

# Equivariant Cohomology, Localization and Gromov-Witten Invariants

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## 1. Part 1

Our main references are [3], [1], and [5].

**1.1. Equivariant Cohomology.** Equivariant cohomology was introduced to help explore the geometry of quotients  $X/G$ , where  $X$  is a space equipped with a  $G$  action. Here we are thinking either  $X$  could be a smooth manifold and  $G$  a compact Lie group, or  $X$  could be projective algebraic and  $G$  could be a reductive algebraic group.

The main difficulty in studying the quotient  $X/G$  comes from the possibility that  $G$  acts on  $X$  in a non-free way. Borel's idea to overcome this was to modify  $X$  by enlarging it with a contractible, free  $G$ -space.

DEFINITION 1.1. *We say that a space  $E$  is a universal  $G$  space, if*

- $E$  carries a free  $G$  action.
- $E$  is contractible

It is not hard to show that  $E$  is unique up to homotopy equivalence. Thus we pick a particular universal  $G$ -space, and denote it  $EG$ .

We then consider the free  $G$ -space  $X \times EG$  as a replacement for  $X$ . The quotient space  $X_G := (X \times EG)/G$  is now a nice object (e.g. smooth manifold)

DEFINITION 1.2. *The equivariant cohomology of  $X$ , denoted  $H_G(X)$ , is defined to be the ordinary cohomology of  $X_G$ . i.e.*

$$H_G^*(X) = H^*(X_G)$$

The equivariant cohomology inherits many natural structures since it is defined as the ordinary cohomology of a space associated to  $X$ .

- Since  $EG$  carries a free  $G$ -action, it has the structure of a principal  $G$ -bundle over the quotient space  $BG := EG/G$ . The space  $BG$  is known to be a classifying space for principal  $G$ -bundles, i.e.  $\text{Bun}_G(Y) = [Y, BG]$ .
- If  $G$  acts freely on  $X$ , then  $H_G(X) = H(X \times_G EG) = H(X/G \times EG) = H(X/G)$ . So we recover the cohomology of the quotient if the action is free.
- if the  $G$  action on  $X$  is trivial, then  $H_G(X) = H(X \times EG/B) = H(X) \otimes H_G(\{\text{pt}\})$ , which is a free module over  $H_G(\text{pt})$ .
- The projection maps from  $X \times EG$  onto the two factors induce maps  $\sigma : X_G \rightarrow X/G$ , and  $\pi : X_G \rightarrow BG$ . These spaces fit into the 'mixing diagram' of Borel and Cartan

$$\begin{array}{ccccc} X & \xleftarrow{pr_1} & X \times EG & \xrightarrow{pr_2} & EG \\ \downarrow & & \downarrow & & \downarrow \\ X/G & \xleftarrow{\sigma} & X_G & \xrightarrow{\pi} & BG \end{array}$$

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Using these maps, we can relate the cohomology of the quotient  $X/G$  to the equivariant cohomology via  $\sigma^* : H(X/G) \rightarrow H_G(X)$ . The map  $\pi : X_G \rightarrow BG$  defines a  $H_G(pt) = H(BG)$  module structure on  $H_G(X)$  via  $\pi^*$ .

- The map  $\pi$  is a fibration with fiber  $X$ . For a chosen basepoint  $p \in BG$ , the inclusion of the fiber at  $p$ ,  $\iota : X \rightarrow X_G$  induces a map  $\iota^* : H_G(X) \rightarrow H(X)$ .

$$\begin{array}{ccc} X & \xrightarrow{\iota} & X_G \\ \downarrow & & \downarrow \pi \\ p & \longrightarrow & BG \end{array}$$

- For a subgroup  $K \subset G$ , We have  $H_G(G/K) = H(G/K \times_G EG) = H((G/K)/G \times BK) = H_K(pt)$ .
- If  $V$  is a vector bundle over  $X$ , such that there exists a lift of the the  $G$  action to one of  $V$ , i.e.  $V$  is an equivariant vector bundle, then we can extend  $V$  to  $V_G = (V \times EG)/G$ , which is a vector bundle over  $X_G$ . With this construction, we can define the equivariant characteristic classes of  $V$  to be the characteristic classes of  $V_G$ . i.e for the chern classes  $c_i$ , we define  $c_i^G(V) := c_i(V_G) \in H^{2i}(X_G) = H_G^{2i}(X)$ .

