RT Pset 3 solutions

Problem 1. a) By induction, we can assume that $[L_i, \mathcal{H}_v(i-1)]=0, \forall\, i<n$. This shows that $[L_n, T_i]=0$ for $i<n-2$. Now $T_{n-2}L_n = v^{-n}T_{n-2}T_{n-1}T_{n-2}\ldots$
$= v^{-n}T_{n-1}T_{n-2}T_{n-1}T_{n-3}\ldots T_{n-2}T_{n-1} = v^{-n}T_{n-1}T_{n-2}T_{n-3}\ldots T_{n-1}T_{n-2}T_{n-1} = v^{-n}T_{n-1}\ldots T_{n-2}T_{n-1}T_{n-2}$
$= L_nT_{n-2}$. This shows $[L_n, T_i]=0$ for all generators T_i of $\mathcal{H}_v(n-1)$.

b) By (a), we have an epimorphism $\mathcal{H}_v^{aff}(n) \twoheadrightarrow \mathcal{H}_v(n)$ given by $T_i \mapsto T_i$, $X_i \mapsto L_i$. It factors through $\mathcal{H}_v^{aff}(n)/(X_1-1) \twoheadrightarrow \mathcal{H}_v(n)$. The latter is an isomorphism.

c) Set $L_n' := \frac{L_n-1}{v-1}$. Then $L_n' = v^{-1}T_{n-1}L_{n-1}'T_{n-1} + v^{-1}\frac{T_{n-1}^2 - v}{v-1}$
$= v^{-1}T_{n-1}L_{n-1}'T_{n-1} + v^{-1}\frac{(v-1)T_{n-1}+v-v}{v-1} = v^{-1}T_{n-1}L_{n-1}'T_{n-1} + v^{-1}T_{n-1}$.
Modulo $v-1$, T_{n-1} equals $(n-1,n)$. By induction, we check that modulo $v-1$, L_n' is the Jucys-Murphy element $\sum_{i=1}^{n-1}(in)$.

Problem 2.

a) Note that all non-degenerate Hermitian forms over $\mathbb{F}_{q^2}$ are equivalent. Indeed, the image of $x \mapsto x\bar{x} = x^{q+1}$ is the subfield $\mathbb{F}_q \subset \mathbb{F}_{q^2}$. In particular, -1 is in the image and the usual argument implies the claim above.

The number $|GU_n(\mathbb{F}_q)|$ coincides with that of orthonormal bases in $\mathbb{F}_{q^2}^n$, i.e. the number of solutions of $\sum_{i=1}^n x_i^{q+1} = 1$. Let A_n be the number of solutions of this equation, and B_n be the number of solutions for $\sum_{i=1}^n x_i^{q+1}=0$. We have $\sum_{i=1}^n x_i^{q+1} \in \mathbb{F}_q$. Moreover, the map $(x_1, \ldots x_n) \mapsto (ax_1, \ldots, ax_n)$ establishes a bijection between the sets of solutions for $\sum_{i=1}^n x_i^{q+1} = 1$ and $\sum_{i=1}^n x_i^{q+1} = a^{q+1}$. We conclude that

$$(1) \quad (q-1)A_n + B_n = q^{2n}$$

Moreover, a solution $(x_1 \ldots x_n)$ to $\sum_{i=1}^n x_i^{q+1}=0$ can either have $x_n=0$ (B_{n-1} solutions) or $x_n \neq 0$. In the latter case we have $(q^2-1)A_{n-1}$ solutions. So

(2) $B_n = (q^2-1)A_{n-1} + B_{n-1}$, $n>1$.

Using (1) and (2), we get $B_n = (q^2-1)A_{n-1} + q^{2(n-1)} - (q-1)A_{n-1}$. Plug this into (1) and get

$$(q-1)A_n = q^{2n} - (q^2-q)A_{n-1} - q^{2(n-1)} \Rightarrow A_n = (q+1)q^{2(n-1)} - qA_{n-1}$$

By induction on i we now prove that $A_i = q^{i-1}(q^i - (-1)^i)$. Since $|G| = A_1 A_2 \ldots A_n$, we are done.

b) Thanks to our assumptions on q, there is a diagonal element in $GU_n(\mathbb{F}_q)$ with pair-wise distinct entries. It follows that any element in the normalizer is a monomial matrix. Any diagonal matrix takes the form $(x_1, \ldots x_n, (-1), \bar{x}_n^{-1}, \ldots, \bar{x}_1^{-1})$. It follows that an element of $N_G(T)/T$ permutes $x_1 \ldots x_n$ and switches some x_i to $\bar{x}_i^{-1}$. So we get an embedding $N_G(T)/T \hookrightarrow W(B_{\lfloor n/2 \rfloor})$. All permutations of $x_1 \ldots x_n$ are unitary. It is easy to realize the transformation that fixes $x_1 \ldots x_{n-1}$ and swaps $x_n, \bar{x}_n^{-1}$ by a unitary matrix.

