WALL-CROSSING FUNCTORS

IVAN LOSEV

1. WALL-CROSSING FUNCTORS AND DIMENSIONS OF SUPPORT

1.1. **Wall-crossing functors.** We are dealing with algebras $\mathcal{A}_\lambda(v)$ and sheaves $\mathcal{A}_\lambda^\theta(v)$. Below we will often omit v from the notation and write $\mathcal{A}_\lambda, \mathcal{A}_\lambda^\theta$.

Here we are going to produce a derived equivalence

$$D^b(\mathrm{Coh}(\mathcal{A}_\lambda^{\theta'})) \xrightarrow{\sim} D^b(\mathrm{Coh}(\mathcal{A}_\lambda^\theta))$$

for two different generic θ, θ'. These derived equivalences will play a crucial role in proving that $\mathrm{Im}\,\mathsf{CC}_v \subset L_\omega^{\mathfrak{a}}[\nu]$. They were introduced in [BPW, Section 6] under the name of twisting functors.

Let us recall from Lecture 3 that abelian localization holds for $(\lambda + n\theta, \theta)$ for all n sufficiently large, see Proposition 2.2 there. Also recall that $\mathrm{Coh}(\mathcal{A}_\lambda^?) \xrightarrow{\sim} \mathrm{Coh}(\mathcal{A}_{\lambda+n\theta'}^?)$ via tensoring with $\mathcal{A}_{\lambda,n\theta'}^?$, where ? stands for θ or θ'. So we can set

$$\mathfrak{WC}_{\theta\leftarrow\theta'} = L\,\mathrm{Loc}_{\lambda+n\theta'}^\theta \circ R\Gamma_{\lambda+n\theta'}^{\theta'},$$

where $\mathrm{Coh}(\mathcal{A}_\lambda^?)$ is identified with $\mathrm{Coh}(\mathcal{A}_{\lambda+n\theta'}^?)$ as above. Note that the equivalence $\mathfrak{WC}_{\theta\leftarrow\theta'}$ depends only on the coset of λ modulo $\mathbb{Z}^{Q_0}$. Also, for $M \in \mathrm{Coh}(\mathcal{A}_\lambda^{\theta'})$ we have $H_i(\mathfrak{WC}_{\theta\leftarrow\theta'}M) = 0$ for $i < 0$ because $\Gamma_{\lambda+n\theta}^\theta$ is exact.

One can present this equivalence on the level of global sections as well. Namely, suppose that λ and λ' are such that abelian localization holds for (λ, θ) and (λ', θ') and $\lambda'-\lambda \in \mathbb{Z}^{Q_0}$. We want an equivalence $D^b(\mathcal{A}_{\lambda'}\text{-mod}) \to D^b(\mathcal{A}_\lambda\text{-mod})$. We set

$$\mathfrak{WC}_{\lambda\leftarrow\lambda'} := R\Gamma_\lambda^\theta \circ (\mathcal{A}_{\lambda',\lambda-\lambda'}^\theta \otimes_{\mathcal{A}_\lambda^\theta} \bullet) \circ L\,\mathrm{Loc}_{\lambda'}^\theta\,.$$

Note that, by the very definition, under the identifications $\mathcal{A}_\lambda\text{-mod} \cong \mathrm{Coh}(\mathcal{A}_\lambda^\theta), \mathcal{A}_{\lambda'}\text{-mod} \cong \mathrm{Coh}(\mathcal{A}_\lambda^{\theta'})$, the equivalence $\mathfrak{WC}_{\lambda\leftarrow\lambda'}$ becomes $\mathfrak{WC}_{\theta\leftarrow\theta'}$.

1.2. **Long wall-crossing.** Fix θ. Let λ and $\lambda^- \in \lambda + \mathbb{Z}^{Q_0}$ be such that abelian localization holds for (λ, θ) and $(\lambda^-, -\theta)$. So we have the long crossing functor $\mathfrak{WC}_{\lambda^-\leftarrow\lambda} : D^b(\mathcal{A}_\lambda\text{-mod}) \xrightarrow{\sim} D^b(\mathcal{A}_{\lambda^-}\text{-mod})$. The reason for "long" here is that θ and $-\theta$ lie in opposite chambers (compare with longest elements in Weyl groups).

The following theorem characterizes the finite dimensional $\mathcal{A}_\lambda$-modules in terms of the long wall-crossing functor. In order to state it, we need to define holonomic $\mathcal{A}_\lambda$-modules.

Definition 1.1. A finitely generated $\mathcal{A}_\lambda$-module M is called <u>holonomic</u> if $\varphi^{-1}(\mathrm{Supp}\,M)$ is a lagrangian subvariety of $\mathcal{M}^\theta(v)$. This can be shown to be equivalent to the condition that the intersection of $\mathrm{Supp}\,M$ with any symplectic leaf in $\mathcal{M}(v)$ is lagrangian and so is independent of the choice of θ, see the appendix to [L2].

Note that any finite dimensional module is holonomic.

Theorem 1.2. *Let M be a simple holonomic $\mathcal{A}_\lambda$-module. Then $\frac{1}{2}\dim\mathcal{M}^\theta(v)-\dim\mathrm{Supp}(M)$ is the maximal number i such that $H_j(\mathfrak{WC}_{\lambda^-\leftarrow\lambda}M)=0$ for all $j<i$. In particular, M is finite dimensional if and only if $H_j(\mathfrak{WC}_{\lambda^-\leftarrow\lambda}M)=0$ for all $j<\frac{1}{2}\dim X$.*

The number i in the theorem will often be called the *homological shift* (under the functor $\mathfrak{WC}_{\lambda^-\leftarrow\lambda}$). Note that this number exists because $\mathfrak{WC}_{\lambda^-\leftarrow\lambda}$ is an equivalence.