We will mainly be interested in the case where  $G = T = (\mathbb{C}^*)^n$ , an algebraic torus. Consider  $\mathbb{C}^\infty - \{0\}$ , this space is contractible and possesses a free  $\mathbb{C}^*$  action, thus  $E\mathbb{C}^* = \mathbb{C}^\infty - \{0\}$  and  $B\mathbb{C}^* = \mathbb{P}^\infty$ . We often write this line bundle as  $L \rightarrow \mathbb{P}^\infty$ . We can easily see that  $B(\mathbb{C}^*)^n = (\mathbb{P}^\infty)^{\times n}$ , and  $E(\mathbb{C}^*)^n = L_1 \oplus \dots \oplus L_n$ , where  $L_i$  is the pullback of this canonical bundle on the  $i$ th component. We have  $H_T(pt) = H(BT) = \mathbb{Q}[\lambda_1, \dots, \lambda_n]$ , where  $\lambda_i = -c_1(L_i) \in H^2(BT) \cong \text{Hom}(T, \mathbb{C}^*)$ .

It is not hard to show that for a maximal torus  $T \subset G$ , we have  $H_G(pt) = H_T(pt)^W$ , where the Weyl group acts by the regular representation on the  $\lambda_i$ .

**1.2. Example:**  $H_T(\mathbb{P}^n)$ . Here we will compute the  $T$ -equivariant cohomology of  $\mathbb{P}^n$ , where  $T = (\mathbb{C}^*)^{n+1}$  acting on  $[z_0 : \dots : z_n] \in \mathbb{P}^n$  via

$$(t_0, \dots, t_n) \cdot [z_0 : \dots : z_n] = [t_0^{-1} z_0 : \dots : t_n^{-1} z_n]$$

We compute  $\mathbb{P}_T^n = \mathbb{P}^n \times_T ET = \mathbb{P}^n \times_T (L_0 \oplus \dots \oplus L_n) = \mathbb{P}(L_0 \oplus \dots \oplus L_n)$ .

$$\begin{array}{ccc} \mathbb{P}^n & \longrightarrow & \mathbb{P}(L_0 \oplus \dots \oplus L_n) = \mathbb{P}_T^n \\ & & \downarrow \\ & & BT \end{array}$$

Thus we see that  $\mathbb{P}_T^n$  is a bundle over  $BT$  with fiber  $\mathbb{P}^n$ , as expected. We have the tautological bundle  $U = \mathcal{O}_{\mathbb{P}_T^n}(-1)$ , with chern class  $\tilde{h} = c_1(U) \in H_T^2(\mathbb{P}^n)$ . This fits into an exact sequence of bundles over  $\mathbb{P}_T^n$

$$U \rightarrow L_0 \oplus \dots \oplus L_n \rightarrow Q$$

We thus have the relation

$$c_{n+1}(U^* \otimes (L_0 \oplus \dots \oplus L_n)) = 0$$

or

$$\prod_{i=0}^n (\tilde{h} + \lambda_i) = 0$$

And we find

$$H_T(\mathbb{P}^n) = \mathbb{Q}[\tilde{h}, \lambda_0, \dots, \lambda_n] / \prod_{i=0}^n (\tilde{h} + \lambda_i)$$

notice that this is a non-free module over  $H_T(pt) = \mathbb{Q}[\lambda_0, \dots, \lambda_n]$ . If we restrict to the fiber, we have  $\iota^* \tilde{h} = h, \iota^* \lambda_i = 0$ , and so

$$\iota^* H_T(\mathbb{P}^n) = \mathbb{Q}[h]/h^{n+1} = H(\mathbb{P}^n)$$

in particular,  $\iota^*$  is a surjection in this case.

