LECTURE 7: CATEGORY $\mathcal{O}$ AND REPRESENTATIONS OF ALGEBRAIC GROUPS

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Introduction

We continue our study of the representation theory of a finite dimensional semisimple Lie algebra $\mathfrak{g}$ by introducing and studying the category $\mathcal{O}$ of $\mathfrak{g}$-modules that has appeared in the seminal paper by Bernstein, Israel and Sergei Gelfand, [BGG]. We establish a block decomposition for this category and use this to prove the Weyl character formula for finite dimensional irreducible $\mathfrak{g}$-modules.

Then we proceed to studying the representations of reductive algebraic groups both in zero and positive characteristic. Our main result is the classification of irreducible rational representations.

1. Category $\mathcal{O}$

1.1. **Definition.** By definition, the category $\mathcal{O}$ consists of all finitely generated $U(\mathfrak{g})$-modules M such that $\mathfrak{h}$ acts on M diagonalizably and $\mathfrak{n}$ acts locally nilpotently, meaning that for each $v \in M$ there is $k \in \mathbb{Z}_{\geqslant 0}$ such that $e_{\gamma_1} \dots e_{\gamma_\ell} v = 0$ for any $\ell \geqslant k$ and any positive roots $\gamma_1, \dots, \gamma_\ell$.

Lemma 1.1. $\Delta(\lambda) \in \mathcal{O}$.

Proof. $\Delta(\lambda)$ is generated by a single vector, v_λ. We have the weight decomposition $\Delta(\lambda) = \bigoplus_{\nu \leqslant \lambda} \Delta(\lambda)_\nu$ so $\mathfrak{h}$ acts on $\Delta(\lambda)$ diagonalizably. Also $e_{\gamma_1} \dots e_{\gamma_k} \Delta(\lambda)_\nu = 0$ provided $\sum_i \gamma_i > \lambda - \nu$ (where the order $\leqslant$ is defined by $\nu' \leqslant \nu$ if $\nu - \nu'$ is the sum of positive roots). □

Lemma 1.2. *The irreducible objects in $\mathcal{O}$ are precisely the irreducible quotients $L(\lambda)$ of $\Delta(\lambda), \lambda \in \mathfrak{h}^*$.*

Proof. Any object $M \in \mathcal{O}$ has a vector v annihilated by $\mathfrak{n}$. Indeed, take any $u \in M$ and let $v = e_{\gamma_1} \dots e_{\gamma_k} u \neq 0$ with maximal possible k. It follows that there is a nonzero homomorphism $\Delta(\lambda) \to M$ for some $\lambda \in \mathfrak{h}^*$. Completing the proof is now an exercise. □

To get more examples of objects in $\mathcal{O}$, we note that if $M \in \mathcal{O}$ and V is a finite dimensional $\mathfrak{g}$-module, then $V \otimes M \in \mathcal{O}$.

1.2. **Structure of the center.** In order to proceed further with our study of the category $\mathcal{O}$, we need to understand the structure of the center of $U(\mathfrak{g})$, let us denote this center by Z. It was described by Harish-Chandra. Set $\rho := \sum_{\alpha > 0} \alpha/2$, where the sum is taken over all positive roots.

Theorem 1.3. *We have an isomorphism $z \mapsto f_z : Z \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^W$ (the algebra of W-invariant polynomials on $\mathfrak{h}^*$) with the property that z acts on $\Delta(\lambda)$ by $f_z(\lambda + \rho)$.*

We will sketch the proof below after providing an example.

Example 1.4. Let $\mathfrak{g} = \mathfrak{sl}_2$. Here $\mathfrak{h}^* = \mathbb{C}$ (with $\epsilon_1 = 1$). So the only positive root α equals 2 and ρ is 1. One can show that the Casimir element $C = 2fe + \frac{1}{2}h^2 + h$ generates the center of $U(\mathfrak{g})$. It acts on $\Delta(z)$ by $\frac{1}{2}z^2 + z = \frac{1}{2}((z+1)^2 - 1)$. The subalgebra $\mathbb{C}[\mathfrak{h}^*]^W \subset \mathbb{C}[\mathfrak{h}^*]$ is $\mathbb{C}[x^2] \subset \mathbb{C}[x]$. We can take $f_C(x) = \frac{1}{2}(x^2+1)$. This actually proves the Harish-Chandra theorem for $\mathfrak{g} = \mathfrak{sl}_2$.

