LECTURE 3: TENSORING WITH FINITE DIMENSIONAL MODULES IN CATEGORY $\mathcal{O}$

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Abstract. These are notes for a seminar talk given at the MIT-Northeastern Category $\mathcal{O}$ and Soergel Bimodule seminar (Autumn 2017).

Contents

1. Goals

The purpose of this document is to discuss the operation of tensoring with a finite dimensional module in the category $\mathcal{O}$ associated to a finite-dimensional semisimple Lie algebra $\mathfrak{g}$. In particular we will investigate the behaviour of certain functors between infinitesimal blocks of the category that arise from these tensor products, and relations to projective objects in an infinitesimal block $\mathcal{O}_\chi$.

2. Review of Notation

We will use the following notation:

- A finite-dimensional semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$.
- A triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. In particular $\mathfrak{h}$ is (a choice of) a Cartan subalgebra, and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ is the (positive) Borel subalgebra.
- Simple roots α_i, with roots R and positive roots R_+ and negative roots R_-. We write $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$.
- We write Λ for the weight lattice.
- We write W for the Weyl group associated to $\mathfrak{g}$, which acts on $\mathfrak{h}^*$ in the usual way, and also via the dotted action: $w \cdot v = w(v+\rho) - \rho$ ($w \in W$). We write $\mathrm{Stab}_W(\lambda)$ for the stabiliser of λ with respect to the usual action. W is generated by the simple reflections s_i which correspond to the simple roots α_i.
- The category of finitely generated $\mathfrak{g}$-modules that are $\mathfrak{h}$-semisimple and locally nilpotent for the action of $\mathfrak{n}_+$ is denoted $\mathcal{O}$. It has a duality operation $M \mapsto M^\vee$.
- Verma modules of highest weight λ, $\Delta(\lambda)$. These have unique simple quotient $L(\lambda)$. Both of these belong to $\mathcal{O}$. We also have dual Verma modules $\nabla(\lambda) = \Delta(\lambda)^\vee$.
- There is a decomposition of categories $\mathcal{O} = \sum_{\lambda \in \mathfrak{h}^*/W} \mathcal{O}_\lambda$. The quotient is by the dotted action of W. The summand $\mathcal{O}_\lambda$ has simple objects indexed by $L(w\cdot\lambda)$ for $w \in W$. There are $|W/\mathrm{Stab}_W(\lambda+\rho)|$ of them.

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3. Tensor Product with Finite-Dimensional Modules

Henceforth, V is a finite dimensional $\mathfrak{g}$-module.

Proposition 3.1. *Tensoring with V defines a functor $\mathcal{O} \to \mathcal{O}$. This functor is exact, and has left and right adjoints both equal to tensoring with V^*.*

Proof. It was shown in the previous lecture that tensoring with a finite dimensional module takes modules in category $\mathcal{O}$ to modules in category $\mathcal{O}$.

To prove adjointness properties, we use the property that $\mathrm{Hom}_{\mathbb{C}}(V, M) = V^* \otimes M$ (which requires V to be finite dimensional). We argue that:

$$\mathrm{Hom}_{\mathbb{C}}(M \otimes_{\mathbb{C}} V, N) = \mathrm{Hom}_{\mathbb{C}}(M, \mathrm{Hom}_{\mathbb{C}}(V, N)) = \mathrm{Hom}_{\mathbb{C}}(M, N \otimes V^*)$$
$$\mathrm{Hom}_{\mathbb{C}}(N, M \otimes_{\mathbb{C}} V) = \mathrm{Hom}_{\mathbb{C}}(N, \mathrm{Hom}_{\mathbb{C}}(V^*, M)) = \mathrm{Hom}_{\mathbb{C}}(M \otimes V^*, N)$$

Certainly, the tensor-hom adjunction guarantees this equality for the underlying vector spaces. It remains to check that this map respects $\mathfrak{g}$-module structure. This is well known and easy to check. $\square$

Proposition 3.2. *Let ν_i ($i \in \{1, 2, \cdots, \dim(V)\}$) be the weights of V with multiplicity, ordered such that if $\nu_i \geq \nu_j$ in the usual partial order on $\mathfrak{h}^*$, then $i \leqslant j$. Then $M = \Delta(\lambda) \otimes V$ has a filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{\dim(V)} = 0$ such that $M_i/M_{i+1} \cong \Delta(\lambda + \nu_i)$.*

Proof. We use the tensor-hom adjunction, and the fact that $\Delta(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ (where $\mathbb{C}_\lambda$ is the one dimensional representation of $\mathfrak{b}$ associated to the weight λ). Let N be a $\mathfrak{g}$-module.

