{\mathbb{P}^n}$ is stable.*

Proof. By the uniqueness of HN-filtration, it is invariant under the $SL(V)$-action on $\mathbb{P}^n = \mathbb{P}(V)$. In particular, every subsheaf in the filtration is a subbundle. However, since $SL(V)$ acts transitively on $\mathbb{P}^n$, and the induced action on the cotangent vectors at a fixed point is irreducible, the only nontrivial invariant subbundle is $\Omega_{\mathbb{P}^n}$. Hence the HN-filtration is trivial and $\Omega_{\mathbb{P}^n}$ is semistable. Now $\gcd(\operatorname{rank}\Omega_{\mathbb{P}^n}, \deg\Omega_{\mathbb{P}^n}) = 1$, so it is μ-stable, and hence stable. □

2.2. $\mathbb{P}^1 \times \mathbb{P}^1$ and change of polarization.

On $\mathbb{P}^1 \times \mathbb{P}^1$, it is easy to compute that

$$\mathrm{Ext}^1(\mathcal{O}(0,3), \mathcal{O}(1,-3)) \cong k^{10}.$$

So we can consider the sheaf E given by a non-trivial extension

$$0 \to \mathcal{O}(1,-3) \to E \to \mathcal{O}(0,3) \to 0.$$

Note that $c_1(E) = (1,0)$, $c_2(E) = 3$. Let $L = \mathcal{O}(1,5)$, $L' = \mathcal{O}(1,7)$. We claim:

Proposition 2.2.1. (i) *E is not L'-semistable.*
(ii) *E is L-stable.*

Proof. (i) $\mu_{L'}(\mathcal{O}(1,-3)) = 4 > \mu_{L'}(E) = \frac{7}{2}$.

(ii) We need to show that for any rank 1 subbundle $\mathcal{O}(D)$ of E, we have $D \cdot L < \frac{5}{2} = \mu_L(E)$.There are two cases:
(a) $\mathcal{O}(D) \hookrightarrow \mathcal{O}(1,-3)$, or
(b) $\mathcal{O}(D) \hookrightarrow \mathcal{O}(0,3)$.

For case (a), $D \cdot L \leq \mathcal{O}(1,-3) \cdot \mathcal{O}(1,5) = 2$.

For case (b), let $D = (\alpha, \beta)$, then $\alpha \leq 0$ and $\beta \leq 3$. $(\alpha,\beta) \neq (0,3)$ since the extension is nontrivial. Hence $D \cdot L = 5\alpha + \beta \leq 2$.

□