Sketch of proof of Theorem 1.3. Step 1. Let us construct a homomorphism $Z \hookrightarrow S(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]$. Note that $Z \subset U(\mathfrak{g})_0$, the zero weight space for the adjoint action of $\mathfrak{h}$ on $U(\mathfrak{g})$. Note that $U(\mathfrak{g})\mathfrak{n} \cap U(\mathfrak{g})_0$ is a two-sided ideal in $U(\mathfrak{g})_0$ with $U(\mathfrak{g})_0/(U(\mathfrak{g})_0 \cap U(\mathfrak{g})\mathfrak{n}) = U(\mathfrak{h})(= \mathbb{C}[\mathfrak{h}^*])$. The homomorphism we need is $Z \hookrightarrow U(\mathfrak{g})_0 \twoheadrightarrow \mathbb{C}[\mathfrak{h}^*]$. Denote the image of z by $\tilde{f}_z$. By the construction, z acts on $\Delta(\lambda)$ by $\tilde{f}_z(\lambda)$.

Step 2. We need to show that $\tilde{f}_z(\lambda + \rho)$ is W-invariant. This is a consequence of the following fact about Verma modules: suppose that $\lambda \in P$ and α_i be such that $m := \lambda(\alpha_i^\vee) \geqslant 0$. Then $e_j f_i^{m+1} v_\lambda = 0$ for any j (for $j \neq i$ we use $[e_j, f_i] = 0, e_j v_\lambda = 0$ and for $i = j$ we use the representation theory of $\mathfrak{sl}_2$). We get a nonzero homomorphism $\Delta(\lambda - (m+1)\alpha_i) \to \Delta(\lambda)$. It follows that $\tilde{f}_z(\lambda) = \tilde{f}_z(\lambda - (m+1)\alpha_i)$. Note that $\lambda - (m+1)\alpha_i = s_i(\lambda + \rho) - \rho$. Therefore $\tilde{f}_z(\lambda) = \tilde{f}_z(w(\lambda + \rho) - \rho)$. The claim in the beginning of the step follows. We set $f_z(\lambda) := \tilde{f}_z(\lambda - \rho)$.

Step 3. Now let us show that $z \mapsto f_z$ is injective. The algebra $U(\mathfrak{g})$ is filtered, $U(\mathfrak{g}) = \bigcup_{m \geqslant 0} U(\mathfrak{g})^{\leqslant m}$, where $U(\mathfrak{g})^{\leqslant m}$ has basis $x_1^{d_1} \ldots x_n^{d_n}$ with $d_1 + \ldots + d_n \leqslant m$. This is an algebra filtration meaning that $U(\mathfrak{g})^{\leqslant m} U(\mathfrak{g})^{\leqslant m'} \subset U(\mathfrak{g})^{\leqslant m+m'}$. So we can consider the associated graded algebra $\operatorname{gr} U(\mathfrak{g}) := \bigoplus_{m \geqslant 0} U(\mathfrak{g})^{\leqslant m}/U(\mathfrak{g})^{\leqslant m-1}$, where the multiplication is given by $(a + U(\mathfrak{g})^{\leqslant m-1})(b + U(\mathfrak{g})^{\leqslant k-1}) := ab + U(\mathfrak{g})^{\leqslant k+m-1}$. Recall that $(x_1^{d_1} \ldots x_n^{d_n})(x_1^{e_1} \ldots x_n^{e_n})$ equals $x_1^{d_1+e_1} \ldots x_n^{d_n+e_n}$ plus lower degree terms. In other words, $\operatorname{gr} U(\mathfrak{g}) = S(\mathfrak{g})$. We note that Z coincides with $U(\mathfrak{g})^G$. Since the G-module $U(\mathfrak{g})$ is completely reducible, we see that $\operatorname{gr} U(\mathfrak{g})^G = S(\mathfrak{g})^G$. The right hand side is $\mathbb{C}[\mathfrak{g}^*]^G \cong \mathbb{C}[\mathfrak{g}]^G$ (here we use the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ induced by $(\cdot, \cdot)$) and we have the restriction (from $\mathfrak{g}$ to $\mathfrak{h}$) homomorphism $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]$. This homomorphism is injective because $G\mathfrak{h}$ is Zariski dense in $\mathfrak{g}$ (see Proposition 1.3 in Lecture 5). On the other hand, the associated graded of the homomorphism $Z \to \mathbb{C}[\mathfrak{h}^*] = \mathbb{C}[\mathfrak{h}]$ coincides with the restriction homomorphism $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]$. So the homomorphism $z \mapsto f_z : Z \to \mathbb{C}[\mathfrak{h}^*]$ is injective.