To prove Theorem 1.2 one needs to relate the functor $\mathfrak{WC}_{\lambda^-\leftarrow\lambda}$ to the homological duality functor to be discussed in the next section.

1.3. **Homological duality.** Let $\mathcal{A}$ be an associative algebra. Then we can consider a contravariant equivalence $D := R\mathrm{Hom}_{\mathcal{A}}(\bullet,\mathcal{A}) : D^-(\mathcal{A}\text{-mod}) \xrightarrow{\sim} D^+(\mathcal{A}^{opp}\text{-mod})$, the *homological duality* functor, a generalization of taking the dual space in Linear algebra. When $\mathcal{A}$ has finite homological dimension, D restricts to an equivalence $D^b(\mathcal{A}\text{-mod}) \xrightarrow{\sim} D^b(\mathcal{A}^{opp}\text{-mod})$. Note that $D^2 = \mathrm{id}$ (or, without abusing the notation, $D_{\mathcal{A}^{opp}} \circ D_{\mathcal{A}} = \mathrm{id}$ and $D_{\mathcal{A}} \circ D_{\mathcal{A}^{opp}} = \mathrm{id}$).

The functor D is closely related to the supports. First, let us consider the commutative situation.

Lemma 1.3. *Let $A := \mathbb{C}[X]$, where X is an affine Cohen-Macaulay variety. Let M be a finitely generated A-module with support of codimension d. Then $H^j(DM) = 0$ for $j < d$, $H^d(DM)$ has support of codimension d, and for $j > d$, $H^j(DM)$ has support of codimension strictly bigger than d.*

For the proof see [Ei, Proposition 18.4]. We will also need another general lemma.

Lemma 1.4. *Let $\mathcal{A}$ be a complete and separated $\mathbb{Z}$-filtered associative algebra and let M, N be finitely generated modules over $\mathcal{A}$. Equip M, N with complete and separated filtrations. Then the space $\mathrm{Ext}^i_{\mathcal{A}}(M,N)$ acquires a complete and separated filtration with $\mathrm{gr}\,\mathrm{Ext}^i_{\mathcal{A}}(M,N) \hookrightarrow \mathrm{Ext}^i_{\mathrm{gr}\,\mathcal{A}}(\mathrm{gr}\,M,\mathrm{gr}\,N)$.*

Let us deduce two corollaries of the previous two lemmas.

Corollary 1.5. *Let λ be such that the localization holds for (λ,θ) and $\mathcal{A} := \mathcal{A}_\lambda$. Let M be a finitely generated $\mathcal{A}$-module with $\mathrm{Supp}(M)$ having codimension d. Then $H^i(DM) = 0$ for $i < d$, $H^i(DM)$ has support of codimension d, $\mathrm{codim}\,\mathrm{Supp}(H^i(DM)) > d$ for $i > d$ and $H^i(DM) = 0$ for $i > \dim\mathcal{M}^\theta(v)$.*

Proof. Let us prove that $H^i(DM) = 0$ for $i > \dim\mathcal{M}^\theta$. This follows from the more general statement that the homological dimension of $\mathcal{A}_\lambda$-mod does not exceed $\dim\mathcal{M}^\theta$. Indeed, as was noticed in Section 2.2 of Lecture 3, the homological dimension of $\mathrm{Coh}(\mathcal{A}^\theta_\lambda)$ does not exceed that of $\mathrm{Coh}(\mathcal{M}^\theta)$ equal to $\dim\mathcal{M}^\theta$. Since abelian localization holds for (λ,θ), our claim follows.

The claims that $H^i(DM) = 0$ for $i < d$ and $\mathrm{codim}\,\mathrm{Supp}\,H^i(DM) > d$ for $i > d$ follow from Lemmas 1.3,1.4. Let us prove that the support of $H^d(DM)$ has codimension exactly d. Assume the converse, then the codimension is strictly bigger than d. The spectral sequence for the composition of derived functors (in our case, of D with itself) shows that the support of all homologies of D^2M has codimension bigger than d. But $D^2M = M$, a contradiction. $\square$

We also have the homological duality functor $D^\theta : D^b(\mathrm{Coh}(\mathcal{A}^\theta_\lambda)) \to D^b(\mathrm{Coh}(\mathcal{A}^{\theta,opp}_\lambda))$ given by $R\mathcal{H}om_{\mathcal{A}^\theta_\lambda}(\bullet,\mathcal{A}^\theta_\lambda)$, where we take the sheaf Hom.

Corollary 1.6. *Let N be a holonomic coherent $\mathcal{A}_\lambda^\theta$-module. Then $H^i(D^\theta M) = 0$ if $i \neq \frac{1}{2}\dim \mathcal{M}^\theta(v)$.*

Proof. We have $H^i(D^\theta M) = 0$ for $i < \frac{1}{2}\dim \mathcal{M}^\theta$. Further, $\dim \operatorname{Supp} H^i(D^\theta M) < \frac{1}{2}\dim \mathcal{M}^\theta$ for $i > \frac{1}{2}\dim X$. But the Gabber theorem (Theorem 1.5 in Lecture 3) implies that the dimension of the support of a nonzero module cannot be smaller than $\frac{1}{2}\dim \mathcal{M}^\theta$. $\square$

1.4. **Sketch of proof of Theorem 1.2.** Now we are almost ready to prove the theorem. The main point of the proof is that, up to composing with abelian equivalences, both $\mathfrak{WC}_{\lambda^- \leftarrow \lambda}$ and $D[\frac{1}{2}\dim \mathcal{M}^\theta]$ are $L\operatorname{Loc}_\lambda^{-\theta}$ at which point we can apply Corollary 1.5.