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{g}}(V \otimes \Delta(\lambda), N) &= \mathrm{Hom}_{\mathfrak{g}}(V \otimes (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda), N) \\
&= \mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda, V^* \otimes N) \\
&= \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, V^* \otimes N) \\
&= \mathrm{Hom}_{\mathfrak{b}}(V \otimes \mathbb{C}_\lambda, N) \\
&= \mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (V \otimes \mathbb{C}_\lambda), N)
\end{aligned}$$

Now we observe that $V \otimes \mathbb{C}_\lambda$ is filtered as a $\mathfrak{b}$-module by $W_j = \mathrm{Span}(\{v_1, v_2, \cdots, v_j\})$ ($j = 1, 2, \cdots, \dim(V)$). Note that W_j/W_{j+1} is one dimensional of weight ν_j. Since $U(\mathfrak{g})$ is free over $U(\mathfrak{b})$ (by the PBW theorem), tensoring up to $U(\mathfrak{g})$ is an exact functor, so the filtration of $V \otimes \mathbb{C}_\lambda$ passes to a filtration of $V \otimes \Delta(\lambda)$ with subquotients $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda+\nu_i} = \Delta(\lambda + \nu_i)$. This proves the proposition. $\square$

Remark 3.3. *Recall that $\mathcal{O} = \oplus_{\chi \in \mathfrak{h}^*/W} \mathcal{O}_\chi$ (decomposition into infinitesimal blocks). It was shown in the previous talk that $\mathcal{O}_\chi$ has enough projectives, and that projectives admit a filtration by Verma modules. This makes $\mathcal{O}_\chi$ into a highest-weight category, where the standard objects are Verma modules.*

4. Projective Functors, Translation Functors, and Equivalences of Categories

Definition 4.1. *Let $\mu \in \mathfrak{h}^*/W$ and $pr_\mu : \mathcal{O} \to \mathcal{O}_\mu$ be the functor projecting onto the summand $\mathcal{O}_\mu$. A projective functor is a functor of the form $pr_\mu(V \otimes \iota_\lambda(-)) : \mathcal{O}_\lambda \to \mathcal{O}_\mu$, where ι_λ is the inclusion $\mathcal{O}_\lambda \hookrightarrow \mathcal{O}$.*

Remark 4.2. *Observe that $pr_\mu(V \otimes \iota_\lambda(-))$ is biadjoint to $pr_\lambda(V^* \otimes \iota_\mu(-))$. This is immediate from Proposition 3.1.*

These functors will allows us to prove equivalences of infinitesimal blocks associated to dominant integral weights, and describe the structure of the categories in the non-dominant integral case.

Proposition 4.3. *If $M \mapsto M^\vee$ is the duality operation defined in the previous lecture, then we have that $pr_\mu(V \otimes \iota(M^\vee)) = pr_\mu(V \otimes \iota_\lambda(M))^\vee$.*

Proof. Firstly, note that duality preserves infinitesimal blocks. To see this, it is enough to know that simple objects are self dual under duality (shown last lecture), then the infinitesimal blocks are generated by successive extensions of the simple objects in the infinitesimal block. Then, the claim reduces to showing that $V \otimes M^\vee = (V \otimes M)^\vee$. It is easy to see that $(V \otimes M)^\vee = V^\vee \otimes M^\vee$, so it is enough to show that finite dimensional representations are self-dual. The category of finite-dimensional representations of $\mathfrak{g}$ is semisimple, so it is enough to see that V and $V^\vee$ have the same character, which is trivial. $\square$

The following statement is immediate from the Proposition 3.2 and the characterisation of the infinitesimal blocks $\mathcal{O}_\mu$.