Step 4. Note that Step 3 implies that the image of the restriction homomorphism $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]$ lies in $\mathbb{C}[\mathfrak{h}]^W$. This also can be checked directly using (3) of Theorem 3.4 in Lecture 6.

To show that $Z \hookrightarrow \mathbb{C}[\mathfrak{h}^*]^W$ is surjective, we need to check that $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]^W$ is surjective. We will do this in the case when $\mathfrak{g} = \mathfrak{sl}_n$, where this is very explicit. In the general case one needs to use some further structure theory of $\mathfrak{g}$ and some Algebraic geometry.

For $\mathfrak{g} = \mathfrak{sl}_n$, the algebra $\mathbb{C}[\mathfrak{h}]^{S_n}$ is generated by the power symmetric functions $\sum_{i=1}^n \epsilon_i^k$, where $k = 2, \ldots, n$. This function is the restriction of $\operatorname{tr}(x^k) \in \mathbb{C}[\mathfrak{g}]^G$. This shows that the restriction homomorphism $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]^W$ is surjective and completes the proof. □

Corollary 1.5. *An element $z \in Z$ acts on $L(\lambda)$ by $f_z(\lambda + \rho)$.*

1.3. **Infinitesimal blocks.** First of all, let us prove the following lemma.

Lemma 1.6. *Any object of $\mathcal{O}$ has finite length (i.e., has finite composition series).*

Proof. First of all, let us check that $\Delta(\lambda)$ has finite length. Set $w \cdot \lambda := w(\lambda + \rho) - \rho$. If $L(\mu)$ is a composition factor of $\Delta(\lambda)$, then $f_z(\mu + \rho) = f_z(\lambda + \rho)$ for any $z \in Z$. By Theorem

1.3, $\mu = w\cdot\lambda$. So only finitely many different simples can occur in the composition series of $\Delta(\lambda)$. The multiplicity of $L(\mu)$ is bounded by $\dim \Delta(\lambda)_\mu$. So $\Delta(\lambda)$ indeed has finite length.

Now we claim that every module in $\mathcal{O}$ has a finite filtration whose successive quotients are quotients of Verma modules. Indeed, by above, M has a sub of required form. The algebra $U(\mathfrak{g})$ is Noetherian. This is because $S(\mathfrak{g})$ is Noetherian, $U(\mathfrak{g})$ is $\mathbb{Z}_{\geqslant 0}$-filtered, and $\operatorname{gr} U(\mathfrak{g}) = S(\mathfrak{g})$. This establishes the claim in the beginning of the paragraph and completes the proof of the lemma. □

For $\lambda \in \mathfrak{h}^*$, define $\mathcal{O}_\lambda$ as a full subcategory in $\mathcal{O}$ consisting of all modules M such that every $z \in Z$ acts on M with generalized eigenvalue $f_z(\lambda+\rho)$. Note that $\mathcal{O}_\lambda = \mathcal{O}_{w\cdot\lambda}$, by definition and $L(w\cdot\lambda), w \in W$, are precisely the irreducible objects in $\mathcal{O}_\lambda$. In particular, the number of simples equals $|W/W_{\lambda+\rho}|$, where $W_{\lambda+\rho}$ is the stabilizer of $\lambda+\rho$ in W.