c) Recall that $q_i = \frac{|Bs_iB|}{|B|}$. Now for s_i we define a parabolic subgroup $P_i = Bs_iB \cup B$. This subgroup can be realized as the stabilizer of an isotropic flag: $\{0\} \subset \mathbb{F}_{q^2} \subset \ldots \subset \mathbb{F}_{q^2}^{i-1} \subset \mathbb{F}_{q^2}^{i+1} \subset \ldots \mathbb{F}_{q^2}^{\lfloor n/2 \rfloor}$

Let us observe that if a block upper-triangular matrix is in $GU_n(\mathbb{F}_q)$, then its block-diagonal part is also in $GU_n(\mathbb{F}_q)$. Note

$? = \lfloor \frac{n}{2} \rfloor - 1$ that s_i lies in the block-diagonal part that is $(\mathbb{F}_{q^2}^{\times})^{?} \times GL_2(\mathbb{F}_{q^2})$ if $i>0$ and $(\mathbb{F}_q^{\times})^{\lfloor n/2 \rfloor - 1} \times GU_m(\mathbb{F}_q)$, where $m=2$ or 3. Moreover, s_i lies in the second factor. So this reduces the problem for a computation for $GL_2(\mathbb{F}_q)$ (the answer is q^2) and for $GU_m(\mathbb{F}_q)$.

Let $m=2$. We need to compute $|B|$. ~~We note that any strictly upper triangular matrix lies in $GU_2(\mathbb{F}_q)$.~~ ~~So $|B|$~~

$x^q+x=0$. The number of such x's $= q$

we have $\begin{pmatrix}1&x\\0&1\end{pmatrix}\in GU_2(\mathbb{F}_q) \iff \cancel{x\in\mathbb{F}_q}$. So $|B|=(q^2-1)q$. So we get
$|BsB| = |GU_2(\mathbb{F}_q)| - |B| = q(q+1)(q^2-1) - (q^2-1)q = q^2(q^2-1)$. We get
$q_0=q$

Let us consider the case $n=3$. For $\begin{pmatrix}1&x_{12}&x_{13}\\0&1&x_{23}\\0&0&1\end{pmatrix}\in GU_3(\mathbb{F}_q)$
we need to have

$$\begin{pmatrix}1&x_{23}^q&x_{13}^q\\0&1&x_{12}^q\\0&0&1\end{pmatrix} = \begin{pmatrix}1&-x_{12}&x_{12}x_{23}-x_{13}\\0&1&-x_{23}\\0&0&1\end{pmatrix} \iff x_{12}=-x_{23}^q,\ x_{13}^q+x_{13}=-x_{23}^{q+1}$$

The map $a\mapsto a^q+a:\mathbb{F}_{q^2}\to\mathbb{F}_q$ is an $\mathbb{F}_q$-linear projection so any fiber has cardinality q. So the cardinality of the set of strictly upper triangular matrices in $GU_3(\mathbb{F}_q)$ is q^3 and $B=(q^2-1)(q+1)q^3$
Now $|BsB|=|G|-|B| = q^3(q+1)(q^2-1)(q^3+1)-q^3(q^2-1)(q+1) = q^3(q+1)(q^2-1)q^3$
So $q_0=q^3$

3) It's enough to prove that $\mathbb{C}[B\backslash_{s_i\chi}G]\simeq\mathbb{C}[B\backslash_{\chi}G]$
Let P_i be the parabolic subgroup corresponding to s_i. The coinduction functor is transitive, so $\mathbb{C}[B\backslash_\chi G] = Hom_P(\mathbb{C}G, Hom_B(\mathbb{C}P,\mathbb{C}_\chi))$ and the similar equality holds for $\mathbb{C}[B\backslash_{s_i\chi}G]$. So it is enough to prove that $Hom_B(\mathbb{C}P,\mathbb{C}_\chi)\simeq Hom_B(\mathbb{C}P,\mathbb{C}_{s_i\chi})$ (an isomorphism of P-modules). Let L, N denote the subgroup of block diagonal matrices and block upper triangular ones:

$$P=\left\{\begin{pmatrix}*&*&*&*\\0&*&*&*\\0&*&*&*\\0&0&0&*\end{pmatrix}\right\},\quad L=\left\{\begin{pmatrix}*&0&0&0\\0&*&*&0\\0&*&*&0\\0&0&0&*\end{pmatrix}\right\},\quad N=\left\{\begin{pmatrix}1&*&*&*\\0&1&0&*\\0&0&1&*\\0&0&0&1\end{pmatrix}\right\}$$

N acts trivially on $\mathbb{C}_\chi, \mathbb{C}_{s_i\chi}$ so $Hom_B(\mathbb{C}P,\mathbb{C}_?) = Hom_{B\cap L}(\mathbb{C}L,\mathbb{C}_?)$
Now let $T_0=\left\{\begin{pmatrix}*&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&*\end{pmatrix}\right\}$ and $G_0=\begin{pmatrix}1&0&0&0\\0&*&*&0\\0&*&*&0\\0&0&0&1\end{pmatrix}$ so that $L=G_0\times T_0$,