## 2. Part 2

**2.1. Localization.** Consider  $T$  a torus acting on  $X$ . Suppose the fixed point locus splits as a disjoint union of connected components  $\bigcup F_i$ . The inclusion maps  $\iota_{F_i} : F_i \rightarrow X$  are  $T$ -equivariant, and induce maps

$$\iota_{F_i}^* : H_G(X) \rightarrow H_T(F_i) = H(F) \otimes H_T(pt)$$

Since  $F_i^T \rightarrow X_T$  is an inclusion, we can also construct pushforwards

$$\iota_{F_i^T *} : H_T^k(F_i) \rightarrow H_T^{k+r}(X)$$

where  $r$  is the codimension of  $F$  in  $X$ . Composing these two maps gives us the gysin operation

$$\iota_{F_i^T}^* \iota_{F_i^T *} : H_T^k(F_i) \rightarrow H_T^{k+r}(F_i)$$

which is known to coincide with the cup product with the equivariant Euler class (top Chern class) of the normal bundle

$$(1) \quad \iota_{F_i^T}^* \iota_{F_i^T *} \alpha = \alpha \cup Euler^T(N_{F_i} X)$$

Since  $H_T := H_T(pt) = \mathbb{Q}[\lambda_0, \dots, \lambda_n]$  is a polynomial ring, we consider its associated field of fractions, i.e. rational functions, denoted  $\mathcal{R}_T$ .

**PROPOSITION 2.1** (Atiyah-Bott [1]). *The equivariant Euler class  $Euler^T(N_{F_i} X) \in H(F_i) \otimes H_T$  of the normal bundle  $N_{F_i} X$  of a component  $F_i$  of the fixed point locus is invertible once we localize to the field of fractions  $\mathcal{R}_T$ . i.e.*

$$(Euler^T(N_{F_i} X))^{-1} \text{ exists in } H(F_i) \otimes_{\mathbb{Q}} \mathcal{R}_T$$

**PROOF.** Since the action of  $T$  on  $F_i$  is trivial, for any point  $x \in F_i$ , the normal bundle  $N_{F_i} X|_x$  at  $x$  carries a representation of  $T$ . Thus we get a decomposition into eigen space corresponding to the characters of  $T$

$$N_{F_i} X|_x = \bigoplus_{\rho \in \text{Hom}(T, \mathbb{C}^*)} V_{\rho}$$

Equivariantly, this means that

$$N_{F_i}^T X|_{x_T} = \bigoplus_{\rho \in \text{Hom}(T, \mathbb{C}^*)} V_{\rho} \otimes L_{\rho}$$

where  $L_{\rho}$  is the line bundle on  $BT$  with character  $\rho$ . However, since the characters of  $T$  are rigid, this decomposition must hold globally on all of  $F_i$ , thus

$$N_{F_i}^T X = \bigoplus_{\rho \in \text{Hom}(T, \mathbb{C}^*)} \mathcal{V}_{\rho} \otimes L_{\rho}$$

where  $\mathcal{V}_{\rho}$  is the eigen-subbundle of  $N_{F_i} X$  with character  $\rho$ . Now if we let  $\{x_j^{\rho}\}$  be the chern roots of  $\mathcal{V}_{\rho}$ , then we have

$$Euler^T(N_{F_i} X) = \prod_{\rho} \prod_j (x_j^{\rho} + \lambda_{\rho})$$

If we can invert the  $\lambda_{\rho}$ , we find

$$Euler^T(N_{F_i} X) = \prod_{\rho} \lambda_{\rho}^{rk \mathcal{V}_{\rho}} \prod_j \left( 1 + \frac{x_j^{\rho}}{\lambda_{\rho}} \right)$$

and we find

$$Euler^T(N_{F_i} X)^{-1} = \prod_{\rho} \lambda_{\rho}^{-rk \mathcal{V}_{\rho}} \prod_j \left( 1 - \frac{x_j^{\rho}}{\lambda_{\rho}} + \left( \frac{x_j^{\rho}}{\lambda_{\rho}} \right)^2 - \dots \right) \in H(F_i) \otimes \mathcal{R}_T$$

where the expression in the brackets only contains finitely many terms since  $x_j^{\rho} \in H^2(F)$  is nilpotent.  $\square$

Using this, it is now easy to show (using eqn 1) that the map

$$\phi : \bigoplus_i H(F_i) \otimes \mathcal{R}_T \rightarrow H_T(X)^\vee =: H_T(X) \otimes_{H_T} \mathcal{R}_T$$

given by

$$\phi : \{\alpha_i\} \mapsto \sum_i \iota_{F_i^T}^* \alpha_i$$

is an isomorphism of  $\mathcal{R}_T$  modules, with inverse

$$\phi^{-1} : \alpha \mapsto \{\alpha_i := (\iota_{F_i^T}^* \alpha) \cup Euler^T(N_{F_i} X)^{-1}\}$$

Thus we have shown

**THEOREM 2.2** (Atiyah-Bott Localization Formula). *For  $\tilde{\alpha} \in H_T(X)$ , we have*

$$\tilde{\alpha} = \phi \circ \phi^{-1}(\alpha) = \sum_i \iota_{F_i^T}^* \left( \frac{\iota_{F_i^T}^* \tilde{\alpha}}{Euler^T(N_{F_i} X)} \right)$$

This localization statement expresses that any equivariant cohomology class  $\tilde{\alpha} \in H_T(X)$  can be reconstructed from its restrictions to the fixed point loci.