Proposition 1.7. *Any $M \in \mathcal{O}$ splits as $\bigoplus_{\lambda\in\mathfrak{h}^*/W} M^\lambda$, where $M^\lambda \in \mathcal{O}_\lambda$.*

Proof. This is the decomposition into the generalized eigenspaces for Z (that exists because M has finite length). □

The point of the previous proposition is that the study of $\mathcal{O}$ reduces to that of $\mathcal{O}_\lambda$'s. These subcategories are called the *infinitesimal blocks*.

1.4. **Characters.** Let $M \in \mathcal{O}$. All weight spaces in the simples $L(\lambda)$ are finite dimensional (this is true even for $\Delta(\lambda)$). Since M has finite length, we have $\dim M_\nu < \infty$ for all ν. So we can consider the formal character $\mathsf{ch}M = \sum_{\nu\in\mathfrak{h}^*} \dim M_\nu e^\nu$, where e^ν is ν viewed as an element of the group algebra of $\mathfrak{h}^*$. The sum $\mathsf{ch}M$ is finite if and only if M is finite dimensional.

Example 1.8. Let us compute $\mathsf{ch}\Delta(\lambda)$. As we have seen in the proof of (1) of Lemma 2.3 in Lecture 6, we have a basis $\prod_{\alpha>0} f_\alpha^{m_\alpha} v_\lambda, m_\alpha \in \mathbb{Z}_{\geqslant 0}$. The latter element has weight $\lambda - \sum_{\alpha>0} m_\alpha \alpha$. It follows that

$$\mathsf{ch}\Delta(\lambda) = e^\lambda \prod_{\alpha>0}\sum_{i=0}^{\infty} e^{-i\alpha} = e^\lambda \prod_{\alpha>0}(1-e^{-\alpha})^{-1}.$$

Using this example and Section 1.3, we are going to compute the characters of the finite dimensional irreducible modules $L(\lambda)$ (the Weyl character formula). For this we need a notation. Set $F(\lambda) := \sum_{w\in W} \det(w) e^{w\lambda}$ (here we take the determinant of w in $\mathfrak{h}$).

Theorem 1.9. *For $\lambda \in P^+$, we have $\mathsf{ch}L(\lambda) = F(\lambda+\rho)/F(\rho)$.*

Sketch of proof. First of all, we have the following formula:

$$F(\rho) = e^\rho \prod_{\alpha>0}(1-e^{-\alpha}). \tag{1.1}$$

On the other hand, recall that each $\Delta(w\cdot\lambda)$ admits an epimorphism onto $L(w\cdot\lambda)$ such that the kernel is filtered with $L(w'\cdot\lambda)$, where $w'\cdot\lambda < w\cdot\lambda$. It follows, in particular, that

$$\mathsf{ch}L(\lambda) = \sum_{w\in W} n_w \mathsf{ch}\Delta(w\cdot\lambda), \tag{1.2}$$

where all $n_w \in \mathbb{Z}$ and $n_1 = 1$. Combining (1.1) with (1.2), we get

$$F(\rho)\mathsf{ch}L(\lambda) = \sum_{w\in W} n_w e^{w(\lambda+\rho)}. \tag{1.3}$$

Also note that $\mathsf{ch}L(\lambda)$ is W-invariant because W acts on the set of weights of $L(\lambda)$ preserving the dimensions of weight spaces. So $w(F(\rho)\mathsf{ch}L(\lambda)) = \det(w)F(\rho)\mathsf{ch}L(\lambda)$. So $n_w = \det(w)n_1 = \det(w)$. □

One can ask how to compute $L(\lambda)$ for general $\lambda \in \mathfrak{h}^*$. This computation is, basically, in three steps.

(1) To do the case when λ in integral and $\lambda+\rho$ is regular (meaning that the stabilizer of $\lambda+\rho$ in W is trivial). This is a very nontrivial problem. The answer is expressed in terms of so called Kazhdan-Lusztig bases in the Hecke algebra of W to be covered later in this class. Finding this answer (Kazhdan-Lusztig) and proving it (Brylinski-Kashiwara and Beilinson-Bernstein) is one of the most significant achievements of Representation theory of the second half of the 20th century.
(2) The case when λ is integral but $\lambda+\rho$ is not regular is reduced to the previous case by means of so called translation functors. This is relatively easy.
(3) The case when λ is not integral is reduced to the integral case for a smaller Weyl group. This reduction, due to Soergel, is not so easy but is not nearly as hard as Step 1.