Proof. First of all, note that

$$(1) \qquad D \cong R\Gamma_\lambda^{-\theta,opp} \circ D^{-\theta} \circ L\operatorname{Loc}_\lambda^{-\theta},$$

where $R\Gamma_\lambda^{-\theta,opp}$ is the derived global section functor for right modules. To prove that equality note that $R\Gamma_\lambda^{-\theta,opp} \circ D^{-\theta} = R\operatorname{Hom}_{\mathcal{A}_\lambda^{-\theta}}(\bullet, \mathcal{A}_\lambda^{-\theta})$ (the (derived) Hom in the category of coherent modules is the (derived) global sections of the (derived) sheaf Hom) and

$$R\operatorname{Hom}_{\mathcal{A}_\lambda^{-\theta}}(L\operatorname{Loc}_\lambda^{-\theta}(\bullet), \mathcal{A}_\lambda^{-\theta}) = R\operatorname{Hom}_{\mathcal{A}_\lambda}(\bullet, R\Gamma_\lambda^{-\theta}(\mathcal{A}_\lambda^{-\theta})) = R\operatorname{Hom}_{\mathcal{A}_\lambda}(\bullet, \mathcal{A}_\lambda),$$

where we use that $L\operatorname{Loc}_\lambda^{-\theta}$ and $R\Gamma_\lambda^{-\theta}$ are adjoint and the equality $R\Gamma_\lambda^{-\theta}(\mathcal{A}_\lambda^{-\theta}) = \mathcal{A}_\lambda$.

As we have seen, Corollary 1.6, $D^{-\theta}[\frac{1}{2}\dim \mathcal{M}^{-\theta}(v)]$ is an abelian equivalence. The functor $R\Gamma_\lambda^{-\theta,opp}$ is an abelian equivalence as well. Indeed, as was mentioned in Lecture 2, $(\mathcal{A}_\lambda^{sym,\theta})^{opp} \cong \mathcal{A}_{-\lambda}^{sym,\theta}$. We can replace λ with $\lambda + n\theta$, where $n \gg 0$, and achieve that abelian localization holds for $(-\lambda - 2\rho(v), -\theta)$. So we conclude that $D[\frac{1}{2}\dim \mathcal{M}^\theta(v)]$ is a composition of $\operatorname{Loc}_\lambda^{-\theta}$ and some abelian equivalence. But the same is true for $\mathfrak{WC}_{\lambda \to \lambda^-}$. Now the claim of the theorem follows from Corollary 1.5. $\square$

2. How the counting conjecture is proved

2.1. **Composition.** One can ask if the composition of two wall-crossing functors is again a wall-crossing functor. We will start by establishing a natural functor morphism $\mathfrak{WC}_{\theta \leftarrow \theta'} \circ \mathfrak{WC}_{\theta' \leftarrow \theta''} \to \mathfrak{WC}_{\theta \leftarrow \theta''}$. For this we will need another realization of the wall-crossing functor.

Lemma 2.1. *The functor $\pi_\lambda^{\theta'} : D(R)\text{-mod}^{G,\lambda} \twoheadrightarrow \operatorname{Coh}(\mathcal{A}_\lambda^{\theta'})$ admits the derived left adjoint $L(\pi_\lambda^{\theta'})^!$. We have $\mathfrak{WC}_{\theta \leftarrow \theta'} = \pi_\lambda^\theta \circ L(\pi_\lambda^{\theta'})^!$.*

Note that the right hand side is more or less the only natural derived right exact functor that one can write between two quotient categories of the same category.

Proof. To prove the claim about the left adjoint, we can replace λ with $\lambda + n\chi'$, where χ' lies in the chamber of θ' and $n \gg 0$. We achieve that $\mathcal{A}_\lambda^0(v) = \mathcal{A}_\lambda(v)$ and the localization theorem holds for (λ, θ'). Because of this, $\pi_\lambda^0 \cong \pi_\lambda^{\theta'}$ (Lemma 2.5 in Lecture 3). Since π_λ^0 has a (derived) left adjoint functor, we are done.

Now let us prove $\mathfrak{WC}_{\theta \leftarrow \theta'} = \pi_\lambda^\theta \circ L(\pi_\lambda^{\theta'})^!$. Under the identification of $\operatorname{Coh}(\mathcal{A}_\lambda^{\theta'})$ with $\mathcal{A}_\lambda$-mod, the left hand side becomes $L\operatorname{Loc}_\lambda^\theta = \mathcal{A}_\lambda^\theta \otimes_{\mathcal{A}_\lambda^0}^L \bullet$. The right hand side is

$$\pi_\lambda^\theta(D(R)/\mathcal{I}_\lambda \otimes_{\mathcal{A}_\lambda^0}^L \bullet) = (D(R)/\mathcal{I}_\lambda \otimes_{\mathcal{A}_\lambda^0}^L \bullet|_{T^*R^{\theta-ss}})^G = (D(R)/\mathcal{I}_\lambda|_{T^*R^{\theta-ss}})^G \otimes_{\mathcal{A}_\lambda^0}^L \bullet$$

$$= \mathcal{A}_\lambda^\theta \otimes_{\mathcal{A}_\lambda^0}^L \bullet.$$

This completes the proof. $\square$

Now to obtain the morphism $\mathfrak{WC}_{\theta \leftarrow \theta'} \circ \mathfrak{WC}_{\theta' \leftarrow \theta''} \to \mathfrak{WC}_{\theta \leftarrow \theta''}$ we compose the adjunction unit $L(\pi_\lambda^{\theta'})^! \circ \pi_\lambda^{\theta'} \to \mathrm{id}$ with $L(\pi_\lambda^{\theta''})^!$ on the right and π_λ^θ on the left.

