HODGE-RIEMANN BILINEAR RELATIONS FOR SOERGEL BIMODULES

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ABSTRACT. This article is a set of notes for a talk given in a graduate seminar Category $\mathcal{O}$ and Soergel Bimodules seminar held jointly in MIT and Northeastern during Fall 2017. In this talk, we elaborate on the inductive machine used by Elias and Williamson in [EW] to prove Soergel's conjecture. We first prove an embedding theorem for Hom spaces between Soergel bimodules. Subsequently, we construct a deformation of the Lefschetz form on every Bott-Samelson bimodule and prove the first part of a limiting argument that gives us the Hodge-Riemann bilinear relations. These relations are the last piece in the induction machine and hence, modulo details we leave for the next talk, this finishes the [EW] proof of the Soergel conjecture.

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1. Introduction

Let us fix a Coxeter system (W, S). We begin by recalling some of the induction machine for the proof of Soergel's conjecture that Seth explained in his talk last week. There were three statements in this induction machine.

1. $S(x)$: Soergel's conjecture holds for B_x.

2a. $hL(x)$: hard Lefschetz holds for $\overline{B_x}$

2b. $hL(\underline{x})$: hard Lefschetz holds for the restriction of the trace form from $BS(\underline{x})$ to B_x.

3a. $HR(x)$: Hodge-Riemann holds for $\overline{B_x}$

3b. $HR(\underline{x}, s)$: Hodge-Riemann holds for $\overline{B_x B_s}$ (with $\underline{x}$ usually a reduced expression) where $xs > x$.

From Seth's talk, the key remaining piece in the induction machine was to show that for any $x \in W$ and $s \in S$ with $xs > x$, $S(\leq x)$ and $HR(\leq x)$ imply $HR(\underline{x}, s)$. Let us give an outline of how this statement is proved.

(1) Construct a one parameter deformation of the Lefschetz operator L_ζ on $\overline{B_x B_s}$ for $\zeta \geq 0$, with L_0 the original Lefschetz operator.

(2) Show $hL(\underline{x}, s)_\zeta$ for each L_ζ.

(3) Show $HR(\underline{x}, s)_\zeta$ for some L_ζ with $\zeta >> 0$.

Hard Lefschetz is equivalent to the non-degeneracy of the Lefschetz form. Since the signature of a continuous family of non-degerate form is constant, the latter two statements above imply $HR(\underline{x}, s)$ for L_0, our original operator.

The talk is organized as follows. First, we will prove the embedding theorem stated in Seth's talk. This is needed for some of the machinery that Seth explained to work. Next, we will construct the deformation of the Lefschetz operator and prove a theorem that gives us $HR(x, s)_\zeta$ for some $\zeta \gg 0$.. Finally, we will explain how to approach the proof of $hL(x, s)_\zeta$ for all $\zeta \geq 0$ using Rouquier complexes.

2. Notation

Fix a Coxeter system (W, S) and pick a reflection faithful representation $\mathfrak{h}$ with linearly independent coroots $\{\alpha_s^\vee : s \in S\}$ and roots $\{\alpha_s : s \in S\}$. Pick some $\rho \in \mathfrak{h}^*$ such that $\rho(\alpha_s^\vee) > 0$ for all $s \in S$. R will denote the ring of functions on $\mathfrak{h}$, $\mathbb{R}[\mathfrak{h}^*]$ and for any $w \in W$, R_w will denote the functions symmetric with respect to w.

For $s \in S$, let B_s denote the Bott-Samelson bimodule $R \otimes_{R_s} R(1)$. We use $BS(\underline{x})$ to denote the Bott-Samelson bimodule associated to an expression $\underline{x}$ of x and B_x to denote the associated Soergel bimodule. Given an arbitrary R-bimodule B, we use $\overline{B}$ to denote the left R-module obtained by quotienting on the right by positive degree polynomials.

We have a basis for B_s as a left or right R-module given by $c_{\text{id}} = 1 \otimes 1$ and $c_s = \frac{\alpha_s \otimes 1 + 1 \otimes \alpha_s}{2}$. Given an expression $\underline{x}$ of length l, we have a basis for $BS(\underline{x})$ given by

$$\{c_\epsilon : \epsilon \in \{0, 1\}^l\}$$

where $c_\epsilon = c_{s_1^{\epsilon_1}} \otimes \cdots \otimes c_{s_m^{\epsilon_m}}$. We use c_{bot} to denote $c_{0\cdots 0}$ and c_{top} to denote $c_{1\cdots 1}$.