2. Representation theory of reductive groups

2.1. **Structure theory in arbitrary characteristic.** Let G be a connected (i.e., irreducible as an algebraic variety) reductive algebraic group over an algebraically closed field $\mathbb{F}$, $T \subset B \subset G$ be a maximal torus and a Borel subgroup. The subgroups T and B were introduced in Lecture 6 in the case of characteristic 0 field but the results there hold in arbitrary characteristic (when we do not refer to Lie algebras). We can speak about simple (resp., semisimple) groups as well: these are the connected reductive groups without any (resp., solvable) connected normal subgroups.

For an algebraic group H, let $X(H)$ denote the set $\operatorname{Hom}(H, \mathbb{F}^\times)$ of group homomorphisms (a.k.a. characters). The set $X(H)$ carries a natural group structure.

Lemma 2.1. *Let $T \cong (\mathbb{F}^\times)^n$. Then $X(T) = \mathbb{Z}^n$ and all representations of T are completely reducible.*

Proof. This is a direct generalization (together with a proof) of the corresonding claims for $\mathbb{F}^\times$, see Example 1.2 in Lecture 3, Lemma 2.2 in Lecture 4. □

We can develop the theory of roots in arbitrary characteristic using the adjoint T-action on $\mathfrak{g}$. We have the Weyl group $W = N_G(T)/T$ acting on $\mathbb{R} \otimes_{\mathbb{Z}} X(T)$ and it behaves as in characteristic 0. For each root α, we have the corresponding group homomorphism $\mathrm{SL}_2 \to G$.

Example 2.2. For $G = \mathrm{SL}_n(\mathbb{F})$, $\alpha = \epsilon_i - \epsilon_j$, we consider $\mathrm{SL}_2(\mathbb{F}) \subset \mathrm{SL}_n(\mathbb{F})$ "located" in the rows and columns i, j.

We can decompose B into the semidirect product $T \ltimes U$, where U is the maximal normal unipotent subgroup. In the examples we consider ($G = \mathrm{SL}_n(\mathbb{F}), \mathrm{Sp}_{2n}(\mathbb{F}), \mathrm{SO}_n(\mathbb{F})$ – for the latter two groups we assume $\operatorname{char}\mathbb{F} > 2$ – for U we take the subgroup of all unitriangular matrices in G). The group U is generated by the images of $\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\} \subset \mathrm{SL}_2(\mathbb{F})$ under all homomorphisms associated to the positive roots (one can restrict to the simple roots). This is easy to see for $\mathrm{SL}_n(\mathbb{F})$ using Example 2.2 and can be checked by hand for other classical groups. The decomposition $B = T \ltimes U$ yields an identification $X(B) \cong X(T)$ because $X(U) = \{1\}$.

Using this and emulating the classification theorem for simple Lie algebras in characteristic 0 one can prove the following result.

Theorem 2.3. *The following is true.*

(1) *The simple simply connected algebraic groups are in bijection with the Dynkin diagrams A_n-G_2. Any semisimple simply connected group is the direct product of simple ones.*
(2) *Any semisimple algebraic group is the quotient of a simple one by a finite central subgroup.*

Recall that we say that G is semisimple if there are no etale covers $\tilde{G} \to G$, where $\tilde{G}$ is an algebraic group.

2.2. **Representation theory of algebraic groups.** Here G is connected and reductive. Let V be a rational representation of G. We have the weight decomposition $V = \bigoplus_{\nu \in X(T)} V_\nu$. We say that $\lambda \in X(T)$ is dominant if $\lambda(\alpha^\vee) \in \mathbb{Z}_{\geqslant 0}$ for any positive root α. The definition of a highest weight of an irreducible G-module V is given in the same way as for Lie algebras in characteristic 0.