The full power of this theorem is realized when we try to compute integrals of certain cohomology classes on  $X$ .

**DEFINITION 2.3.** *We say a cohomology class  $\alpha \in H(X)$  has an equivariant extension if there exists a class  $\tilde{\alpha} \in H_T(X)$  such that  $\iota^* \tilde{\alpha} = \alpha$ , where  $\iota$  is the usual inclusion of a fiber.*

We are interested in computing integrals of the form  $\int_X \alpha$ , where  $\alpha$  has an equivariant extension. Recall the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\iota} & X_T \\ \downarrow & & \downarrow \pi \\ p & \longrightarrow & BT \end{array}$$

Given a class  $\tilde{\alpha} \in H_T(X)$ , we have a ‘push-pull’ formula for the above diagram

$$ev_0 \int_{X_T/BT} \tilde{\alpha} = \int_X \iota^* \tilde{\alpha} = \int_X \alpha$$

where  $ev_0$  is the ‘evaluation at 0’ map  $ev_0 = \iota_p^* : \mathcal{R}_T \rightarrow H(pt) = \mathbb{Q}$ , however this map is not well defined for all rational functions in  $\mathcal{R}_T$ , only for those which are constant away from a discrete set of points where they are undefined. From now on, we omit mention of  $ev_0$ .

Next, we use the localization theorem, to pull back this integral to the fixed point set.

$$\int_{X_T/BT} \tilde{\alpha} = \int_{X_T/BT} \sum_i \iota_{F_i^T}^* \left( \frac{\iota_{F_i^T}^* \tilde{\alpha}}{Euler^T(N_{F_i} X)} \right)$$

we can easily see that

$$\int_{X_T/BT} \iota_{F_i^T}^* \beta = \int_{F_i^T/BT} \beta$$

so we arrive at

**PROPOSITION 2.4** (Atiyah-Bott Integration formula).

$$\int_X \alpha = \int_{X_T/BT} \tilde{\alpha} = \sum_i \int_{F_i^T/BT} \left( \frac{\iota_{F_i^T}^* \tilde{\alpha}}{Euler^T(N_{F_i} X)} \right)$$

**2.2. Warm-up Examples.** To see how this formula is useful, we consider a simple example

PROPOSITION 2.5. *Let  $T$  act on  $X$  with fixed point loci  $\bigcup F_i$ , then*

$$\chi(X) = \sum_i \chi(F_i)$$

PROOF. We have  $\chi(X) = \int_X c_{top}(TX)$ . Since  $X$  carries a  $T$  action, there is a natural lift to an action on  $TX$ , thus we have

$$\int_X c_{top}(TX) = \int_{X_T/BT} c_{top}^T(TX)$$

Localizing at the fixed point loci we have

$$\int_{X_T/BT} c_{top}^T(TX) = \sum_i \int_{F_i^T/BT} \frac{\iota_{F_i^T}^* c_{top}^T(TX)}{c_{top}^T(N_{F_i} X)}$$

but naturality of the Chern classes we have

$$\iota_{F_i^T}^* c_{top}^T(TX) = c_{top}^T(\iota_{F_i^T}^* TX)$$

The exact sequence of equivariant bundles  $TF_i \rightarrow \iota_{F_i^T}^* TX \rightarrow N_{F_i} X$  tells us that

$$c_{top}^T(\iota_{F_i^T}^* TX) = c_{top}^T(TF_i) c_{top}^T(N_{F_i} X)$$

and so

$$\sum_i \int_{F_i^T/BT} \frac{\iota_{F_i^T}^* c_{top}^T(TX)}{c_{top}^T(N_{F_i} X)} = \sum_i \int_{F_i^T/BT} c_{top}^T(TF_i) = \sum_i \int_{F_i} c_{top}(TF_i) = \sum_i \chi(F_i)$$

□

For the previous example, we avoided the question of studying the equivariant geometry of the fixed point sets by using an exact sequence. In general, we will have to compute the structure of the equivariant normal bundles. For this next example, we reproduce a simple result using localization techniques.