3. The Embedding Theorem

In this section, we will prove the embedding theorem that Seth used in his talk. Fix $x \in W$ and $s \in S$ with $xs > x$ and assume $S(< xs), HR(< xs)$. Pick some $y < xs$. Recall that we have a local intersection form on $Hom(B_y, B_x B_s)$ given by

$$(f, g)_y^{x,s} := g^* \circ f \in \text{End}(B_y) = \mathbb{R}.$$

Here, g^* is the adjoint of g and we have identified B_y and $B_x B_s$ with their duals using their global intersection forms that arise from $S(x)$ and $S(y)$.

Theorem 3.1. The map

$$i : \text{Hom}(B_y, B_x B_s) \to (\overline{B_x B_s})^{-l(y)},$$

defined by sending $f \mapsto \overline{f(c_{\text{bot}})}$, is injective, with image contained in the primitives

$$P_\rho^{-l(y)} \subseteq (\overline{B_x B_s})^{-l(y)}.$$

Moreover, i is an isometry with respect to the Lefschetz form up to a positive scalar factor.

Proof. We first prove the last two statements of the theorem.

Primitivity:

Pick $f \in \mathrm{Hom}(B_y, B_x B_s)$. Since f is a bi-module map, it descends to a map of left R-modules

$$\overline{B_y} \to \overline{B_x B_s}.$$

Now, $\rho^{l(y)+1}$ annihilates $\overline{c_{bot}} \in \overline{B_y}^{-l(y)}$ because $\overline{B_y}$ is concentrated in degrees $\leq l(y)$. Hence, $\rho^{l(y)+1}$ also annihilates $\overline{f(c_{\mathrm{bot}})}$, proving the statement regarding the primitives.

Isometry:

Next, let us prove the statement regarding the isometry. Let $N > 0 \in \mathbb{R}$ be the value of

$$\langle c_{\mathrm{bot}}, \rho^{l(y)} c_{\mathrm{bot}} \rangle$$

which is positive by $HR(y)$. Pick $f, g \in \mathrm{Hom}(B_y, B_x B_s)$. Then,

$$\begin{aligned} (f,g)_y^{x,s} &= g^* \circ f \\ &= \frac{1}{N} \langle \overline{g^* \circ f(c_{\mathrm{bot}})}, \overline{c_{\mathrm{bot}}} \rangle_{\overline{B_y}} \\ &= \frac{1}{N} \langle \overline{f(c_{\mathrm{bot}})}, \overline{g(c_{\mathrm{bot}})} \rangle_{\overline{B_x B_s}}. \end{aligned}$$

This proves that i is an isometry. The only thing left to prove is that i is injective. Note that this would be immediate if $HR(x,s)$ were true because an isometry (up to scalar) into a definite space is necessarily an injection. But we are not assuming $HR(x,s)$ so we need to find a different proof of injectivity.

Injectivity:

Recall the standard bi-module Δ_y, which was just R as a left module with right action twisted by y, i.e., $r_1 \cdot r = y(r) r_1$. In the category of R-bimodules, we have an inclusion

$$\Delta_y \to B_y.$$

Recall also that for any R-bimodule M and for any $A \subseteq W$, we have the submodule

$$\Gamma_A M := \{ m \in B : \mathrm{Supp}(m) \subseteq Gr(A) \}$$

where

$$Gr(A) = \bigcup_{x \in A} Gr(x)$$

and

$$Gr(x) = \{ (xv, v) : v \in \mathfrak{h} \} \subseteq \mathfrak{h} \times \mathfrak{h} \}$$

is the twisted graph that is the support of Δ_y as a bimodule. A theorem of Soergel mentioned previously by Seth says that for any Soergel bimodule B there is a finite subset $A \subseteq W$ such that $\Gamma_A B = B$ and for any $x \in A$, we have an isomorphism

