HW 2 Solutions

Problem 1: 1) We just need to check that 

\[ \Delta([x,y]) = \Delta([x,0]) \Delta([y,0]) \]

This is straightforward.

2) \[ \Delta(x^p) = \Delta(x)^p = (x \otimes 1 + 1 \otimes x)^p = \sum_{i=0}^{p} \binom{p}{i} x^i \otimes x^{p-i} = x^p \otimes 1 + 1 \otimes x^p \]

3) We have \( S \otimes U \mapsto S \otimes U \) by PBW. Moreover, we can define \( \Delta: S \mapsto S \otimes S \) similarly to \( \Delta \). The homomorphism \( \Delta \) preserves filtrations and the associated graded homomorphism is \( \Delta \). So, it's enough to check that if \( u \in S \otimes S \) is primitive and \( M \leq \mathfrak{p} \), then \( M \leq M \Delta \).

Assume \( M \Delta \leq M \). Let \( q_i(u) \) be defined as the coefficient of \( x_i \).

Then, when \( u \) is a monomial, we see that \( q_i(u) = \frac{\partial u}{\partial x_i} \).

The latter therefore holds for any \( u \). If \( u \) is primitive, then \( q_i(u) = 0 \) for all \( i \). Since \( \deg u = m < p \), this means \( u = 0 \) (contradiction).

4) \( (x+y)^p - x^p - y^p \) is primitive and lies in \( U \otimes S \).

So, \( (x+y)^p - x^p - y^p \in \mathfrak{g} \cap \mathfrak{g}^p \). Under the natural associative algebra homomorphism \( q: U \mapsto \mathfrak{g} \), we get \( q((x+y)^p - x^p - y^p) = (x+y)^p - x^p - y^p \). Hence, \( (x+y)^p - x^p - y^p = (x+y)^p - x^p - y^p \).

Rem: Let \( \mathfrak{g} \) be a free Lie algebra on \( x, y \). So \( U(\mathfrak{g}) \) is a free associative algebra. With suitable modification, 3) works for \( \mathfrak{l} \).

So we see that \( (x+y)^p - x^p - y^p \in \mathfrak{g} \). It follows that for any associative algebra \( A/\mathfrak{p} \) and any \( a \in \mathfrak{a} \), the difference \( (a+b)^p - a^p - b^p \) is a Lie polynomial of \( a, b \) independent of \( \mathfrak{g} \) (just plug \( a = \mathfrak{a} \) and \( b = \mathfrak{b} \) into \( z = 2\mathfrak{g} \)). This gives another proof of the claim of this problem.
Problem: Note that in all 3 cases $\Delta_x(z)$ admits a weight decomposition $\Delta_x(z) = \bigoplus_{i=0}^{\infty} \Delta_x(z)_{-i}$. In cases 1 & 2 ($x = (0, 0)$ or $(0, 0')$) we have $z \in \mathbb{F}$, while in case 3, $z \not\in \mathbb{F}$. Moreover, $f_{\Delta_x(z)_{-i}} = \Delta_z(z)_{-i}$, if $i < p-1$ in all cases while in case 2 we also have $f_{\Delta_x(z)} = \bigoplus_{i=0}^{\infty} \Delta_x(z)_i$.

Claim. If $U \subseteq \Delta_x(z)$, then $U = \bigoplus_{j=0}^{\infty} \Delta_x(z)_{-j}$ for some $j$ with $z-i+1 = 0$.

Proof: $U$ is the sum of weight spaces and contains $\Delta_x(z)_{-i+1}$ with each $\Delta_x(z)_i$; if $i < p-1$. If $U = \Delta_x(z)$, then the claim is trivially true. Otherwise, take minimal $i$ s.t. $\Delta_x(z)_{-i} \subseteq U$.

Case 1: $z-i+1 = 0$. So $\Delta_x(z)$ is irreducible. It has a unique vector annihilated by $e$ with weight $z$. This shows $\Delta_x(z) \cong \Delta_x(z')$ if $z \neq z'$.

Case 2: In the claim, $i = 0$. So $\Delta_x(z)$ is irreducible. It has one vector annihilated by $e$ if $z = p-1$ and two such vectors else (with weights $z$ and $-2z$). This gives a homomorphism $\Delta_x(-2z) \rightarrow \Delta_x(z)$ that has to be iso b/c both modules are irreducible. On the other hand, $\Delta_x(z) \cong \Delta_x(z')$ if $z' \neq z, -2z$ for the same reasons.