One can ask when $\mathfrak{WC}_{\theta \leftarrow \theta'} \circ \mathfrak{WC}_{\theta' \leftarrow \theta''} \to \mathfrak{WC}_{\theta \leftarrow \theta''}$ is an isomorphism. This is true under certain "reduced decomposition" assumption. The following is a part of [BPW, Theorem 6.35].

Proposition 2.2. *Let $\theta, \theta', \theta''$ satisfy the following condition: if θ, θ'' lie to one side of a wall, then θ' lies on the same side. Then $\mathfrak{WC}_{\theta \leftarrow \theta'} \circ \mathfrak{WC}_{\theta' \leftarrow \theta''} \xrightarrow{\sim} \mathfrak{WC}_{\theta \leftarrow \theta''}$.*

This result shows that every wall-crossing functor can be presented as the "reduced" composition of "short" wall-crossing functors (those that intersect only one wall). You should view this as a categorical analog of reduced expressions in Weyl groups (or, even better, braid groups).

2.2. Sketch of the proof of the inclusion $\mathrm{Im}\, \mathsf{CC}_v \subset L_\omega^{\mathfrak{a}}[\nu]$.

Let us briefly (and very informally) explain how the inclusion $\mathrm{Im}\, \mathsf{CC}_v \subset L_\omega^{\mathfrak{a}}[\nu]$ is proved. The finite dimensional modules are those shifted by $\mathfrak{WC}_{-\theta \leftarrow \theta}$ by $\frac{1}{2} \dim \mathcal{M}^\theta(v)$. We decompose $\mathfrak{WC}_{-\theta \leftarrow \theta}$ into the composition of short wall-crossing functors. Some of them can be ignored as the following proposition shows.

Proposition 2.3. *Suppose that θ, θ' are separated by a wall that does not have the form $\ker \alpha$, where $\lambda \cdot \alpha \in \mathbb{Z}$ (recall that, by definition, a wall is the hyperplane $\ker \alpha$, where α is a root with $\alpha \leqslant v$). Then $\mathfrak{WC}_{\theta \leftarrow \theta'}$ is an abelian equivalence.*

The walls $\ker \alpha$ with $\alpha \cdot \lambda \in \mathbb{Z}$ will be called *essential*. One can show that if $\mathrm{Im}\, \mathsf{CC}_v \not\subset L_\omega^{\mathfrak{a}}[\nu]$, then the following is true:

(*) there is a simple finite dimensional module that will not be shifted by the wall-crossing functors through the walls $\ker \alpha$, where α is a real root (conjugate to a simple root by an element of the Weyl group). Very informally, we need to take a "singular" object for $\mathfrak{a}$.

This finishes the proof in the case when Q is of finite type.

When Q is of affine type, then all imaginary (=not real) roots are proportional to δ. So there is only one imaginary wall to be crossed. The main result in this case is that the short wall-crossing functor through $\ker \delta$ cannot shift any object by more than $\frac{1}{2} \dim \mathcal{M}^\theta(v) - 1$. There is an algebro-geometric counterpart of this statetement: for λ Zariski generic on the hyperplane $\ker \delta$, the variety $\mathcal{M}_\lambda(v)$ has no isolated singularities. Ideally, one should be able to deduce the aforementioned property of the wall-crossing functor from the last claim, but, in real life, the proof is a way more technical (it will be briefly discussed below).

This is only a rough sketch, the actual proof is much more technical and involved.

3. Real short wall-crossing

In order to establish (*) one needs to relate the short wall-crossing functor $\mathfrak{WC}_{\theta' \leftarrow \theta}$ through $\ker \alpha$, where α is a real root, to the categorification functors E_α, F_α. We will explain what happens in the case when $\theta = \theta^+$ (the stability condition with positive entries) and $\alpha = \alpha^i$. The general case can be treated using Proposition 2.3 (that allow us to freely switch stability conditions within a single chamber for $\mathfrak{a}$) and quantum LMN isomorphisms explain in Sections 3.1, 3.2 of Lecture 4.

In this case λ_i has to be integral. We can choose λ so that $\lambda_i \geqslant 0$ and abelian localization holds for (λ, θ) and, equivalently, for $(s_i \cdot \lambda, s_i \theta)$ (since λ_i is integral, we see that $s_i \cdot \lambda - \lambda \in \mathbb{Z}^{Q_0}$). If $\lambda_i < 0$, we just replace λ with $\lambda + n\theta$ for a sufficiently large λ. We get the wall-crossing functor

$$\mathfrak{WC}_{s_i\theta \leftarrow \theta} : D^b(\mathrm{Coh}(\mathcal{A}_\lambda^\theta(v))) \xrightarrow{\sim} D^b(\mathrm{Coh}(\mathcal{A}_{s_i\cdot\lambda}^{s_i\theta}(v)))$$

On the other hand, we have a quantum LMN isomorphism $s_i : \mathcal{A}_{s_i\cdot\lambda}^{s_i\theta}(v) \xrightarrow{\sim} \mathcal{A}_\lambda^\theta(s_i \cdot v)$. So the composition $s_{i*} \circ \mathfrak{WC}_{s_i\theta \leftarrow \theta}$ is an equivalence

$$D^b(\mathrm{Coh}(\mathcal{A}_\lambda^\theta(v))) \xrightarrow{\sim} D^b(\mathrm{Coh}(\mathcal{A}_\lambda^\theta(s_i \cdot v))).$$

We emphasize that the $\mathfrak{g}(Q)$-weights corresponding to $v, s_i \cdot v$ are related via the simple reflection corresponding to the $\mathfrak{sl}_2$-subalgebra generated by e_i, f_i.