$$\Gamma_{\geq x} B / \Gamma_{> x} B \cong R_x(-l(x))^{\oplus h_x(B)},$$

where $h_x(B) \in \mathbb{Z}_{\geq 0}[v^{\pm}]$. The character of B is then $\sum_{x \in W} h_x(B) H_x$. By $S(y)$, the character of B_y is $\underline{H_y} \in H_y + \sum_{w<y} v\mathbb{Z}[v] H_w$. Hence, the copy of Δ_y in B_y is unique and any generator $c \in \Delta_y$ must sit inside degree $l(y)$ and must project down to a generator for $\overline{B_y}^{l(y)} \cong \mathbb{R}$. Now, since $HR(y)$ holds, so does $hL(y)$ and hence $\rho^{l(y)} c_{\mathrm{bot}}$ has nonzero image in $\overline{B_y}^{l(y)}$ and is hence a nonzero multiple of $\overline{c}$. Hence, it suffices to prove that

$$i' : \mathrm{Hom}(B_y, B_x B_s) \to \overline{B_x B_s}^{l(y)}$$

defined by $f \mapsto \overline{f(c)}$ is injective. To show this, consider the exact sequence

$$0 \to \Delta_y \to B_y \to B_y / \Gamma_{\geq y} B_y \to 0.$$

Since this is part of the Δ-flag for B_y used to define the character, we have $\mathrm{ch}(\Delta_y) = H_y$, $\mathrm{ch}\, B_y = \underline{H}_y$, $\mathrm{ch}(B/\Gamma_{\geq y} B_y) = \underline{H}_y - H_y$. Since the characters add up correctly (which need not be the case since the exact sequence doesn't necessarily split), Soergel's Hom formula tells us that we also have an exact sequence

$$0 \to \mathrm{Hom}^\bullet(B_y/\Gamma_{\geq y} B_y, B_x B_s) \to \mathrm{Hom}^\bullet(B_y, B_x b_s) \to \mathrm{Hom}^\bullet(\Delta_y, B_x B_s) \to 0.$$

Computing characters again, we have $\mathrm{ch}(B_x B_s) = \underline{H}_x \underline{H}_s$. Hence, the Hom formula tells us that

$$\mathrm{rk}\, \mathrm{Hom}^\bullet(B_y/\Gamma_{\geq y} B_y, B_x B_s) = \overline{(\underline{H}_y - H_y, \underline{H}_x \underline{H}_s)} \in v^{-1}\mathbb{Z}[v^{-1}].$$

The last inclusion follows from the fact that

$$\underline{H}_y - H_y \in \bigoplus_{w<y} v\mathbb{Z}[v] H_w$$

and

$$\underline{H_x H_s} \in \bigoplus_{w<xs} \mathbb{Z}[v] H_w.$$

Hence, $\mathrm{Hom}^{\leqslant 0}(B_y/\Gamma_{\geq y} By, B_x B_s) = 0$ and we have an isomorphism

$$\mathrm{Hom}(B_y, B_x B_s) \to \mathrm{Hom}(\Delta_y, B_x B_s) = \Gamma_y(B_x B_s)(l(y)).$$

The latter equality comes from evaluating at c. Now, using the Hom formula again, we see that $\mathrm{Hom}^{<0}(B_y, B_x B_s) = 0$ and hence the same holds for $\mathrm{Hom}^{<0}(\Delta_y, B_x B_s)$. Thus, $\Gamma_y(B_x B_s)$ is concentrated in degrees $\geq l(y)$.

At this point, we use a result of Soergel (see proof of Prop 6.4 in [Soe]). This says that $\Gamma_y(B_x B_s)$ is s direct summand of $B_x B_s$ as a right R-module. Since the latter is free as a right module, if $m \in B_x B_s$ and $mr \in \Gamma_y(B_x B_s)$, then $m \in \Gamma_y(B_x B_s)$. Combine this statement with the fact that $\Gamma_y(B_x B_s)$ site in degrees $l(y)$ and higher, the induced map

$$\Gamma_y(B_x B_s)^{l(y)} \to \overline{B_x B_s}^{l(y)}$$

is injective. Composing this statement with the isomorphism above finishes the proof, as this composite map is $f \mapsto \overline{f(c)}$.

□

4. Deformation of the Lefschetz operator

Fix $x \in W$ and $s \in S$ with $xs > x$. For the rest of the talk, we assume $S(\leq x)$ and $HR(\leq x)$ and work towards proving the key inductive statement of $HR(\underline{x}, s)$. As mentioned in the introduction, we need to use a limiting argument. We begin by defining a deformation of the Lefschetz operator.