Corollary 4.4. *The object $pr_\mu(V \otimes \Delta(w \cdot \lambda))$ admits a filtration by $\Delta(w \cdot \lambda + \nu_i)$, where $w \cdot \lambda + \nu_i \in W \cdot \mu$ (each with multiplicity 1).*

Remark 4.5. *In view of Proposition 4.3, Corollary 4.4 holds with Verma modules replaced by dual Verma modules.*

Proposition 4.6. *Let $\lambda, \mu \in P$ be such that $\lambda, \mu + \rho, \lambda - \mu$ are all dominant, and $w \in W$. Let V be the finite dimensional simple $\mathfrak{g}$-module of highest weight $\lambda - \mu$. We have that $pr_\mu(V^* \otimes \Delta(w \cdot \lambda)) = \Delta(w \cdot \mu)$, and also that $pr_\lambda(V \otimes \Delta(w \cdot \mu))$ is filtered by modules of the form $\Delta(w' \cdot \lambda)$, where $w' \in wStab_W(\mu + \rho)$ (each with multiplicity 1).*

Proof. We consider $pr_\mu(V^* \otimes \Delta(w \cdot \lambda))$, using the Proposition 3.2 to find all candidates for terms arising in a filtration by Verma modules. The weights of V^* are precisely the negatives of those in V. For each i, obtain a Verma module of highest weight $w \cdot \lambda - \nu_i$ provided that $w \cdot \lambda - \nu_i \in W \cdot \mu$ (and none otherwise). We must find w and ν_i such that $w \cdot \lambda - \nu_i = w' \cdot \mu$ (where $w' \in W$). We write the dotted action in terms of the usual one, and act by w^{-1} to get $\lambda + \rho - w^{-1}\nu_i = w^{-1}w'(\mu + \rho)$. Since $\lambda - \mu$ is dominant, $\lambda - \mu \geq w^{-1}\nu_i$ (recall that ν_i is a weight of V, which has highest weight $\lambda - \mu$), and equality holds if and only if $w^{-1}\nu_i$ is a the highest weight $\lambda - \mu$. Similarly, $\mu + \rho \geq w^{-1}w'(\mu + \rho)$, and equality holds if and only if $w^{-1}w' \in \mathrm{Stab}_W(\mu + \rho)$. We use these inequalities to get:

$$\lambda + \rho - w^{-1}\nu_i \geq \lambda + \rho - (\lambda - \mu) = \mu + \rho \geq w^{-1}w'(\mu + \rho)$$

The condition $\lambda + \rho - w^{-1}\nu_i = w^{-1}w'(\mu + \rho)$ holds if and only if both inequalities were equalities. This gives $\nu_i = w(\lambda - \mu)$ and $w' \in w\mathrm{Stab}_W(\mu + \rho)$. Note that the highest weight space of V is one-dimensional, and therefore the weight space of weight $w(\lambda - \mu)$ is also one-dimensional as it is in the same Weyl group orbit as the highest weight space; this means that there is exactly one choice of i satisfying the condition. We obtain a one-term filtration by $\Delta(w \cdot \lambda - w(\lambda - \mu)) = \Delta(w \cdot \mu)$. Hence, $pr_\mu(V^* \otimes \Delta(w \cdot \lambda)) = \Delta(w \cdot \mu)$.

To see the filtration statement, we perform an analogous analysis, but with $-\nu_i$ replaced by ν_i. We need to consider $w \cdot \mu + \nu_i = w' \cdot \lambda$, which becomes ${w'}^{-1}w(\mu + \rho) + {w'}^{-1}\nu_i = \lambda + \rho$. The inequalities ${w'}^{-1}w(\mu + \rho) \leq \mu + \rho$ and ${w'}^{-1}\nu_i \leq \lambda - \mu$ give:

$${w'}^{-1}w(\mu + \rho) + {w'}^{-1}\nu_i \leq \mu + \rho + {w'}^{-1}\nu_i \leq \mu + \rho + (\lambda - \mu) = \lambda + \rho$$

We have equality only when ${w'}^{-1}w \in \mathrm{Stab}_W(\mu + \rho)$ and $\nu_i = w'(\lambda - \mu)$. The first condition is equivalent to $w^{-1}w' \in \mathrm{Stab}_W(\mu + \rho)$, so $w' \in w\mathrm{Stab}_W(\mu + \rho)$. The second condition determines the terms in the filtration:

$$\Delta(w \cdot \mu + \nu_i) = \Delta(w \cdot \mu + w'(\lambda - \mu)) = \Delta(w' \cdot \mu + w'(\lambda - \mu)) = \Delta(w' \cdot \lambda)$$