It turns out that an equivalence between these categories can be produced from the categorical $\mathfrak{sl}_2$-action. Namely, we have the Rickard complexes of Chuang and Rouquier that categorify the simple reflection for $\mathfrak{sl}_2$. A version that works for our purposes is in [CDK, Section 8]. We get an equivalence

$$D^b(\mathrm{Coh}(\mathcal{A}_\lambda^\theta(v))) \xrightarrow{\sim} D^b(\mathrm{Coh}(\mathcal{A}_\lambda^\theta(s_i \cdot v)))$$

that we denote by Θ.

Theorem 3.1. *We have an isomorphism of functors* $\Theta^d \cong s_{i*} \circ \mathfrak{WC}_{s_i\theta \leftarrow \theta}$.

An important feature of the theorem is that it relates the wall-crossing functor $\mathfrak{WC}_{s_i\theta \leftarrow \theta}$ to the $\mathfrak{sl}_2$-categorification functors E_i, F_i. What it shows, in particular, is that a simple $M \in \mathcal{A}_\lambda^\theta\text{-mod}_{\varphi^{-1}(0)}$ has homological shift zero under $\mathfrak{WC}_{s_i\theta \leftarrow \theta}$ if and only if it is singular under E_i (or F_i depending on the sign of $\langle \nu, \alpha^{i\vee} \rangle$) in some precise sense, see [BL, Section 6] for details.

Besides being useful in the study of the wall-crossing functors and ultimately proving (*), this theorem allows to prove the claim in the previous lecture (Section 3.3) on the action of s_* on the middle homology. Namely, one can show that the wall-crossing functor applied to an object in $D^b_{\varphi^{-1}(0)}(\mathrm{Coh}(\mathcal{A}_?^?(v)))$ does not change the characteristic cycle. From here one can deduce that s_{i*} acts on the middle homology as the reflection from the corresponding $\mathfrak{sl}_2$ (up to a sign).

Let us say a couple of words about the proof of Theorem 3.1. This is again achieved using reduction in stages. More precisely, we use Lemma 2.1 and the decomposition of the quotient functor $\pi_\lambda^\theta(v)$ from the previous lecture. So we first prove the isomorphism of functors from Theorem 3.1 on the equivariant derived categories for the grassmanians and then deduce the isomorphism in general by passing to quotient categories.

4. IMAGINARY SHORT WALL-CROSSING

4.1. **Main result.** Here we are going to state the main result regarding the short imaginary wall-crossing functor (crossing the wall $\ker \delta$). This result presents a general pattern in the behavior of many wall-crossing functors as we will mention below. The result is that the functor is a *perverse equivalence* with respect to a filtration by annihilators of two-sided ideals that are flat with respect to a parameter on a hyperplane.

Let us start by explaining the definition of a perverse equivalence (due to Chuang and Rouquier), a.k.a. filtered Morita equivalence. Let $\mathcal{C}, \mathcal{C}'$ be two abelian categories equipped with descending filtrations by Serre subcategories: $\mathcal{C} = \mathcal{C}_0 \supsetneq \mathcal{C}_1 \supsetneq \ldots \supsetneq \mathcal{C}_q \neq \{0\}, \mathcal{C}' =$

$\mathcal{C}'_0 \supsetneqq \mathcal{C}'_1 \supsetneqq \ldots \supsetneqq \mathcal{C}'_q \supsetneqq \{0\}$. Let $0 = d_0 < d_1 < d_2 < \ldots < d_q$ be nonnegative integers (to be called homological shifts). Consider an equivalence $\mathcal{F} : D^b(\mathcal{C}) \to D^b(\mathcal{C}')$.

We say that $\mathcal{F}$ is perverse with respect to the filtrations (and shifts) above if the following holds:

(P1) For all i, the functor $\mathcal{F}$ restricts to an equivalence $D^b_{\mathcal{C}_i}(\mathcal{C}) \xrightarrow{\sim} D^b_{\mathcal{C}'_i}(\mathcal{C}')$, where $D^b_{\mathcal{C}_i}(\mathcal{C})$ stands for the category of all complexes in $D^b(\mathcal{C})$ whose homology lie in $\mathcal{C}_i$.

(P2) If $M \in \mathcal{C}_i$, then $H_j(\mathcal{F}M) = 0$ for $j < d_i$.

(P3) The functor $M \mapsto H_{d_i}(\mathcal{F}M)$ gives rise to an equivalence $\mathcal{C}_i/\mathcal{C}_{i+1} \xrightarrow{\sim} \mathcal{C}'_i/\mathcal{C}'_{i+1}$ of abelian categories. Moreover, we have $H_j(\mathcal{F}M) \in \mathcal{C}'_{i+1}$ for $j > d_i$.

Here is our main result concerning the wall-crossing functor through the imaginary wall.

Theorem 4.1. *Let θ, θ' be two stability conditions separated by the affine wall $\ker \delta$. Let λ, λ' be quantization parameters such that abelian localization holds for $(\lambda, \theta), (\lambda', \theta')$ and $\lambda' - \lambda \in \mathbb{Z}^{Q_0}$. Set $r := w \cdot \delta$, where δ is the indecomposable imaginary root, and let m denote the denominator of $\langle \lambda, \delta \rangle$. Then there are filtrations*

$$\mathcal{A}_\lambda(v)\text{-mod} \supsetneqq \mathcal{C}_1 \supset \mathcal{C}_2 \ldots \supset \mathcal{C}_q \neq \{0\}, \mathcal{A}_{\lambda'}(v)\text{-mod} \supsetneqq \mathcal{C}'_1 \supset \mathcal{C}'_2 \ldots \supset \mathcal{C}'_q \neq \{0\},$$

where $q := \lfloor \frac{\dim \mathcal{M}^\theta(v)}{2rm} \rfloor$ and homological shifts $d_i := i(rm - 1)$ making $\mathfrak{WC}_{\lambda' \leftarrow \lambda}$ a perverse equivalence.