Definition 4.1. For $\zeta \geq 0$, define

$$L_\zeta := (\rho \cdot -)_{B_x} \mathrm{id}_{B_s} + \mathrm{id}_{B_x} (\zeta\rho \cdot -)_{B_s}.$$

Here, $(\rho \cdot -)_{B_x}$ and $(\zeta\rho \cdot -)_{B_s}$ depict multiplication in the corresponding tensor factor and we view L_ζ as an endomorphism of $B_x B_s$.

We recall the deformed versions of hard Lefschetz and Hodge-Riemann mentioned in the introduction.

1. $$hL(x,s)_\zeta : L_\zeta^i : (\overline{B_x B_s})^{-i} \to (\overline{B_x B_s})^i$$
   is an isomorphism.

2. $HR(\underline{x}, s)_\zeta$: For any embedding $B_x \subseteq BS(\underline{x})$, the Lefschetz form

$$(\alpha, \beta)^{\zeta}_{-i} := \langle \alpha, L^i_\zeta \beta\rangle_{\overline{B_x B_s}}$$

is $(-1)^{(l(x)+1-i)/2}$-definite on the primitive subspace $P^{-i}_{L_\zeta}$.

3. $HR(x, s)$: $HR(\underline{x}, s)_\zeta$ holds for any reduced expression $\underline{x}$ for x.

Our goal is to prove $HR(x, s)_0$. To show this, it suffices to show that $hL(x, s)_\zeta$ holds for all $\zeta \geq 0$ and $HR(x, s)_\zeta$ holds for some $\zeta \geq 0$ because signatures cannot change in a continuous family of non-degenerate forms. We begin by showing that $HR(x, s)_\zeta$ holds for some $\zeta \geq 0$.

5. Deformed Hodge-Riemann Relations

Instead of working with just B_x, we will work a little more generally. Let B be any summand of $BS(\underline{x})$. We can define an invariant form on B and BB_s in exactly the same as we do for $B_x B_s$ and we can similarly define the standard Lefschetz operators on $\overline{B}$ and $\overline{BB_s}$ and the deformed versions of the latter. For the rest of this section, we use $(-,-)^{-i}_\rho$ to refer to the Lefschetz form on $\overline{B}$ and $(-,-)^{-i}_{L_\zeta}$ to refer to the deformed Lefschetz form on $\overline{BB_s}$. The goal of the section is to prove the following theorem.

Theorem 5.1. Suppose that $\overline{B}$ satisfies hard Lefschetz and Hodge-Riemann with the standard sign. Then for $\zeta \gg 0$, the induced action of L_ζ on $\overline{BB_s}$ satisfies hard Lefschetz and Hodge-Riemann with the standard sign.

To prove this theorem, we will use the following Lemma, whose proof is an elementary problem of counting dimensions of vector spaces.

Lemma 5.2. Let V and W be two finite dimensional graded vector spaces, equipped with graded non-degenerate symmetric forms and Lefschetz operators satisfying hard Lefschetz. Assume that W is even or odd and $\underline{\dim} V = (v + v^{-1})\underline{\dim} W$. Suppose W satisfies Hodge-Riemann with standard sign. Then V satisfies Hodge-Riemann with standard sign if and only if for all $i \geq 0$, the signature of the Lefschetz form on $P^{-i+1} \subseteq W^{-i+1}$ is equal to the signature of the Lefschetz form on all of V^{-i}. (By convention $P^1 = 0$.)

Proof of Theorem 5.1. First, recall the maps α and β from B to BB_s introduced in Seth's talk, that were used to inductively construct the intersection form. We had $c_{\mathrm{id}} = 1 \otimes 1$ and $c_s = \frac{1\otimes\alpha_s + \alpha_s\otimes 1}{2}$ in B_s. $\alpha(b) = bc_{\mathrm{id}}$ and $\beta(b) = bc_s$. The inductive construction of the form was based on the formulas

$$\langle \alpha(b), \alpha(b')\rangle_{BB_s} = \partial_s(\langle b, b'\rangle_B)$$

$$\langle \alpha(b), \beta(b')\rangle_{BB_s} = \langle b, b'\rangle_B = \langle \beta_b, \alpha(b')\rangle_{BB_s}$$