Conversely, each $w' \in w\mathrm{Stab}_W(\mu + \rho)$ gives rise to a term corresponding to $\nu_i = w'(\lambda - \nu)$. This proves the claim about multiplicity. $\square$

Theorem 4.7. *Suppose $\lambda_1, \lambda_2 \in P$ are dominant integral weights. There is an equivalence $\mathcal{O}_{\lambda_1} \xrightarrow{\sim} \mathcal{O}_{\lambda_2}$ that takes $\Delta(w \cdot \lambda_1)$ to $\Delta(w \cdot \lambda_2)$. Similarly, it takes $\nabla(w \cdot \lambda_1)$ to $\nabla(w \cdot \lambda_2)$ and $L(w \cdot \lambda_1)$ to $L(w \cdot \lambda_2)$.*

Proof. Firstly, we reduce to the case $\lambda_1 - \lambda_2$ is dominant. This implies the general case, for it would imply that $\mathcal{O}_{\lambda_1}$ and $\mathcal{O}_{\lambda_2}$ are each equivalent to $\mathcal{O}_{\lambda_1 + \lambda_2}$. Now we apply proposition 4.6 with $\lambda = \lambda_1$ and $\mu = \lambda_2$. If λ is the finite dimensional irreducible of highest weight $\lambda_1 - \lambda_2$, we obtain functors $\varphi = pr_{\lambda_1}(V \otimes \iota_{\lambda_2}(-)) : \mathcal{O}_{\lambda_2} \to \mathcal{O}_{\lambda_1}$ and $\varphi^* = pr_{\lambda_2}(V^* \otimes \iota_{\lambda_1}(-)) : \mathcal{O}_{\lambda_1} \to \mathcal{O}_{\lambda_2}$. Recall that by Remark 4.2, φ and φ^* are mutually adjoint (on both sides).

Note that proposition 4.6 implies that $\varphi(\Delta(w \cdot \lambda_2)) = \Delta(w \cdot \lambda_1)$. In fact, we also have $\varphi^*(\Delta(w \cdot \lambda_1)) = \Delta(w \cdot \lambda_2)$. This is because $\lambda_2 + \rho$ is strictly dominant, so $\mathrm{Stab}_W(\lambda_2 + \rho) = \{1\}$; thus $w' = w$. In the notation of the proof of the proposition, this means that $v_i = \lambda - \mu$ (the highest weight, of which there is only one). Thus there is only one term in the filtration, $\Delta(w' \cdot \lambda_2) = \Delta(w \cdot \lambda_2)$. Hence, the module is isomorphic to $\Delta(w \cdot \lambda_2)$.

Now, let $\mathcal{G} = \varphi^* \circ \varphi$, an endofunctor of $\mathcal{O}_{\lambda_1}$. Since φ and φ^* are adjoint, there is a adjunction unit: a natural transformation $\eta : \mathrm{Id} \to \mathcal{G}$. We obtain a map $\eta_{\Delta(w\cdot\lambda_1)} : \Delta(w\cdot\lambda_1) \to \varphi^* \circ \varphi(\Delta(w\cdot\lambda_1))$. This map is nonzero for the following reason. Consider the following composition, which is required to be Id_φ by adjunction properties:

$$\varphi \xrightarrow{\varphi\circ\eta} \varphi\circ\varphi^*\circ\varphi \xrightarrow{\varepsilon\circ\varphi} \varphi$$

Here ε is the adjunction counit. The first arrow is $\varphi\circ\eta$, which must therefore be a monomorphism for each object M (it has a right inverse). In particular, $\varphi(\eta_{\Delta(w\cdot\lambda_1)})$ is injective, and hence nonzero. This means that $\eta_{\Delta(w\cdot\lambda_1)}$ must be nonzero. Since $\mathrm{End}_{\mathfrak{g}}(\Delta(\nu)) = \mathbb{C}$ (for any $\nu \in \mathfrak{h}^*$), it follows that a nonzero map is an isomorphism; in particular $\varphi^*\circ\varphi$ defines an isomorphism on Verma modules. We now show that this implies we have an equivalence of categories.

