Classical Hodge theory and the Decomposition theorem via Hodge theory

1) Hodge theory and Lefschetz linear algebra
2) Semismall maps and Hard Lefschetz theorem
3) Intersection cohomology and Decomposition theorem

1) Hodge theory and Lefschetz linear algebra

Classical Hodge theory for smooth projective complex varieties starts with the Hodge decomposition:

\[ H^i(X; \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X) \]

For our purpose, we will always assume that

\[ H^{p,q}(X) = 0 \quad \text{when} \quad p \neq q \]

The structure of interest to us is the total cohomology

\[ H = \bigoplus H^i(X; \mathbb{R}) \]
We start with the axiomatized setup:

Fix: \( H = \oplus H^i \) a finite dim graded \( \mathbb{R} \)-vector space.

\( \langle -,- \rangle: H \times H \to \mathbb{R} \) a symmetric, non-degenerate, graded form, \( \langle H^i, H^j \rangle = 0 \) if \( i \neq j \).

Hence, if \( b_i = \dim H^i \), then \( b_i = b_{-i} \), \( \forall i \in \mathbb{Z} \).

Example: If \( M \) is a compact manifold of dim \( 2n \), set \( H^i = H^{i+n}(M; \mathbb{R}) \). Let \( \langle -,- \rangle \) be

\[
\langle w_1, w_2 \rangle = \int_M w_1 \wedge w_2.
\]

If \( H^{2k+1}(M; \mathbb{R}) = 0 \) for any \( k \), \( \langle -,- \rangle \) is symmetric.

A Lefschetz operator is a map \( L: H^* \to H^{*-2} \) s.t.

\[
\langle Lx, y \rangle = \langle x, Ly \rangle \quad \text{for \( \forall x, y \in H \)}.
\]

Example: With \( M \) as above, and \( \alpha \in H^2(M; \mathbb{R}) \), \( \cdot \alpha \) gives a Lefschetz operator.
Def: A Lefschetz operator $L$ satisfies the hard Lefschetz theorem (hL), if $L^i : H^{-i} \to H^i$ is an isomorphism for all $i$.

Exercise: Let $\mathfrak{sl}_2(\mathbb{R}) = \mathbb{R}[f, h, e]$. A Lefschetz operator satisfies (hL) $\iff$ there exists an action of $\mathfrak{sl}_2(\mathbb{R})$ on $H$ such that $e = L$ and $hx = ix$ for all $x \in H^i$. Moreover, this action is unique.

Example: If $X = \mathbb{C}P^n$ is a smooth projective variety, then $L = \cup c_1(\mathcal{O}(1))$ satisfies (hL).

If $L$ satisfies (hL), then we have the primitive decomposition

$$H = \bigoplus_{i \geq 0} \left( \bigoplus_{i \geq j > 0} L^j P^{-i}_L \right),$$

where $P^{-i}_L = \ker L^{i+1} \subset H^{-i}$ is the isotypic component "lowest weight" pair $H^i$ and $H^{-i}$, $L^i$ identifies them.
Lefschetz form: \((\alpha, \beta)_L^{-i} := \langle \alpha, L^i \beta \rangle\) (symmetric)

\((hL) \iff\) non-degeneracy of \((-,-)_L^{-i} \forall i \geq 0\)

Exercise: \((L\alpha, L\beta)_L^{-i+2} = (\alpha, \beta)_L^{-i} \quad i \geq 2\)

\((hL)\): \(H^{-i} = P^{-i}_L \oplus LP^{-i-2}_L \oplus \cdots\) is orthogonal w.r.t \((-,-)_L\)

Hodge–Riemann bilinear relations: Assume \(H^{\text{odd}} = 0\) or \(H^{\text{even}} = 0\)

Let \(\min\) be s.t. \(H^{\text{min}} \neq 0\) but \(H^{j} = 0\ \forall j < \min\).

\((H, \langle-,-\rangle, L)\) satisfies the Hodge–Riemann bilinear relations \((HR)\) if the restriction of \((-,-)_L^{\text{min+2i}}\) to \(P^{\text{min+2i}}_L\) is \((-1)^i\)-definite.

\(H^{\text{min+2i}} = L^i P^{\text{min}}_L \oplus L^{i-1} P^{\text{min+2}}_L \oplus \cdots \oplus P^{\text{min+2i}}_L\) (orthogonal)

\[\Rightarrow\] signature of \((-,-)_L^{\text{min+2i}} = \sum_{i \geq 0} (-1)^i \dim P^{\text{min+2i}}_L\]

Example: See first part of Page 12
2) Semismall maps and the hard Lefschetz theorem

The reference is [dCM] “The hard Lefschetz theorem and the topology of semismall maps”.

In this section, we always consider a \( f: X \to Y \)

where \( X \) is smooth projective, and \( X, Y \) both irreducible.