PROPOSITION 2.6. *Let  $h \in H^2(\mathbb{P}^n)$  be the hyperplane class, i.e  $h = c_1(\mathcal{O}(-1))$ . Then*

$$\int_{\mathbb{P}^n} h^n = 1$$

PROOF. Consider the action of the torus  $T = (\mathbb{C}^*)^{n+1}$  on  $\mathbb{C}^{n+1}$ , given by

$$(t_0, \dots, t_n) \cdot (z_0, \dots, z_n) = (t_0^{-1} z_0, \dots, t_n^{-1} z_n)$$

This descends to an action of  $T$  on  $\mathbb{P}^n$  described in the previous section. The fixed points of this action on  $\mathbb{P}^n$  are the lines in  $\mathbb{C}^{n+1}$  given by

$$q_i : z \rightarrow (0, \dots, \overset{i^{th}}{z}, \dots, 0)$$

Thus there are  $n + 1$  fixed points, and so  $\chi(\mathbb{P}^n) = \sum_i \chi(pt) = n + 1$  as expected. Now since the fixed points are isolated points, we have  $N_{q_i} \mathbb{P}^n = T_{q_i} \mathbb{P}^n$ . Now we recall that the tangent bundle to  $\mathbb{P}^n$  is given by the bundle  $\text{Hom}(U, \mathbb{C}^{n+1}/U)$ , where  $U = \mathcal{O}(-1)$ . Thus  $T_{q_i} \mathbb{P}^n = \text{Hom}(\mathbb{C}q_i, \mathbb{C}^{n+1}/\mathbb{C}q_i)$ . The weight of the  $T$ -representation of  $\mathbb{C}q_i$  is  $\lambda_i$ , i.e.  $q_i^* U = L_i^\vee$ , and so the weights of  $\text{Hom}(\mathbb{C}q_i, \mathbb{C}^{n+1}/\mathbb{C}q_i) = (\mathbb{C}q_i)^\vee \otimes (\mathbb{C}^{n+1}/\mathbb{C}q_i)$  are  $\{\lambda_i - \lambda_j\}_{j \neq i}$ . Furthermore, we have

$$\iota_{q_i}^* \tilde{h} = \iota_{q_i}^* c_1(\mathcal{O}_{\mathbb{P}^n}(-1)) = c_1(\iota_{q_i}^* \mathcal{O}_{\mathbb{P}^n}(-1)) = c_1(L_i^\vee) = \lambda_i$$

. Now applying our localization theorem, we have

$$\int_{\mathbb{P}^n} h^n = \int_{\mathbb{P}^n/BT} \tilde{h}^n = \sum_i \frac{\iota_{q_i}^* \tilde{h}^n}{\text{Euler}^T(T_{q_i} \mathbb{P}^n)} = \sum_i \frac{\lambda_i^n}{\prod_{j \neq i} (\lambda_i - \lambda_j)} \in \mathcal{R}_T$$

Now this expression gives us a degree 0 rational function, however, we know it must be equivalent to a constant function. So we are free evaluate it by restricting to the subtorus  $\lambda_i = \mu_i$ , where  $\{\mu_i\}$  are a set of  $n + 1$  roots of unity. On this subtorus, we have the identity

$$\prod_{j \neq i} (\mu_i - \mu_j) = (n + 1) \mu_i^n$$

and so we find

$$\sum_i \frac{\lambda_i^n}{\prod_{j \neq i} (\lambda_i - \lambda_j)} = \sum_i \frac{\mu_i^n}{(n+1)\mu_i^n} = 1$$

□

### 2.3. Gromov-Witten Invariants of a Quintic Hypersurface in a quintic threefold.

2.3.1. *Set up.* Here we want to compute some particular Gromov-Witten invariants of a generic smooth quintic  $V$  in  $\mathbb{P}^4$ . This example is particularly interesting, as 3-dimensional Calabi-Yau manifolds are important for 10d superstring theory compactifications.