The functor $\mathcal{G}$ must be an isomorphism on any object that admits a standard filtration. This can be proved by induction on the length of the filtration (the base case being length 1), by considering a short exact sequence coming from the filtration. Suppose we have $0 \to F \to M \to \Delta(\nu) \to 0$ for some weight ν and module F with standard filtration. We obtain the following diagram by applying $\mathcal{G}$:

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r] \arrow[d, "\eta_F"] & M \arrow[r] \arrow[d, "\eta_M"] & \Delta(\nu) \arrow[r] \arrow[d, "\eta_{\Delta(\nu)}"] & 0 \\
0 \arrow[r] & \mathcal{G}(F) \arrow[r] & \mathcal{G}(M) \arrow[r] & \mathcal{G}(\Delta(\nu)) \arrow[r] & 0
\end{tikzcd}$$

The diagram commutes because η is a natural transformation. By induction we know that $\mathcal{G}$ is an isomorphism on the first and last terms, and it is exact. Therefore the five-lemma allows us to conclude that the middle map is an isomorphism; this completes the induction. Since projective objects admit standard filtrations (as $\mathcal{O}_\lambda$ is a highest-weight category), $\mathcal{G}$ is an isomorphism on projective objects.

It now follows that $\mathcal{G}$ is an isomorphism on all objects, by applying a similar argument to a truncated projective resolution of an arbitrary object (recall that $\mathcal{O}_\lambda$ has enough projectives), $P_1 \to P_0 \to M \to 0$. This means that η defines a natural isomorphism between $\varphi^*\circ\varphi$ and the identity functor. The same proof works in the opposite direction (i.e. for $\varphi\circ\varphi^*$), as φ and φ^* are biadjoint. Thus φ and φ^* define an equivalence of categories.

To check the behaviour on dual Verma modules, we simply use Proposition 4.3. The statement about behaviour on simples follows by taking maximal quotients of Verma modules (which are unique). $\square$

Even in the case where λ_2 is not dominant, we can say some things.

Theorem 4.8. *Take λ_1, λ_2 to satisfy the conditions of Proposition 4.6 with $\lambda = \lambda_1$ and $\mu = \lambda_2$. We have that $pr_\mu(V^*\otimes L(w\cdot\lambda_1))$ is zero unless w is equal to the (unique) longest element in the coset $w\mathrm{Stab}_W(\lambda_2+\rho)$, in which case it is $L(w\cdot\lambda_2)$.*

Proof. We continue to write $\varphi(-) = pr_\mu(V\otimes\iota_\lambda(-))$. We choose a coset $w\mathrm{Stab}_W(\lambda_2+\rho)$. Let u be the longest element of this coset, and assume $w \neq u$. Then $\varphi(\Delta(w\cdot\lambda_1)) = \Delta(u\cdot\lambda_2) = \varphi(\Delta(u\cdot\lambda_1))$. We also have (by the results of the first lecture) that $\Delta(u\cdot\lambda_1) \hookrightarrow \Delta(w\cdot\lambda_1)$. Let C be the cokernel of this inclusion, so that we have a short exact sequence:

$$0 \to \Delta(u\cdot\lambda_1) \to \Delta(w\cdot\lambda_1) \to C \to 0$$

Applying the exact functor φ gives:

$$0 \to \Delta(u\cdot\lambda_2) \to \Delta(w\cdot\lambda_2) \to \varphi(C) \to 0$$

But the first two objects are the same ($u\cdot\lambda_2 = w\cdot\lambda_2$), and an injective endomorphism of a Verma module is an isomorphism. This means that the first map is an isomorphism (it is injective by exactness). This in turn forces $\varphi(C) = 0$. Hence, φ annihilates any subquotient of C, including $L(w\cdot\lambda_2)$.

On the other hand, every $\varphi(\Delta(w\cdot\lambda_1)) = \Delta(w\cdot\lambda_2)$ so that every Verma module is in the image of φ. The classes of Verma modules span $K_0(\mathcal{O}_{\lambda_2})$, so the exact functor φ induces a surjection of Grothendieck groups $[\varphi] : K_0(\mathcal{O}_{\lambda_1}) \twoheadrightarrow K_0(\mathcal{O}_{\lambda_2})$. By rank considerations, it follows that $\varphi(L(u\cdot\lambda_1))$ must be nonzero when u is the longest element in its coset; $K_0(\mathcal{O}_{\lambda_2})$ has rank equal to the number of such simples, and the images of the other simples are zero. It remains to determine $\varphi(L(u\cdot\lambda_1))$.