This theorem is proved in [BL, Section 7] for a particular choice of w and in [L1, Section 6] in general. A more general result is proved in [L3, Theorem 3.1]. Namely, let θ, θ' be two stability conditions that are *opposite with respect to a face* meaning that the chambers C, C' containing θ, θ' satisfy the following condition: they have a common face Γ and there is an interval passing through the interior of Γ whose end-points are in the interiors of C, C'. Then the corresponding wall-crossing functor $\mathfrak{WC}_{\theta' \leftarrow \theta}$ is perverse and one can describe the filtrations generalizing the description given below in the setting of Theorem 4.1.

4.2. **Wall-crossing via HC bimodules.** There is a description of the equivalence $\mathfrak{WC}_{\lambda' \leftarrow \lambda}$, which is slightly different from the original one, that is going to play a crucial role in the study of these functors. Assume that (λ', θ') satisfies abelian localization. We set $\mathcal{A}^{(\theta')}_{\lambda, \lambda' - \lambda} := \Gamma(\mathcal{A}^\theta_{\lambda, \lambda' - \lambda})$, where we write $\mathcal{A}^\theta_{\lambda, \lambda' - \lambda}$ for the $\mathcal{A}^{\theta'}_{\lambda'}$-$\mathcal{A}^{\theta'}_\lambda$-bimodule quantizing the line bundle $\mathcal{O}(\lambda' - \lambda)$ on $\mathcal{M}^{\theta'}(v)$. So $\mathcal{A}^{(\theta')}_{\lambda, \lambda - \lambda'}$ is an $\mathcal{A}_{\lambda'}$-$\mathcal{A}_\lambda$-bimodule.

Lemma 4.2 (Section 6.4 in [BPW])**.** *We have $\mathfrak{WC}_{\lambda' \leftarrow \lambda} = \mathcal{A}^{(\theta')}_{\lambda, \lambda' - \lambda} \otimes^L_{\mathcal{A}_\lambda} \bullet$.*

Proof. These functors are equal when applied to $\mathcal{A}_{\lambda'}$ and so to every complex of free modules. $\square$

There are two features of the bimodule $\mathcal{A}^{(\theta)}_{\lambda', \chi}$ that make Lemma 4.2 useful. First, this bimodule is *Harish-Chandra*. Second, it is included into a flat (with respect to λ) family.

Let us define Harish-Chandra bimodules. Recall that the algebra $\mathcal{A}_\lambda$ is filtered, the filtration is inherited from the filtration on $D(R)$ by the order of a differential operator. We have $[\mathcal{A}_{\lambda, \leqslant i}, \mathcal{A}_{\lambda, \leqslant j}] \subset \mathcal{A}_{\lambda, \leqslant i+j-1}$. Now we say that an $\mathcal{A}_{\lambda'}$-$\mathcal{A}_\lambda$-bimodule $\mathcal{B}$ is Harish-Chandra (shortly, HC) if it can be equipped with a bimodule filtration $\mathcal{B}_{\leqslant j}$ (to be called *good*) such that the following two properties hold:

- If a is a homogeneous degree i element in $\mathbb{C}[\mathcal{M}(v)] = \operatorname{gr} \mathcal{A}_\lambda = \operatorname{gr} \mathcal{A}_{\lambda'}$ and $\tilde{a}, \tilde{a}'$ are its lifts to $\mathcal{A}_\lambda, \mathcal{A}_{\lambda'}$, then $\tilde{a}'b - b\tilde{a} \in \mathcal{B}_{\leqslant i+j-1}$. In particular, $\operatorname{gr} \mathcal{B}$ is a graded $\mathbb{C}[\mathcal{M}(v)]$-module (meaning that the left and the right actions coincide).
- $\operatorname{gr} \mathcal{B}$ is finitely generated as a $\mathbb{C}[\mathcal{M}(v)]$-module.

For example, $\mathcal{A}_\lambda$ viewed as a module over itself is Harish-Chandra. Obviously, Harish-Chandra modules form an abelian category. Another useful (and easy) property is that the tensor product of two HC bimodules is HC. What is less easy to see (but is still true) is that all Tor's and Ext's (in the categories of the left modules or of the right modules) are HC. This is left as an exercise (hint: use Lemma 1.4 or a similar statement).

Lemma 4.3. *The wall-crossing bimodule $\mathcal{A}_{\lambda',\chi}^{(\theta)}$ (where $\chi \in \mathbb{Z}^{Q_0}$) is HC.*

Proof. Recall that $\mathcal{A}_{\lambda,\chi}^{(\theta)} = \Gamma(\mathcal{A}_{\lambda,\chi}^\theta)$ and $\operatorname{gr} \mathcal{A}_{\lambda,\chi}^\theta = \mathcal{O}(\chi)$. So $\mathcal{A}_{\lambda,\chi}^{(\theta)}$ comes with a natural filtration and $\operatorname{gr} \mathcal{A}_{\lambda,\chi}^{(\theta)} \subset \Gamma(\mathcal{O}(\chi))$. The latter is a finitely generated $\mathbb{C}[\mathcal{M}(v)]$-module. Our claim follows. $\square$

Let us also recall that, for a HC $\mathcal{A}_{\lambda'}$-$\mathcal{A}_\lambda$ bimodule $\mathcal{B}$ we can assign its associated variety $\mathrm{V}(\mathcal{B})$ that is the support of $\operatorname{gr} \mathcal{B}$. We will view it as a subvariety of $\mathcal{M}(v)$.