We first compute the expected dimension of the moduli space, using

$$\dim \overline{\mathcal{M}}_{g,n}(X, \beta) = (g-1)(\dim X - 3) - \int_{\beta} \omega_X + n$$

So for our quintic three fold we have  $\dim V = 3$ ,  $[\omega_V] = 0$  since  $V$  is CY, and so the expected dimension is

$$\dim \overline{\mathcal{M}}_{g,n}(V, \beta) = n$$

Our first example will be concerning  $\overline{\mathcal{M}}_1 = \overline{\mathcal{M}}_{0,0}(V, 1)$  where  $1 = [line] \in H^2(V)$ . This is known to be a zero-dimensional smooth variety, consisting of finitely many points. The relevant GW invariant is

$$N_1 := \langle \cdot \rangle_{0,1}^V = \int_{\overline{\mathcal{M}}_{0,0}(V,1)} 1$$

Since the moduli space is smooth and equidimensional, this GW invariant agrees with the classical enumerative invariant

$$n_1 = \# \text{ lines on smooth generic quintic}$$

We will compute this number using localization techniques. We do so by using the embedding  $i : V \rightarrow \mathbb{P}^4$ , and the equivariant geometry of an associated Grassmanian of lines. We consider the induced map on the moduli spaces of stable maps

$$i : \overline{\mathcal{M}}_{0,0}(V, 1) \rightarrow \overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, 1)$$

given by sending a curve  $C$  in  $V$  to its image  $i(C)$  in  $\mathbb{P}^4$ . From an earlier talk, we know that the compactified moduli space of degree 1 curves in  $\mathbb{P}^n$  is given by the Grassmanian

$$\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, 1) = \text{Gr}(2, 5)$$

Our goal is to pushforward the calculation of  $N_1$  from  $\overline{\mathcal{M}}_1$  to  $\text{Gr}(2, 5)$

$$\int_{\overline{\mathcal{M}}_1} 1 = \int_{\text{Gr}(2,5)} i_* 1$$

To compute  $i_* 1$ , we need to express  $\overline{\mathcal{M}}_1$  as the zero set of some section of a vector bundle over  $\text{Gr}(2, 5)$ . Consider the tautological sequence of bundles on  $Y = \text{Gr}(2, 5)$

$$0 \rightarrow U \xrightarrow{q} \mathcal{O}_Y^5 \rightarrow Q \rightarrow 0$$

Where  $[U \rightarrow \mathcal{O}_Y^5]_{p \in Y} = [U_p \rightarrow \mathbb{C}^5]$  is the inclusion of the 2-plane  $U_p$  in  $\mathbb{C}^5$  defined by  $p$ . Now  $V$  is given as the zero set of a degree 5 polynomial, i.e., a section  $s$  of  $\text{Sym}^5(\mathcal{O}_Y^*)$ . Using the tautological sequence, we can restrict the polynomial  $s$  to the fibers of  $U$

$$\bar{s} = q^* s \in \Gamma(\text{Sym}^5(U^*))$$

Now by construction, the zero set of  $\bar{s}$  is the subvariety of  $Y$  that consists of lines in  $\mathbb{P}^4$  on which the defining polynomial of  $V$  vanishes identically, i.e. the space of lines in  $V$ .

$$Z(\bar{s}) = \overline{\mathcal{M}}_1 \subset Y$$

With this construction, and with the additional knowledge that  $\bar{s}$  intersects the zero section of  $\text{Sym}^5(U^*)$  transversally (which is true when  $V$  smooth), we have the standard formula

$$i_* 1 = c_{top}(\text{Sym}^5(U^*)) \in H^{top}(Y)$$

I.e., the Euler class of the bundle  $F = \text{Sym}^5(U^*)$  is Poincare dual to the homology class of the zero locus of a generic section of  $F$ . Thus, we have set up the calculation we need to perform

$$\langle \cdot \rangle_{0,1}^V = \int_{\overline{\mathcal{M}}_{0,0}(V,1)} 1 = \int_{\text{Gr}(2,5)} i_* 1 = \int_{\text{Gr}(2,5)} c_{top}(\text{Sym}^5(U^*))$$