$$\langle \beta(b), \beta(b')\rangle_{BB_s} = \langle b, b'\rangle_B \alpha_s.$$

We want to use this inductive construction to relate the forms on $\overline{B}$ with that on $\overline{BB_s}$ and then use the previous Lemma. The first fact that is easy to see is that

$$(\beta(b), \beta(b'))^{-i}_{L_\zeta} = 0$$

for any $b, b' \in \overline{B}_{-i-1}$ because of the α_s that shows up on the right. Additionally, left multiplication by $\zeta\rho$ is the same as right multiplication by $\zeta\rho$ on $c_s \in B_s$. Hence, the defining formulas also show that, for $b \in \overline{B}^{-i+1}, b' \in \overline{B}^{-i-1}$, we have

$$\begin{aligned}(\alpha(b), \beta(b'))^{-i}_{L_\zeta} &= (\alpha(b), \beta(b'))^{-i}_{L_0} \\ &= \langle \alpha(b), \rho^i \beta(b') \rangle_{\overline{BB_s}} \\ &= (b, \rho b')^{-i+1}_{\rho}.\end{aligned}$$

Let us now pick a suitable basis for $(\overline{BB_s})^{-i}$ and compare signatures. Choose elements $e_1, \ldots, e_n \in B^{-i-1}$ that project to an orthogonal basis (for the Lefschetz form) of $\overline{B}^{-i-1}$. Choose elements $p_1, \ldots, p_n \in B^{-i+1}$ projecting to an orthogonal basis for the primitives P^{-i+1} inside $\overline{B}^{-i+1}$. Then,

$$\rho e_1, \ldots \rho e_n, p_1, \ldots, p_m$$

project to an orthogonal basis for $(\overline{B})^{-i+1}$ and hence, since α applied to a basis for $\overline{B}^{-i+1}$ and β applied to a basis for $\overline{B}^{-i-1}$ give a basis for $(\overline{BB_s})^{-i}$,

$$\alpha(\rho e_1), \ldots, \alpha(\rho e_n), \beta(e_1), \ldots, \beta(e_n), \alpha(p_1), \ldots, \alpha(p_m)$$

project to a basis of $(\overline{BB_s})^{-i}$. The Gram matrix of $(-,-)^{-i}_{L_\zeta}$ in this basis has the form

$$M^{-i}_\zeta := \begin{pmatrix} * & J & * \\ J & 0 & 0 \\ * & 0 & Q_\zeta \end{pmatrix}.$$

Here, the $*$ and Q_ζ are unknown entries representing our lack of knowledge about $(\alpha(b), \alpha(b'))^{-i}_{L_\zeta}$. J is a non-degenerate diagonal matrix as

$$(\alpha(\rho e_i), \beta(e_j))^{-i}_{L_\zeta} = (\rho e_i, \rho e_j)^{-i+1}_\rho = (e_i, e_j)^{-i-1}_\rho$$

by our computations above. The 0's come from the fact that the form is 0 on the image of β. By expanding along the first row, we see that the determinant of this matrix only depends on the entries of J and the determinant of Q_ζ. Additionally, M^{-i}_ζ is invertible if Q_ζ is. Hence, we can deform the $*$ entries away by finding a path in the space of invertible symmetric matrices to the matrix

$$M := \begin{pmatrix} 0 & J & 0 \\ J & 0 & 0 \\ 0 & 0 & Q_\zeta \end{pmatrix}.$$

Recall that the signature of a non-degenerate symmetric matrix is the difference between the number of its positive and negative eigenvalues. The signature of M is thus the signature of Q_ζ, as the signature of

$$\begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$$

must be 0 as it is traceless. Since signatures of non-degenerate symmetric matrices cannot change, we conclude that the signature of M^{-i}_ζ is the signature of Q_ζ. What remains is to check that for $\zeta \gg 0$, Q_ζ is non-degenerate and has signature equal to the signature of $P^{-i+1}_\rho \subseteq \overline{B}^{-i+1}$. Since M^{-i}_ζ is the signature of $(\overline{BB_s})^{-i}$, this puts us in the framework of the previous lemma with $V = \overline{BB_s}$ and $W = \overline{B}$ and finishes the proof.