In view of Remark 4.5, the costandard objects $\nabla(w\cdot\lambda_1)$ behave similarly to $\Delta(w\cdot\lambda_1)$ under tensoring with V. Analogously to $\Delta(u\cdot\lambda_2) \twoheadrightarrow \varphi(L(u\cdot\lambda_1))$, we have $\varphi(L(u\cdot\lambda_1)) \hookrightarrow \nabla(u\cdot\lambda_2)$. Composing these two maps, and using the fact from the previous lecture that $\dim(\mathrm{Hom}_{\mathcal{O}}(\Delta(\lambda),\nabla(\mu))) = \delta_{\lambda,\mu}$, we see that a (nonzero) map must be (a scalar multiple of) projection onto $L(u\cdot\lambda_2)$ followed by inclusion into $\nabla(u\cdot\lambda_2)$. It immediately follows that $\varphi(L(u\cdot\lambda_1)) = L(u\cdot\lambda_2)$. □

Corollary 4.9. *Let $\lambda+\rho$ be dominant, and w be the longest element in the coset $w Stab_W(\lambda+\rho)$. We have the following equality of multiplicities:*

$$[\Delta(w'\cdot\lambda) : L(w\cdot\lambda)] = [\Delta(w'\cdot 0) : L(w\cdot 0)]$$

Proof. This follows by exactness of φ and the characterisation of the action of φ on simple objects in the previous theorem (we take $\lambda_1 = 0$, $\lambda_2 = \lambda$). □

Remark 4.10. *The projection functors satisfy the following transitivity property. Suppose that $\lambda_1,\lambda_2,\lambda_3$ are all dominant weights such that $\lambda_1-\lambda_2$ and $\lambda_2-\lambda_3$ are dominant. Assume further that $Stab_W(\lambda_1+\rho) \subseteq Stab_W(\lambda_2+\rho) \subseteq Stab_W(\lambda_3+\rho)$. For $i,j\in\{1,2,3\}$ with $i<j$, let $\varphi_{i,j} : \mathcal{O}_{\lambda_j} \to \mathcal{O}_{\lambda_i}$ be the projection functor $pr_{\lambda_i}(V_{i,j}\otimes \iota_{\lambda_j}())$, where $V_{i,j}$ is the finite-dimensional irreducible $\mathfrak{g}$-module of highest weight $\lambda_i-\lambda_j$. Then we have $\varphi_{1,2}\circ\varphi_{2,3} \cong \varphi_{1,3}$. This is left as an exercise.*

5. Projections to the Walls and Reflection Functors

In this section we assume that λ_1 and λ_2 satisfy the conditions of Proposition 4.6 with $\lambda=\lambda_1$ and $\mu=\lambda_2$, but we further require that there is a unique i such that $\langle\alpha_i,\lambda_2+\rho\rangle = 0$ (the inner products with other simple roots are positive). This means that $\lambda_2+\rho$ is on the wall of the dominant Weyl chamber associated to α_i.

Proposition 5.1. *Let $\varphi : \mathcal{O}_{\lambda_2} \to \mathcal{O}_{\lambda_1}$ and $\varphi^* : \mathcal{O}_{\lambda_2} \to \mathcal{O}_{\lambda_1}$ be as in Proposition 4.6. Then we have $\varphi\circ\varphi^* : \mathcal{O}_{\lambda_1} \to \mathcal{O}_{\lambda_1}$. If $w\cdot\lambda_1 \nleq ws_i\cdot\lambda_1$, we have the following short exact sequence:*

$$0 \to \Delta(w\cdot\lambda_1) \to \varphi\circ\varphi^*(\Delta(w\cdot\lambda_1)) \to \Delta(ws_i\cdot\lambda_1) \to 0$$

Otherwise we have:

$$0 \to \Delta(ws_i\cdot\lambda_1) \to \varphi\circ\varphi^*(\Delta(w\cdot\lambda_1)) \to \Delta(w\cdot\lambda_1) \to 0$$

Proof. This immediately follows from Proposition 4.6, noting that $\mathrm{Stab}_W(\lambda_2+\rho) = \{1, s_i\}$, and that weight spaces in the same Weyl group orbit have the same dimension (so the $s_i(\lambda_1-\lambda_2)$ weight space is one dimensional). Note that the Verma with lower weight is the quotient, and the Verma with larger weight is the submodule, because extensions can only exist in one direction. □