Let us proceed to families of HC bimodules. Set $\mathfrak{p} := \mathbb{C}^{Q_0} = \mathfrak{g}^{*G}$. Set $\mathcal{A}_\mathfrak{p}^0 := [D(R)/D(R)\Phi([\mathfrak{g},\mathfrak{g}])]^G$, this is a filtered $\mathbb{C}[\mathfrak{p}]$-algebra whose specialization to $\lambda \in \mathfrak{p}$ coincides with $\mathcal{A}_\lambda^0$. We can introduce the notion of a HC bimodule for this algebra similarly to the above. An example is provided by $\mathcal{A}_{\mathfrak{p},\chi}^0 = [D(R)/D(R)\Phi([\mathfrak{g},\mathfrak{g}])]^{G,\chi}$. The specialization of this bimodule to λ from the right coincides with $\mathcal{A}_{\lambda,\chi}^0$ introduced in Section 1.4 of Lecture 3. Note that there is a natural homomorphism $\mathcal{A}_{\lambda,\chi}^0 \to \mathcal{A}_{\lambda,\chi}^{(\theta)}$. Informally speaking, this homomorphism is an isomorphism "every time we actually need this". Here is an example of a precise statement.

Lemma 4.4 (Lemma 5.26 in [BL]). *Assume that $\mathcal{A}_\lambda = \mathcal{A}_\lambda^0, \mathcal{A}_{\lambda+\chi} = \mathcal{A}_{\lambda+\chi}^0$ (that is the case when λ is Zariski generic). Further, assume that abelian localization holds for $(\lambda + \chi, \theta)$. Then $\mathcal{A}_{\lambda,\chi}^0 = \mathcal{A}_{\lambda,\chi}^{(\theta)}$.*

Here is a basic fact that makes considering families as above actually useful. For a HC $\mathcal{A}_\mathfrak{p}^0$-bimodule $\mathcal{B}_\mathfrak{p}$ we can define the notion of the (right) support in $\mathfrak{p}$, the set of points λ such that $\mathcal{B}_\lambda := \mathcal{B}_\mathfrak{p} \otimes_{\mathbb{C}[\mathfrak{p}]} \mathbb{C}_\lambda \neq \{0\}$. One can show that the support is a constructible set (a union of locally closed subvarieties). Moreover, if $\mathfrak{p}_0 \subset \mathfrak{p}$ is an affine subset, then $\mathcal{B}_{\mathfrak{p}_0} := \mathcal{B}_\mathfrak{p} \otimes_{\mathbb{C}[\mathfrak{p}]} \mathbb{C}[\mathfrak{p}_0]$ is generically free (of infinite rank, in general) as a $\mathbb{C}[\mathfrak{p}_0]$-module (and, moreover, the support is closed, Proposition 5.16). What this allows to do is to transfer properties of $\mathcal{B}_\lambda$ from λ Weil generic to λ Zariski generic. The basic property checked this way is whether a HC bimodule is zero or not.

4.3. Deformation/degeneration argument.

Let C, C' be the chambers containing the stability conditions θ, θ', they share the wall $\ker \delta$. We may assume that

(**) abelian localization holds for $(\lambda + (C \cap \mathbb{Z}^{Q_0}), \theta)$ and $(\lambda' + (C \cap \mathbb{Z}^{Q_0}), \theta')$, note that both parameter loci are Zariski dense.

Set $\chi := \lambda' - \lambda$, $\mathfrak{p}_0 := \lambda + \ker \delta$, $\mathfrak{p}_0' := \lambda' + \ker \delta$. Consider the $\mathcal{A}_{\mathfrak{p}_0'}^0$-$\mathcal{A}_{\mathfrak{p}_0}^0$-bimodule $\mathcal{B}_{\mathfrak{p}_0} := \mathcal{A}_{\mathfrak{p}_0,\chi}^0$ and the $\mathcal{A}_{\mathfrak{p}_0}^0$-$\mathcal{A}_{\mathfrak{p}_0'}^0$-bimodule $\mathcal{B}_{\mathfrak{p}_0}' := \mathcal{A}_{\mathfrak{p}_0',-\chi}^0$. When specialized to a point $\lambda_1 \in \lambda + (C \cap \ker \delta \cap \mathbb{Z}^{Q_0})$, the bimodule $\mathcal{B}_{\mathfrak{p}_0}$ becomes the wall-crossing bimodule $\mathcal{A}_{\lambda_1+\chi \leftarrow \lambda_1}^{(\theta)}$ and a similar claim holds for $\mathcal{B}_{\mathfrak{p}_0}'$.

Now we are going to explain how to produce the filtration on the category $\mathcal{A}_{\lambda_1}$-mod (for a Zariski generic $\lambda_1 \in \mathfrak{p}_0$). When λ_1 is Weil generic, we just take the filtration by the GK-dimension of the annihilator. More precisely, one can show, [L2], that the algebra $\mathcal{A}_{\lambda_1}$ has finite length as a bimodule. So, for every $i \in \mathbb{Z}_{\geqslant 0}$, there is a minimal ideal $\mathcal{J}$ with $\dim V(\mathcal{A}_{\lambda_1}/\mathcal{J}) \leqslant 2i$, we denote this ideal by $\mathcal{J}_{\lambda_1,i}$. One can show, see [L1, Section 6], that when λ_1 is Weil generic in $\mathfrak{p}_0$, then the possible codimensions of the varieties $V(\mathcal{A}_{\lambda_1}/\mathcal{J})$ are divisible by $2(rm-1)$ and, moreover, there is exactly one two-sided ideal corresponding to each codimension and the ideals form a chain.