Case 1: Here the only proper submodule $U \subseteq \Delta_x(z)$ is $\Delta_x(-2z)$ by (7), it exists if $z \neq p-1$ while $\Delta_x(z)$ are pairwise non-isomorphic.
Problem 3: Since \( V \) is rational, \((1 \times x) v = \sum v_n(x)\), where \( v_n(x) \in V_n[x]\). From \((1 \times 0) (1 \times 0) (1 \times 0) = (1 \times 1)\), we deduce that \( \mathbb{Z}^{m-n} V_m(x) = v_n(1 \times x) \). So \( v_n(x) = 0 \) for \( n > m\). Also \( v_n(x) = v_n(1 \times x)\), which implies that \( V_n(x) \) is constant. Since \( v_n(1) = v_n\), we see that \((1 \times x) v = v\).

Problem 4: 1) Using the invariant form \((x y) = tr(x y)\) on \( g \) we identify \( g \) with \( g^*\). Then \( \sum_{i \neq i_0} E_i \otimes E_{g^*} \otimes E_{g} \) corresponds to \( id = End(g) = g \otimes g^*\). So this element is in \( (g \otimes g^*)\). The action of \( \sum_{i \neq i_0} E_i \otimes E_{g^*} \) on \( M \otimes M \) commutes with \( g\).

There is also a direct solution.

2) The endomorphisms \( T_{n_0}, T_{n_1} \), satisfy the relations in \( S\).

Let's prove that \( X_n X_i = X_i X_n \) for \( i < j\). Let \( n = M \otimes V^{-1}\). Then \( X_i \) is an endomorphism of this \( g \)-module. So for \( \mathbf{v} \in V, \mathbf{m} \in M \otimes V^{-1}\), we have

\[
X_i X_j (\mathbf{v} \otimes \mathbf{m}) = \sum_{E_j \otimes E_i} X_j (\mathbf{v} \otimes E_i \mathbf{m}) = \sum_{E_j \otimes E_i} X_j (\mathbf{v} \otimes E_i \mathbf{m}) = X_j X_i (\mathbf{v} \otimes \mathbf{m})
\]

It's clear that \( X_i T_{j} = T_{j} X_i \) for \( i > j \geq 1\). In order to prove that \( T_{i} X_{i+1} = T_{i+1} X_{i} + 1\) it's enough to assume that \( i = 1\), let \( \mathbf{v} \in V, \mathbf{m} \in M\)

\[
X_i T_{i+1} (\mathbf{v} \otimes \mathbf{m}) = X_i (\mathbf{v} \otimes T_{i+1} \mathbf{m}) = \sum_{E_j \otimes E_i} X_i (\mathbf{v} \otimes (E_j \mathbf{m}) = \sum_{E_j \otimes E_i} E_j \mathbf{v} \otimes E_i \mathbf{m} + \sum_{E_j \otimes E_i} x_i (E_j \mathbf{v} \otimes E_i \mathbf{m})
\]

\[
X_i T_{i+1} (\mathbf{v} \otimes \mathbf{m})
\]

It remains to prove that \( \sum_{E_j \otimes E_i} E_j \mathbf{v} \otimes E_i \mathbf{m} = \mathbf{v} \otimes \mathbf{m}\). This is checked directly on basis elements \( e_k \otimes e_l\).
Problem 5: 1) \([\cdot, \cdot, \cdot] \) is clearly skew-symmetric. Let's check Jacobi identity
\[
\begin{align*}
&\begin{bmatrix} x & y & z \end{bmatrix} [t^k, t^{l+n}, t^{m+n}] = \begin{bmatrix} x & y & z \end{bmatrix} [t^k, t^{l+n}, t^{m+n}] \\
= & \begin{bmatrix} [x, y, z] & [z, t^k, t^{l+n}] & [y, t^k, t^{m+n}] \end{bmatrix} + (k+l) S_{x y z}, \quad \text{tr} \begin{bmatrix} x, y, z \end{bmatrix} = (k+l) \text{tr} \begin{bmatrix} x, y, z \end{bmatrix},
\end{align*}
\]
So the Jacobi id will follow if we check that, for \(k+l+m=0\),
\[
(k+l) \text{tr} \begin{bmatrix} x, y, z \end{bmatrix} + (l+m) \text{tr} \begin{bmatrix} y, z, x \end{bmatrix} + (m+k) \text{tr} \begin{bmatrix} z, x, y \end{bmatrix} = 0
\]
The l.h.s. is
\[
\begin{align*}
&((k+l) \text{tr}(xyz) + (l+m) \text{tr}(yzx) + (m+k) \text{tr}(zxy)) - \\
&-((k+l) \text{tr}(zxy) + (l+m) \text{tr}(xyz) + (m+k) \text{tr}(yxz)) = \\
&= \text{tr}(xyz) = \text{tr}(zyx) = \text{tr}(yxz) = \text{tr}(xyz).
\end{align*}
\]
So all brackets above are zero as \(k+l+m=0\).