2.3.2. *The Calculation.* We are going to compute  $\int_{\text{Gr}(2,5)} c_{\text{top}}(\text{Sym}^5(U^*))$  using localization techniques, by considering the obvious action of  $T = (\mathbb{C}^*)^5$  on  $Y = \text{Gr}(2,5)$ . The fixed points of this action are given by the planes

$$J_I : (z_1, z_2) \rightarrow (0, \dots, \overset{i_1^{th}}{z_1}, \dots, \overset{i_2^{th}}{z_2}, \dots, 0)$$

where  $I = \{i_1, i_2\}$  is a subset of  $\{1, \dots, 5\}$ ,  $|I| = 2$ . It's easy to see that  $J_I^*U = L_{i_2}^\vee \oplus L_{i_1}^\vee$ , and so the weights of  $J_I^*\text{Sym}^5(U^*)$  are

$$\{a\lambda_{i_1} + (5-a)\lambda_{i_2}\}_{a=0}^5$$

Now since the fixed points are isolated, we know that  $N_{J_I}Y = T_{J_I}Y$ , and again we use  $TY = \text{Hom}(U, \mathbb{C}^5/U)$ , so

$$T_{J_I}Y = ((L_{i_1} \oplus L_{i_2}) \otimes (\mathbb{C}^5 / (L_{i_1}^\vee \oplus L_{i_2}^\vee)))$$

and thus the weights of  $T$  on  $T_{J_I}Y$  are

$$\{\lambda_i - \lambda_j\}_{i \in I, j \notin I}$$

With all of this equivariant data worked out, we can use our localization theorem to compute

$$\int_{\text{Gr}(2,5)} c_{\text{top}}(\text{Sym}^5(U^*)) = \sum_I \frac{t_{J_I}^* c_{\text{top}}(\text{Sym}^5(U^*))}{\text{Euler}^T(T_{J_I}Y)} = \sum_I \frac{\prod_{a=0}^5 (a\lambda_{i_1} + (5-a)\lambda_{i_2})}{\prod_{i \in I, j \notin I} (\lambda_i - \lambda_j)} \in \mathcal{R}_T$$

This time there is no obvious easy tricks to help us evaluate this rational function, however, with computer assistance we can find

$$\sum_I \frac{\prod_{a=0}^5 (a\lambda_{i_1} + (5-a)\lambda_{i_2})}{\prod_{i \in I, j \notin I} (\lambda_i - \lambda_j)} = 2875$$

and so, we find

$$N_1 = \langle \cdot \rangle_{0,1}^V = 2875 = n_1 = \# \text{ lines on smooth generic quintic}$$

**2.4. Comments on higher degree case.** In general, the moduli space  $\overline{\mathcal{M}}_{n,d} = \overline{\mathcal{M}}_{0,n}(V,d)$  is not smooth, or equidimensional. However, Kontsevich [4] observed that is a smooth stack, and a localization theorem could be applied to compute the associated GW invariants. He embeds  $\overline{\mathcal{M}}_{n,d}$  into  $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^4, d)$  as before, but now greater care needs to be taken in identifying the fixed points, as stable maps can degenerate into reducible curves with multiple components. He develops a combinatorial description of the fixed point set, and the weights of the  $T$  action on the normal bundle for each fixed point. He then employs a version of the localization theorem for smooth stacks, and computes (in particular)

$$N_4 = \int_{[\overline{\mathcal{M}}_{0,0}(V,4)]} 1 = \frac{15517926796875}{64}$$

The fractional nature of this invariant reflects the non-equidimensionality and orbifold nature of the moduli space. To relate this number to enumerative invariants, one needs to analyse how multiple degree covers of curves contribute to these numbers.

**THEOREM 2.7 (Aspinwall-Morrison Formula).** *Let  $C \subset V$  be a rigid embedded smooth curve in a CY threefold  $V$ . The the contribution of degree  $d$  multiple covers of  $C$  to  $\langle \cdot \rangle_{0,d[C]}^V$  is  $d^{-3}$ .*

Using this, (and some extra information about instanton numbers in low degree), the following should hold

$$N_4 = \sum_{d|4} d^{-3} n_{(4/d)}$$

where  $n_d$  is the number of smooth curves of degree  $d$  in  $V$ . It was known classically that  $n_1 = 2875$  (as we just computed) and that  $n_2 = 609250$ . Kontsevich's calculation then produces

$$\frac{15517926796875}{64} = n_4 + 2^{-3}609250 + 4^{-3}2875$$

which yields

$$n_4 = 242467530000$$

an enumerative invariant which was not known classically (the Schubert calculus is overwhelming), but was predicted using mirror symmetry in [2]. Thus Kontsevich's result verified the predictions of mirror symmetry in this case.

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