To compute this signature, we go back to the definition of the Lefschetz form. Recall that

$$(x, y)^{-i}_{L_\zeta} = \mathrm{Tr}_{\mathbb{R}}(L^i_\zeta(xy))$$

where the $tr_{\mathbb{R}} : BS(\underline{x})B_s \to \mathbb{R}$ was defined to be taking the c_{top} coefficient in R (as a right R-module) and then mapping down to $\mathbb{R}$. Hence,

$$(\alpha(p),\alpha(q))_{L_\zeta}^{-i} = \mathrm{Tr}_{\mathbb{R}}(L_\zeta^i((pq)c_{\mathrm{id}})) = \mathrm{Tr}_{\mathbb{R}}\left(\sum_{j=0}^{i}\binom{i}{j}\rho^{i-j}pq(\zeta\rho)^j c_{\mathrm{id}}\right).$$

By the formulas for left multiplication by polynomials on c_{id}, we have for $j \geq 1$,

$$\rho^{i-j}pq(\zeta\rho)^j c_{\mathrm{id}} = \rho^{i-j}pqc_s\partial_s((\zeta\rho)^j) + \rho^{i-j}pqc_{\mathrm{id}}s(\zeta\rho)^j.$$

For $j > 1$, we will always have some positive degree polynomial and the right and hence the real trace will be 0. Hence,

$$(\alpha(p),\alpha(q))_{L_\zeta}^{-i} = \mathrm{Tr}_{\mathbb{R}}(\rho^i pqc_{\mathrm{id}}) + i\zeta\rho(\alpha_s^\vee)\mathrm{Tr}_{\mathbb{R}}(\rho^{i-1}pq)$$

as tensoring with c_s on the right sends c_{top} in $\overline{B}$ to c_{top} in $\overline{BB_s}$. Now, note that we can renormalize Q_ζ as Q_ζ/ζ for positive ζ and the signature will not change. But the limit

$$\lim_{\zeta\to\infty}\frac{Q_\zeta}{\zeta} = i\rho(\alpha_s^\vee)Q$$

where Q is the Gram matrix of $(-,-)_\rho^{-i+1}$ in the basis $p_1,\ldots,p_n$. This is non-degenerate by our assumption of Hodge-Riemann for B. Hence, for some $\zeta \gg 0$, Q_ζ is non-degenerate with the same signature as Q, i.e., the same signature as $(-,-)_\rho^{-i+1}$, as $\rho(\alpha_s^\vee) > 0$. This finishes the proof.

□

We have proved the statement we needed to regarding Hodge-Riemann. All that is left is to prove that $hL(x,s)_\zeta$ holds for all $\zeta \geq 0$. This is the hardest part of [EW] and we will give a roadmap of the proof for now, leaving most of the details for the last talk.

6. Hard Lefschetz for Soergel Bimodules

We begin with some motivation from Hodge theory. Recall the weak Lefschetz theorem, which states that for a smooth projective variety X and a general hyperplane section $r : X_H \to X$, the restriction r^* is injective in degrees $\leq \dim_{\mathbb{C}} X - 1$. Using this fact, under the assumption that $H^*(X_H)$ satisfies the Hodge-Riemann relations, we can deduce hard Lefschetz for X. This is done via the following lemma, proved in Xiaolei's talk.

Lemma 6.1. Suppose that we have a map of graded $\mathbb{R}[L]$-modules (with $\deg(L) = 2$)

$$\phi : V \to W(1)$$

such that

(1) ϕ is injective in degrees ≤ -1.
(2) V and W are equipped with graded bilinear forms such that

$$\langle\phi(a),\phi(b)\rangle_W = \langle a, Lb\rangle_V$$

for all $a, b \in V$.
(3) W satisfies the Hodge-Riemann bilinear relations.

Then $L^i : V^{-i} \to V^i$ is injective. If V is finite dimensional with symmetric Betti numbers, then hard Lefschetz holds for V.

Apply this Lemma to $\phi = r^*$ and the form coming from Poincare duality. Weak Lefschetz gives us statement (1) and induction gives us (3). The only thing that needs checking is (2).

$$(r^*a, r^*b)_{X_H} = (a, r_*r^*b)_X = (a, Lb)_X.$$

Hence, we see that Hodge-Riemann for X_H implies hard Lefschetz for X via the weak Lefschetz theorem. However, in the setting of Soergel Bimodules, we do not have an obvious replacement for weak Lefschetz. So, how do we fix this?