2) Relations involving elements \(e_i, h_i, f_{ij} \) with \(i \neq 0\) only follow from the same relations for \(S_n \). So we only need to check relations involving \(h_0, e_0, f_0 \) \([h_0, h_0] = 0 \) is obvious
\[
\begin{bmatrix} e_0 f_0 \end{bmatrix} = [t E_{m}, t E_{m}] = E_{m} - E_{m} + \text{tr}(E_{m} E_{m}) c = E_{m} - E_{m} + c = h_0
\]
All other relations do not have summands of \(c \) and can be checked in \(S_n [t^{\pm 1}] \). They are all homogeneous in \( t \) so we can check them by replacing \( h_0 \) with \( E_{m} - E_{m} \), \( c \) with \( E_{m} \), \( f_0 \) with \( E_{m} \). They now follow from the same relations for \( S_n \) by shifting the basis in \( C^n \).

3) Set \( S = \sum_{i=0}^{n} \). Note that \( S_i (a_j) = \begin{cases} -a_i, & i = j \\ a_i + a_j, & i, j \neq 0 \end{cases} \), else

where we view indices as element of \( W/N \). It follows that \( \bar{S} (a_j) = S_j \).

Let \( \tilde{\alpha} \) denote the free group on basis \( \alpha, \ldots, \alpha \). The group \( \tilde{\alpha} \) acts on \( \tilde{\tilde{\alpha}} \). On \( \tilde{\tilde{\alpha}}/\tilde{\tilde{\alpha}} \) the action becomes that of the Weyl group of \( S_n \), i.e. \( S_n \). So we get an epimorphism \( W \to S_n \). If \( \tau \) is an element in the kernel, then \( \tau (a_i) = a_i + n \). Let us produce an element in the kernel, let \( \omega = S - d_0 \). Set \( \tau = S_0 S_0 \). For \( \omega \in \tilde{\tilde{\alpha}} \) we get
\[
\begin{align*}
\tau (\omega) = S_0 S_0 (\omega) = S_0 (\omega - (\omega, \omega) \omega) = S_0 (\omega + (\omega, \omega) S - (\omega, \omega))
= & \omega - (\omega, \omega) \omega + (\omega, \omega) S - (\omega, \omega) S = \omega + (\omega, \omega) S = \omega - (\omega, \omega) \omega
\end{align*}
\]
For \( \omega \in \tilde{\tilde{\alpha}} \), let us write \( \tau_\alpha \) for \( \tau (\alpha) = \omega - (\omega, \omega) \omega \). Then \( \tau_\alpha \), \( \tau_\alpha \), \( \tau_\alpha \), \( \tau_\alpha \).
and \( e_i e_j = e_{i+j} \) for \( i, j \in \mathbb{Z} \). From here we deduce that we have a homomorphism \( S_n \times \mathbb{Z} \to W \) that sends \( i \in S_n \) to \( e_i \in S_n \times \mathbb{Z} \) and \( 1 \in \mathbb{Z} \) to \( e_0 \). This homomorphism is surjective because \( S_n \) and \( T_\mathbb{Z} = \mathbb{Z} \times \mathbb{Z} \) generate \( W \). Its kernel is a normal subgroup in \( \mathbb{Z} \) stable under \( S_n \). If it is nontrivial, then it has full rank, and hence \( S_n \times \mathbb{Z} / \text{kernel} \) is finite. But \( \langle e_0 \rangle \cong \mathbb{Z} \) is infinite. Contradiction. So \( S_n \times \mathbb{Z} \to W \) is precisely \( \mathbb{Z} \langle x, y \rangle / (x^0 \mathbb{Z}) \).

4) \( \Lambda^{m,n} = W \bigoplus_{i \in \mathbb{Z}} e_i \). Let \( d_\sigma = \tau_\sigma (\sigma) \). Now note that \( m^* \) is a root for \( \Lambda^{m,n} \) (w \( \sigma \mathbb{Z} = \tau^m \mathbb{Z} \)). So it is a root for \( g_i \). The radical of the invariant form \( (\cdot, \cdot) \) on \( \mathbb{Z} \) is spanned by \( m^* \). So \( \Delta^{m,n} = \mathfrak{o}_0 \), \( m^* \mathfrak{o}_0 \).

5) Let \( \mathfrak{g} \) denote the kernel of \( g_i \) \( \mathbb{Z} \to \mathfrak{g} \). We see that \( \Delta(g_i) = m^* \mathfrak{g} \). So \( \mathfrak{g} = \bigoplus_{i \neq j} m^* \mathfrak{g} \). Suppose \( x \in \mathfrak{g} \cap \bigoplus_{i \neq j} m^* \mathfrak{g} \). Then \( e_i x = e_j x = 0 \) \( \forall i, j \in \mathbb{Z} \). On the other hand, \( x = \mathfrak{g} \). \( e_i x = 0 \). Note that \( m^* \mathfrak{g} \cup m^* \mathfrak{g} \) is not a root for \( i \neq j \). So \( 0 = e_i x = e_i e_j y \). But \( e_i e_j = 0 \Rightarrow e_i y = 0 \). So \( x = 0 \) and we are done.