Definition 5.2. *The functor $\varphi^*\circ\varphi$ is an endofunctor of $\mathcal{O}_0$. We call this $\mathcal{P}_i$ (the subscript i corresponds to the simple root α_i). Note that it is a self-adjoint exact functor.*

Proposition 5.3. *The Grothendieck group of $\mathcal{O}_0$ can be identified with $\mathbb{Z}W$. The functor $\mathcal{P}_i$ induces an endomorphism of $K_0(\mathcal{O}_0)$ which is given by right multiplication by $1+s_i$ in $\mathbb{Z}W$.*

Proof. We may take $\Delta(w\cdot 0)$ to be a $\mathbb{Z}$-basis of $K_0(\mathcal{O}_0)$ (this follows from the highest-weight structure discussed in the previous lecture). This provides the identification with $\mathbb{Z}W$ (as abelian groups). An exact functor respect the relations of the Grothendieck group, and therefore defines an endomorphism of the Grothendieck group (as an abelian group). Proposition 5.1 implies that the image of the endomorphism associated to the functor $\mathcal{P}_i$ evaluated on $\Delta(w\cdot 0)$ is $[\Delta(w\cdot 0)] + [\Delta(ws_i\cdot 0)]$. Under our identification, this becomes $w \mapsto w + ws_i = w(1+s_i)$. □

6. Projective Objects in Category $\mathcal{O}$

Proposition 6.1. *The object $\Delta(0)$ is projective in the category $\mathcal{O}_0$.*

Proof. This was in the preceding lecture (it is a consequence of the fact that 0 is maximal in $W \cdot 0$, so any vector of weight 0 is automatically a singular vector; this implies $\mathrm{Hom}_{\mathcal{O}_0}(\Delta(0), -)$ is an exact functor). □

Definition 6.2. *For $w \in W$, choose a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_l}$. Let $P_{\underline{w}} = \mathcal{P}_{i_l} \circ \cdots \circ \mathcal{P}_{i_2} \circ \mathcal{P}_{i_1}(\Delta(0))$.*

The following theorem was proved in the previous lecture, but we include a slightly different proof, which will be important later.

Theorem 6.3. *The category $\mathcal{O}_0$ has enough projectives. The object $P_{\underline{w}}$ is the projective cover of $L(w \cdot 0)$ plus a direct sum of projective covers of $L(w' \cdot 0)$ with w' smaller than w in the Bruhat order.*

Proof. Firstly, since $\mathcal{P}_i$ is self-adjoint and exact, it takes projectives to projectives. This is because if P is a projective object, $M \mapsto \mathrm{Hom}_{\mathcal{O}_0}(P, \mathcal{P}_i(M))$ is the composition of two exact functors, hence exact. But adjointness makes this equal to $M \mapsto \mathrm{Hom}_{\mathcal{O}_0}(\mathcal{P}_i(P), M)$. The exactness of this functor is precisely the statement that $\mathcal{P}_i(P)$ is projective. It immediately follows that each $P_{\underline{w}}$ is projective.

By iterating Proposition 5.1, we find that $P_{\underline{w}}$ is filtered by subquotients equal to Verma modules. Moreover, the Verma modules that appear can be described using the Proposition in the following way. Inductively, we expand $(1 + s_{i_1})(1 + s_{i_2}) \cdots (1 + s_{i_l}) = \sum_{w \in W} m_w w$, the coefficients m_w are precisely the multiplicities of the Verma modules $\Delta(w \cdot 0)$ in this filtration. We observe that we have one instance of $\Delta(w \cdot 0)$ (which is in fact a quotient of $P_{\underline{w}}$, as its highest weight is the lowest among modules appearing in the filtration). Note also that all other terms correspond to elements of W that can be written by removing simple reflections from a reduced expression for w, i.e. lower in the Bruhat order. If P is a projective module, $\dim(\mathrm{Hom}_{\mathcal{O}}(P, L(w \cdot 0)))$ is the multiplicity of the projective cover of $L(w \cdot 0)$ in P. We use this as follows.

Using the fact that in the Grothendieck group, $[P(\lambda)]$ equals $[\Delta(\lambda)]$ plus higher order terms, the only possible projective covers arising in $P_{\underline{w}}$ are $P(\mu)$, where $\mu = w' \cdot 0$, with $w \succeq w'$. Since $\Delta(w \cdot 0)$ occurs exactly once in the filtration, we have exactly one summand isomorphic to $P(w \cdot 0)$. □