There are two things we need replacements for. First, we need to construct a degree 1 map from $\overline{B_x B_s}$ into a space that satisfies Hodge-Riemann. Second, we need to factor the Lefschetz operator that makes the above degree 1 map satisfy property (2) in the lemma. Let us talk first about how to factor the Lefschetz operator.

6.1. Factoring L_ζ.

Fix an expression $\underline{x} = s_1 \cdots s_m$ and for each i, let $\underline{x_{\hat{i}}}$. Let $\mu : B_s \to R$ be the multiplication map on $B_s = R \otimes_{R_s} R(1)$. Let us recall some morphisms introduced by Seth in the previous talk:

$$\mathrm{Br}_i : BS(\underline{x}) \to BS(\underline{x})(2) : b_1 \cdots b_i \cdots b_m \mapsto b_1 \cdots (b_i c_{s_i}) \cdots b_m$$

$$\phi_i : BS(\underline{x}) \to BS(\underline{x_{\hat{i}}})(1) : b_1 \cdots b_m \mapsto b_1 \cdots \mu(b_i) \cdots b_m$$

$$\chi_i : BS(\underline{x_{\hat{i}}}) \to BS(\underline{x})(1) : b_1 \cdots b_{i-1} b_{i+1} \cdots b_m \mapsto b_1 \cdots b_{i-1} c_{s_i} b_{i+1} \cdots b_m.$$

So, $\mathrm{Br}_i = \chi_i \circ \phi_i$. Importantly, we had the following formula explained by Seth.

Proposition 6.2. As endomorphisms of $BS(\underline{x})$, we have:

$$\rho \cdot (-) = \sum_{i=1}^{m} (s_{i-1} \cdots s_1 \rho)(\alpha_{s_i}^\vee) \mathrm{Br}_i + (-) \cdot x^{-1} \rho.$$

Passing to $\overline{BS(\underline{x})}$, the second half of the formula dies, and we get the ordinary Lefschetz operator as a sum $\lambda_i \mathrm{Br}_i$, with λ_i positive if $\underline{x}$ is reduced.

Let us look at Br_i more closely. Use Tr_i and $\langle -, - \rangle_i$ to denote the trace and intersection form on $BS(\underline{x_{\hat{i}}})$. We have the following lemma.

Lemma 6.3. For $bb' \in BS(\underline{x})$, we have $\langle b, \mathrm{Br}_i b' \rangle = \langle \phi_i b, \phi_i b' \rangle_i$.

Proof. We may assume $b = b_1 \cdots b_m$ and $b' = b'_1 \cdots b'_m$. First note that

$$\langle b, \mathrm{Br}_i b' \rangle = \mathrm{Tr}[(b_1 b'_1) \cdots (b_i b'_i c_{s_i}) \cdots (b_m b_{m'})] = \mathrm{Tr}[(b_1 b_1)' \cdots \mu(b_i)\mu(b'_i) c_{s_i} \cdots (b_m b'_m)].$$

To see this, note that the trace only cares about the c_{s_i} coefficient of $b_i b'_i c_{s_i}$ when written in the c_{id}, c_{s_i} right R-basis. Hence, if

$$b_i = c_{\mathrm{id}} r_1 + c_{s_i} r_2, b'_i = c_{\mathrm{id}} r'_1 + c_{s_i} r'_2$$

then, as $c_{s_i}^2 = c_{s_i} \alpha_{s_i}$,

$$b_i b'_i = c_{\mathrm{id}} r_1 r'_1 + c_{s_i}(r_2 r'_1 + r_1 r'_2) + c_{s_i}^2 (r_2 r'_2) = c_{\mathrm{id}} r_1 r'_1 + c_{s_i}(r_2 r'_1 + r_1 r'_2 + r_2 r'_2 \alpha_{s_i}).$$

Hence,

$$\mathrm{Tr}_{s_i}(b_i b'_i c_{s_i}) = (r_1 r'_1 + (r_2 r'_1 + r_1 r'_2)\alpha_{s_i} + r_2 r'_2 \alpha_{s_i}^2)$$

where Tr_{s_i} means the c_{s_i} coefficient. This is exactly $\mu(b_i)\mu(b'_i)$. From here the proof is evident, because

$$\mathrm{Tr}[(b_1 b_1)' \cdots \mu(b_i)\mu(b'_i) c_{s_i} \cdots (b_m b'_m)] = \mathrm{Tr}_i[(b_1 b'_1) \cdots \mu(b_i)\mu(b'_i) \cdots (b_m b'_m)]$$

which is what the lemma states.