Denote \( Y^k := \{ y \in Y \mid \dim f^{-1}(y) = k \} \)

**Def:** We say \( f: X \to Y \) is semismall, if

\[
\dim Y^k + 2k \leq \dim X = n, \quad \forall k
\]

**Rmk:** In this case, \( f \) is generically finite:

\[
\left\{ \begin{array}{l}
\text{For } k > 0 \\
\dim Y^k + k \leq n - k \\
\text{so } f^*(k) \text{ is } X
\end{array} \right.
\]

Again, let \( H = \oplus H^i \), where \( H^i := H^{n+i}(X; \mathbb{R}) \).

Now we consider the Lefschetz operator:

\[
L = \nu c_1(f^*A), \text{ where } A \text{ is ample on } Y
\]
Thm (dC-M): Let $f : X \to Y$, $L$ be as above, assume that $f$ is semismall. Then $(H, L)$ satisfies (hL), (HR).

Example: See Page 12-13

To see why the semismall condition is relevant, consider a birational morphism $f : X \to Y$ between 3-folds which contracts a surface $S$ to a point. In this case, $f$ is not semismall. Now $L([S]) = [S] \cup f^*A \neq 0$, so the (hL) doesn't hold. Completely similar method shows that (hL) of $L$ implies $f$ is semismall.

([dCM]: Prop 2.2.7).

dCM proof strategy:

(hL), (HR) \xrightarrow{\text{weak}} (hL) \text{ in } \dim n \xrightarrow{\text{Lefschetz}} \dim n+1 \xrightarrow{\text{limit}} (HR) \text{ in } \dim (n+1)

Key steps:

1. Weak Lefschetz substitute: Suppose $H, \overset{\cdot}{\to} H, L_H$.

(wL)
W, L_H, L_W are as above, with L_H, L_W Lefschetz operators. Suppose \( \phi : H \to W \) of deg 1 st.

1) \( \phi \) injective in degrees \( \leq -1 \).

2) \( \langle \alpha, L_H^\beta \rangle_H = \langle \phi \alpha, \phi \beta \rangle_W \), \( \phi \circ L_H = L_W \circ \phi \).

3) \( W \) satisfies (HR).

Then \( L_H \) satisfies (hL).

pf. Fix \( 0 \neq h \in H^{-i} \), with \( i \leq -1 \), and consider

\( \phi(h) \in W^{i+1} \). Then either:

1) \( 0 \neq L_H^i(\phi(h)) = \phi(L_H^i(h)) \Rightarrow L_H^i h \neq 0 \), or

2) \( 0 = L_H^i(\phi(h)) \Rightarrow \phi(h) \in \mathcal{P}_L^{-i+1} \Rightarrow \)

\[ 0 \neq (\phi(h), \phi(h))^{-i+1}_L = \langle \phi(h), L_H^{-i+1} \phi(h) \rangle = \langle h, L_H^i h \rangle \]

\[ \square \]

2) Limit lemma: Suppose that \( [0, \infty) \to \text{Hom}(H, H(z)) \)

\[ J \mapsto L_J \]

is a continuous family of Lefschetz operators satisfying (hL). If \( \exists J \in (0, \infty) \) st. \( L_J \) satisfies (HR), then all \( L_J \)
satisfy (HR).

pf: All $L_j$ satisfy $(hL) \iff (-,-)_{L_j}^i$ is a continuous family of symmetric non-degenerate forms.

Hence all have same signature. Hence all satisfy (HR). □

Sketch of pf of Thm (dC-M):

When $n=1$, $L$ is defined by an ample divisor on $X$, so it follows from classical Hodge theory.

Assume $(hL)$ & (HR) in dim $n$. In dim $n+1$, Prop 2.1.5 in [dCM] states that we can find $\phi$ for a smooth divisor $H \in |f^*A|$, the restriction $c^*: H^*(X; \mathbb{R}) \rightarrow H^*(H; \mathbb{R})$ puts us in the situation as in Key Step (i). Hence we have $(hL)$ by induction.

For (HR), just note that $f^*A$ is on the boundary is nef, so by Kleiman's thm, it
of the ample cone of $X$, so $f^*A + \varepsilon B$ is ample, for any ample $B$ and $0 < \varepsilon << 1$. This puts us in the situation of limit lemma, and concludes (HR).

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**Warning**: we never introduce (HR) in general. Our definition of (HR) is only for the case $H^p, g = 0$ when $p \neq q$. This should be enough for our purpose.
3) Intersection cohomology and the Decomposition theorem.

To any complex variety $X$, we consider the intersection cohomology group $IH^*(X)$ ($\mathbb{R}$-coefficients):

1. $IH^*(X)$ is a graded vector space, concentrated in degrees between 0 and $2N$, where $N = \dim_c X$;
2. If $X$ is smooth, then $IH^*(X) = H^*(X)$;
3. If $X$ is projective, then $IH^*(X)$ is equipped with a non-degenerate Poincaré pairing $\langle -, - \rangle$, which is the usual Poincaré pairing for $X$ smooth.