Note that for a Weil generic parameter $\lambda_1 \in \mathfrak{p}_0$, the wall-crossing functor $\mathfrak{WC}_{\lambda_1+\chi \leftarrow \lambda_1}$ becomes a long wall-crossing functor (thanks to Proposition 2.3, only $\ker \delta$ is an essential wall because λ_1 is Weil generic). Then we can use Theorem 1.2 together with restriction functors for HC bimodules to deduce Theorem 4.1 for a Weil generic parameter (see, e.g., [L3, Section 3]).

In order to establish Theorem 4.1 for our initial parameter λ, thanks to (**), it is enough to prove it for a Zariski generic $\lambda_1 \in \mathfrak{p}_0$ (and for functors of derived tensor products by $\mathcal{A}_{\lambda_1,\chi}^0$). The first step here is to show that the ideals $\mathcal{J}_{\lambda_1,i}$ defined for a Weil generic λ_1 are actually specializations of certain ideals $\mathcal{J}_{\mathfrak{p}_0,i}$ that form a chain (again, this is non-trivial and is based on induction for HC bimodules but we do not discuss the proof, see [L3, Section 3]). Similarly, we have ideals $\mathcal{J}_{\mathfrak{p}_0',i} \subset \mathcal{A}_{\mathfrak{p}_0'}$. Now the filtration on $\mathcal{A}_{\lambda_1}$-mod is defined by these ideals (the subcategory $\mathcal{C}_i$ consists precisely of the modules annihilated by $\mathcal{J}_{\lambda_1,i}$, for λ_1 Zariski generic, this is a Serre subcategory because $\mathcal{J}_{\lambda_1,i}^2 = \mathcal{J}_{\lambda_1,i}$, the latter equality is proved similarly to the properties (P1),(P2) whose proof is explained below). Similarly, we have ideals $\mathcal{J}_{\mathfrak{p}_0',i} \subset \mathcal{A}_{\mathfrak{p}_0'}$ and define a filtration on $\mathcal{A}_{\lambda_1+\chi}$-mod.

The strategy of the proof that the functor $\mathcal{A}_{\lambda_1,\chi}^0 \otimes_{\mathcal{A}_{\lambda_1}}^L \bullet$ is perverse is as follows:

(i) Translate (P1)-(P3) to statements involving vanishing of Tor's and Ext's of HC bimodules (recall that those are HC as well).
(ii) Prove the translated statements for Weil generic parameters.
(iii) Then deduce them for Zariski generic parameters.

We are going to illustrate this by sketching proofs of (P1),(P2) (property (P3) is harder).

Sketch of proofs of (P1) and (P2). Let us write $\mathcal{F}_{\lambda_1}$ for $\mathcal{A}_{\lambda_1,\chi}^0 \otimes_{\mathcal{A}_{\lambda_1}}^L \bullet$. (P1) says that if $M \in \mathcal{A}_{\lambda_1+\chi}$ is annihilated by $\mathcal{J}_{\lambda_1,i}$, then all homology of $\mathcal{F}_{\lambda_1}(M)$ are annihilated by $\mathcal{J}_{\lambda_1',i}$ (and a similar claim for $\mathcal{F}_{\lambda_1}^{-1} = R\operatorname{Hom}_{\mathcal{A}_{\lambda_1}'}(\mathcal{A}_{\lambda_1,\chi}^0, \bullet)$).

Note that if M is annihilated by $\mathcal{J}_{\lambda_1,i}$, then it admits a resolution whose terms are direct sums of several copies of $\mathcal{A}_{\lambda_1}/\mathcal{J}_{\lambda_1,i}$. So the claim that all $H_j(\mathcal{F}_{\lambda_1}M)$ are annihilated by $\mathcal{J}_{\lambda_1',i}$ is equivalent to

$$(2) \qquad \mathcal{J}_{\lambda_1',i} \operatorname{Tor}_j^{\mathcal{A}_{\lambda_1}}(\mathcal{A}_{\lambda_1,\chi}^0, \mathcal{A}_{\lambda_1}/\mathcal{J}_{\lambda_1,i}) = 0$$

for all j. Similarly, our claim for $\mathcal{F}_{\lambda_1}^{-1}$ is equivalent to

$$(3) \qquad \operatorname{Ext}_{\mathcal{A}_{\lambda_1'}}^j(\mathcal{A}_{\lambda_1,\chi}^0, \mathcal{A}_{\lambda_1'}/\mathcal{J}_{\lambda_1',i})\mathcal{J}_{\lambda_1,i} = 0, \quad \forall j.$$

We already know (2) and (3) for a Weil generic parameter $\lambda_1 \in \mathfrak{p}_0$. But it is easy to prove that, say, the HC bimodule $\mathcal{J}_{\mathfrak{p}_0',i} \operatorname{Tor}_j^{\mathcal{A}_{\mathfrak{p}_0}}(\mathcal{A}_{\mathfrak{p}_0,\chi}^0, \mathcal{A}_{\mathfrak{p}_0}/\mathcal{J}_{\mathfrak{p}_0,i})$ specializes to the left hand side of (2) for a Zariski generic λ_1. Since the Weil generic specialization is zero, so is the Zariski generic specialization. This finishes the proof of (P1).

Let us proceed to (P2): $H_j(\mathcal{F}_{\lambda_1}(M)) = 0$ for $j < d_i$. This is equivalent to $\text{Tor}_j^{\mathcal{A}_{\lambda_1}}(\mathcal{A}_{\lambda_1,\chi}^0, \mathcal{A}_{\lambda_1}/\mathcal{J}_{\lambda_1}) = 0$ for $j < d_i$. Then we proceed as in the proof of (P1). $\square$

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