□

Assuming $\underline{x}$ is reduced, we rescale the forms on each $BS(\underline{x_{\hat{i}}})$ by multiplying by $(s_{i-1} \cdots s_1 \rho)(\alpha_{s_i}^\vee)$. Then, combining the above proposition and lemma, we get

Lemma 6.4. Consider the induced map

$$\phi : \overline{BS(\underline{x})} \to \overline{BS(\underline{x}_{\hat{i}})}(1)$$

where $\phi = \sum \phi_i$. Then, for all $b, b' \in \overline{BS(\underline{x})}$, we have

$$\langle b, \rho b' \rangle = \langle \phi b, \phi b' \rangle' \in \mathbb{R}$$

with the latter form depicting the sum of the normalized forms on $\oplus_i \overline{BS(\underline{x}_{\hat{i}})}$.

Hence, we have successfully factored the Lefschetz operator L_0 at least. To show that the image of ϕ has suitable Hodge-Riemann properties, we need to compare ϕ with some maps obtained from Rouquier complexes.

6.2. **Rouquier Complexes.** Let us now elaborate on the brief definition of Rouquier complexes that we gave in the introduction and use them to describe the proof of hard Lefschetz. This section is meant to be an outline and many details are missing. The details will be provided in the next talk.

Definition 6.5. For $s \in S$, define the Rouquier complex

$$F_s := (0 \to B_s \to R(1) \to 0)$$

where B_s sits in homological degree 0 and the only nontrivial map is $\mu : B_s \to R(1)$, (which is ϕ for $BS(s) = B_s$). For $x \in W$, pick a reduced expression $s_1 \cdots s_m$. Then,

$$F_x := R_{s_1} \otimes_R \cdots \otimes_R R_{s_m}.$$

The proof of $hL(x, s)_\zeta$ for $\zeta \geq 0$ now breaks into a few cases. The case of $xs < x$ is dealt with in a fairly elementary manner and we won't talk about that today.

For $xs > x$ and $\zeta > 0$, we do the following. As will be shown in the next talk, if $xs > x$, the two nonzero terms of $F_x F_s$ that are lowest in homological degree have the form

$$B_x B_s \to ({}^1F_x) B_s \oplus B_x(1)$$

where the superscript on the top left of F_x denotes the term in homological degree 1. Here, the map is ϕ, so we can write $\phi = (d_1, d_2)$ where d_1 is the differential to the first summand and d_2 to the second summand. In the next talk, we will check that d_1 commutes with L_ζ, where L_ζ is defined in a very analogous manner for ${}^1F_x B_s$ and that

$$d_2(L_\zeta b) = \rho \cdot d_2(b) + d_2(b) \cdot \zeta\rho.$$

Hence, if L is the operator on $\overline{({}^1F_x)B_s} \oplus \overline{B_x}(1)$ defined by L_ζ on the first summand and $\rho \cdot (-)$ on the second summand, then

$$\overline{\phi}(L_\zeta b) = L\overline{\phi}(b)$$

with $\overline{\phi}$ the obvious induced map. To finish the proof, there are three technical steps left:

(1) Check $\overline{\phi}$ is injective in degrees $\leq l(x)$.
(2) Define a suitable form $\langle -, - \rangle^\gamma_{\mathbb{R}}$ on $\overline{({}^1F_x)B_S} \oplus \overline{B_x}(1)$ and check that

$$\langle b, L_\zeta b' \rangle_{\overline{B_x B_s}} = \langle \overline{\phi}(b), \overline{\phi}(b') \rangle^\gamma_{\mathbb{R}}.$$

(3) Check that $\overline{{}^1FxB_S} \oplus \overline{B_x}(1)$ satisfies Hodge-Riemann with respect to L and $\langle -, - \rangle^\gamma_{\mathbb{R}}$.

The constructions and verification needed for these 3 properties will be done in the next talk. With these three properties, we prove hard Lefschetz for $xs > x$ and $\zeta > 0$. The $xs > x$, $\zeta = 0$ is more difficult but follows in a similar manner to the above argument. The key extra step is a further decomposition of ${}^1F_x$ into two summands. This will also be shown in the next talk.

References

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