Theorem 2.4. *The irreducible rational representations of a connected reductive group G are in bijection with the dominant elements in $X(T)$ (an irreducible representation corresponds to its unique highest weight). In characteristic 0, any rational representation of G is completely reducible.*

In characteristic 0, this can be deduced from the corresponding results about semisimple Lie algebras combined with Lemma 2.1. Below we will explain what to do in characteristic p.

Similarly to Problem 3 in Homework 2, we prove the following result.

Lemma 2.5. *Let λ be a highest weight of an irreducible G-module V and $v \in V_\lambda$. Then $bv_\lambda = \lambda(b)v_\lambda$ for any $b \in B$.*

The next step in the proof of Theorem 2.4 is to produce the *Weyl module* $W(\lambda)$ that has the universal property

$$\mathrm{Hom}_G(W(\lambda), V) = \{v \in V | bv = \lambda(b)v, \forall v \in V\}.$$

We start by producing the dual Weyl module $W^\vee(\lambda)$ with lowest weight λ^{*-1}, where we write λ^* for the highest weight of $L(\lambda)^*$ (in characteristic 0). This is done in the same way as in the SL_2-case: we take the line bundle $\mathcal{O}(\lambda^*)$ on G/B that is the homogeneous vector bundle with fiber $\mathbb{F}_{\lambda^{*-1}}$ over $eB \in G/B$. Then we set $W^\vee(\lambda) := \Gamma(\mathcal{O}(\lambda^*)) = \{f \in \mathbb{F}[G] | f(gb) = \lambda^*(b)f(g)\}$. Then we set $W(\lambda) = W^\vee(\lambda^*)^*$. Similarly to the SL_2-case, this module has the required universal property.

Now we need to establish the following two facts: $W(\lambda)_\lambda = \mathbb{F}$ and if $W(\lambda)_\nu \neq \{0\}$ implies $\nu \leqslant \lambda$. This will imply that there is a unique simple quotient $L(\lambda)$ of $W(\lambda)$ and complete the proof of Theorem 2.4. Both claims above follow from the next theorem.

Theorem 2.6. *We have $\mathsf{ch} W(\lambda) = F(\lambda + \rho)/F(\rho)$. In other words, the character is independent of the characteristic.*

Sketch of proof. Note that if M is a finitely generated $\mathbb{Z}$-module, then $\dim \mathbb{Q}\otimes_{\mathbb{Z}} M = \chi(\mathbb{F}_p \otimes^L_{\mathbb{Z}} M)$, where χ is the Euler characteristic. With some standard manipulations along these lines, we conclude that $\bigoplus_{i=0}^{\infty} \mathsf{ch} H^i(G/B, \mathcal{O}(\lambda^*))$ is independent of the characteristic. But $H^i(G/B, \mathcal{O}(\lambda^*)) = 0$ for $i > 0$, this is the Borel-Weil-Bott theorem (in characteristic p it was proved by Kempf). □

2.3. **Characters of simples.** First, let us explain the general form of the Steinberg decomposition. We say that a dominant weight λ is restricted if $\langle \lambda, \alpha_i^\vee \rangle < p$ for all i. So for any dominant weight λ there is a unique p-adic expansion $\lambda = \lambda_0 + p\lambda_1 + \ldots + p^\ell \lambda_\ell$, where all λ_i are restricted. The following theorem (due to Steinberg) reduces the computation of $L(\lambda)$ to that of $L(\lambda_i)$'s.

Theorem 2.7. *Let λ_0 be restricted and $\lambda = \lambda_0 + p\mu$. Then $L(\lambda) \cong L(\lambda_0) \otimes \mathrm{Fr}^* L(\mu)$.*

The question of computation of the characters of $L(\lambda)$ with restricted λ's is wide open. The answer is known (and complicated) when p is very large, [AJS] (there are actual bounds, but they are huge, see [F]). For quite a long time, there was a conjecture on the multiplicities of $L(\lambda)$'s in $W(\mu)$'s when $p \geqslant h$, where h is the so called Coxeter number (it is equal to n for $\mathrm{SL}_n(\mathbb{F})$). Recently, this conjecture was disproved by Williamson, [W]. The problem of computing the multiplicities is wide open even on the level of conjectures.

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