Caution!

1. $X \mapsto IH^*(X)$ is not functorial: in general, $f: X \to Y$ doesn't induce a pull-back on $IH$;
2. $IH^*(X)$ is not a ring, but rather a module over the cohomology ring $H^*(X)$. 
Key properties when $X$ is projective: $(\text{BBD}, \text{Saito}, \text{dCM})$

(1) multiplication by $c_1$ of an ample line bundle on $\text{IH}^i(X)$ satisfies the hard Lefschetz theorem;

(2) the groups $\text{IH}^i(X)$ satisfy the Hodge–Riemann bilinear relations.

According to our convention, here we consider $\text{IH}^i(X)[-N]$. Also (2) should be applied only to the case of pure type $(p, p)$. We will not go through these issues though.

The main theorem on $\text{IH}$ is the following:

Thm (Decomposition theorem) Let $f: \hat{X} \to X$ be a resolution, then $\text{IH}^i(X)$ is a direct summand of $H^i(\hat{X})$, as modules over $H^*(X)$. $(\text{BBD}, \text{Saito}, \text{dCM})$

We will not prove this theorem, but use it to compute one example. At the end, it will be clear how it's related to section 2) in the semismall case.
Example: \( \text{Gr}(2,4) \) \( \dim = 4 \)

Let \( 0 < C < C^2 < C^3 < C^4 \) be the standard coordinate flag on \( C^4 \). For \( a := \{ 0 = a_0 \leq a_1 \leq \ldots \leq a_4 = 2 \} \) with \( a_i \leq a_{i+1} \leq a_i + 1 \), consider

\[
C_a := \{ V \in \text{Gr}(2,4) \mid \dim(V \cap C^i) = a_i \}
\]

It's easy to see that \( C_a \cong C^{d(a)} \), where

\[
d_a = 7 - \frac{4}{\sum_{i=0} a_i}.
\]

This gives the cohomology table of \( \text{Gr}(2,4) \)

\[
\begin{array}{c|cccc}
0 & 2 & 4 & 6 & 8 \\
\hline
R & R & R^2 & R & R
\end{array}
\]

It can be checked the (hL) and (HR) via Schubert calculus.

Now let \( X := \{ V \in \text{Gr}(2,4) \mid \dim(V \cap C^2) \geq 1 \} \)

Then \( X = C_a \) where \( a = \{ 0,0,1,1,2 \} \). Hence, \( X \) decomposes into \( \{ 0,0,1,1,2 \}^6 \{ 0,0,1,2,2 \}^4 \{ 0,1,1,1,2 \}^4 \{ 0,1,1,2,2 \}^2 \{ 0,1,2,2,2 \}^3 \).
The cohomology of $X$ are

\[
\begin{array}{c|c|c|c}
0 & 2 & 4 & 6 \\
\hline
\mathbb{R} & \mathbb{R}^2 & \mathbb{R}^2 & \mathbb{R} \\
\end{array}
\]

$H^i(X)$ doesn't satisfy Poincaré duality or $(hL)$. 

$X$ has a unique singular point $V_0 = \mathbb{C}^2$. To construct a resolution of $X$, consider $f: \tilde{X} \to X$,

\[
\tilde{X} := \{(v, w) \in \text{Gr}(2, \mathbb{C}^4) \times \mathbb{P}(\mathbb{C}^3) \mid w = v \cap \mathbb{C}^3\}
\]

and $f(v, w) = v$. Clearly $f$ is an isomorphism over $X \setminus \{V_0\}$, and has fiber $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^3)$ over $V_0$. The projection $(v, w) \mapsto w$ realizes $\tilde{X}$ as a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$. This gives us the cohomology table of $\tilde{X}$:

\[
\begin{array}{c|c|c|c}
0 & 2 & 4 & 6 \\
\hline
\mathbb{R} & \mathbb{R}^2 & \mathbb{R}^2 & \mathbb{R} \\
\end{array}
\]

Claim: $IH^i(X) = H^i(\tilde{X})$
Pf: Clearly the pull-back morphism $H^i(X) \to H^i(\tilde{X})$ is injective. The Decomposition theorem states that $IH^i(X)$ is a summand of $H^i(\tilde{X})$ (as $H^i(X)$-modules!), hence we now $IH^i(X) = H^i(\tilde{X})$ for $i \neq 2$.

Finally, we must have $IH^3(X) = H^3(\tilde{X})$, since $IH^*(X)$ satisfies the Poincaré duality.

In this case, (hL) and (HR) for $IH^*(X)$ are equivalent to that of $H^*(\tilde{X})$ with $f^*\mathcal{O}_{\tilde{X}}(1)$. Note that $f$ is semi-small in our case, so this follows exactly from Thm(dCM) in Section 2.

Rmk: A large part of this note is directly taken from a lecture note and a survey of Eliás and Williamson.