$T_0\simeq(\mathbb{F}_{q^2}^\times)^{n-2}$, $G_0=GU_2(\mathbb{F}_q)$. We have $\chi|_{T_0}=s_i\chi|_{T_0}$ and so the T_0-actions

(= by the same character)

on $\mathrm{Hom}_{B\cap L}(\mathbb{C}L, \mathbb{C}_\chi)$, $\mathrm{Hom}_{B\cap L}(\mathbb{C}L, \mathbb{C}_{s_i\chi})$ coincide. So we only need to check that these modules are the same over $GL_2(\mathbb{F}_q)$. So we have reduced the proof to the case of $n=2$. Here $\chi = s_i\chi$ or χ is generic. In the latter case, we have seen that both modules are simple and there is a nonzero homomorphism between them. So they are isomorphic.

4) a) $\overline{T_{st} - q(T_s+T_t)+q^2} = \overline{T}_s\overline{T}_t - q^{-1}(\overline{T}_s+\overline{T}_t)+q^{-2} = (T_s+q^{-1}-q)(T_t+q^{-1}-q)$
$-q^{-1}(T_s+T_t+2q^{-1}-2q)+q^{-2} = T_sT_t - q(T_s+T_t)+q^2$

Same argument works for C_{ts}

b) What we need to prove is that $(T_s+q^{-1})C_w=0$ that will follow if we check that C_w lies in the submodule spanned by $T'_u=(T_s-q)T_u$ with $\ell(su)>\ell(u)$. The latter is free over $\mathbb{Z}[q^{\pm 1}]$. We will be done if we check that $\overline{T'_u} = T'_u + \sum_{\substack{x<u \\ \ell(sx)>\ell(x)}} R_u^x(q)T'_x$ (then we just use the argument of Thm 1.4 in Lec 10)

We have $\overline{T'_u} = (\overline{T}_s-\overline{q})\overline{T}_u = (T_s-q)\overline{T}_u$. We deduce the required claim for $\overline{T}_u = \sum_{x\leq u} R_u^x(q)T_x$

c) We know that C_{w_0} is a linear combination of $(T_s-q)T_u$ w. $\ell(su)>\ell(u)$ for all s. This is because $\ell(sw_0)<\ell(w_0)$. So if $C_{w_0} = \sum_w P_w(q)T_w$, then $qP_{su}(q) = -P_u(q)$. We deduce that $C_{w_0} = \sum_w (-q)^{\ell(w_0)-\ell(w)}T_w$

5) a) Assume the converse: the extension $0 \to \Delta(0) \to P\Delta(0) \to \Delta(-2) \to 0$ is split. We have $\dim \mathrm{Hom}(\Delta(0), \Delta(0)) = \dim \mathrm{Hom}(\Delta(-2), \Delta(0)) = 1$. So $\dim \mathrm{Hom}(P\Delta(0), \Delta(0)) = 2$. On the other hand, P is self-adjoint. So $\dim \mathrm{Hom}(P\Delta(0), \Delta(0)) = \dim \mathrm{Hom}(\Delta(0), P(\Delta(0))) = \dim \mathrm{Hom}(\Delta(0), \Delta(0)) + \dim \mathrm{Hom}(\Delta(0), \Delta(-2)) = 1$ because $\mathrm{Hom}(\Delta(0), \Delta(-2)) = 0$. Contradiction. This proves the claim.

b) The dimension of $\mathrm{Hom}(P(-2), P(-2))$ equals to the multiplicity of $L(-2) = \Delta(-2)$ in $P(0)$. The multiplicity of $L(-2) = \Delta(-2)$ in both $\Delta(0), \Delta(-2)$ equals 1. So we conclude that $\dim \mathrm{End}(P(-2)) = 2$. Since $P(-2)$ is indecomposable, the algebra $\mathrm{End}(P(-2))$ is local. We conclude that $\mathrm{End}(P(-2)) = \mathbb{C}[x]/(x^2)$

c) Note that if $\mathbb{V}$ kills a homomorphism φ, then it kills its image. But all subobjects of $\Delta(0), \Delta(-2)$ contain $L(-2)$ so $\mathbb{V}$ cannot kill the image of a homomorphism to an object filtered by $\Delta(0), \Delta(-2)$. We conclude that $\mathbb{V}$ is faithful on projective objects.

Now $\mathbb{V}(P(-2)) = \mathbb{C}[x]/(x^2)$, and $\mathbb{V}(P(0)) = \mathbb{V}(\Delta(0))$ is 1-dim-l so is $\mathbb{C}[x]/(x)$. Comparing dimensions of Hom spaces, we see that $\mathbb{V}$ is fully faithful on the projective objects.