LECTURE 18: DELIGNE-SIMPSON PROBLEM

IVAN LOSEV

Introduction

The Deligne-Simpson problem asks to find a condition on conjugacy classes $C_1, \ldots, C_k \subset \mathrm{Mat}_n(\mathbb{C})$ such that there are matrices $Y_i \in C_i$ with

(1) $\sum_{i=1}^k Y_i = 0$,
(2) and there are no proper subspaces in $\mathbb{C}^n$ stable under all Y_i.

Crawley-Boevey reduced this problem to checking if there is an irreducible representation in $\mathrm{Rep}(\Pi^\lambda(Q), v)$ for suitable Q, λ, v produced from $C_1, \ldots, C_k$. Recall, Section 4.1 of Lecture 17, that this is equivalent to $v \in \Sigma_\lambda$, where $\Sigma_\lambda \subset \mathbb{Z}^{Q_0}$ is a combinatorially defined set.

Crawley-Boevey's approach was strongly motivated the Kraft-Procesi construction who proved that the closures of conjugacy classes of matrices are normal. The proof easily reduces to the case of nilpotent orbits. Kraft and Procesi realized their closures as certain quotients that are special cases of Nakajima quiver varieties. This allowed them to prove the normality.

In the first section we will recall the necessary background from Invariant theory, a field that studies quotients under group actions. Then we will explain the Kraft-Procesi construction. Finally, we will explain Crawley-Boevey's approach to the DS problem.

1. Invariant theory

Let X be an affine algebraic variety and let G be a reductive algebraic group (such as $\mathrm{GL}(n)$ or, more generally, G_v). We assume that G acts on X in such a way that the action map $G \times X \to X$ is a morphism of algebraic varieties. If $X = V$ is a vector space, then a rational representation of G in V provides an example of such an action. In general, we can G-equivariantly embed X into V (as a closed subvariety). Our goal is to study the algebra $\mathbb{C}[X]^G = \{f \in \mathbb{C}[X] | f(g.x) = f(x), \forall x \in X, g \in G\}$.

The first general result here is due to Hilbert.

Theorem 1.1. *The algebra $\mathbb{C}[X]^G$ is finitely generated.*

So we can consider the variety $X//G$ with algebra $\mathbb{C}[X]^G$ of polynomial functions. The inclusion $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$ gives rise to a dominant morphism $\pi : X \to X//G$. It turns out that this morphism has very nice properties (that follow because G is reductive).

Theorem 1.2. *The following is true.*

(1) *The morphism π is surjective.*
(2) *Let $Y_1, Y_2 \subset X$ be closed G-stable subvarieties of X with $Y_1 \cap Y_2 = \varnothing$. Then $\pi(Y_1) \cap \pi(Y_2) = \varnothing$. In particular, every fiber of π contains a unique closed orbit.*
(3) *Let Z be an affine algebraic variety with a G-invariant morphism $\varphi : X \to Z$. Then there is a unique morphism $\psi : X//G \to Z$ such that $\varphi = \psi \circ \pi$.*
(4) *In particular, if $X' \subset X$ is a closed G-stable subvariety, then the induced morphism $X'//G \hookrightarrow X//G$ is a closed embedding with image $\pi(X')$.*

Here is an important example of a computation of $X//G$ and π. Let V, V' be finite dimensional vector spaces, $X = \operatorname{Hom}(V, V') \times \operatorname{Hom}(V', V), G = \operatorname{GL}(V')$ acts on X by $g.(A, B) = (gA, Bg^{-1})$. The following result is traditionally known as the (first and second) "main theorem of invariant theory for $\operatorname{GL}(V')$".

Theorem 1.3. *The quotient $X//G$ is the subvariety of all operators of rank $\leqslant \dim V'$ in $\operatorname{End}(V)$. The quotient morphism $\pi : X \to X//G$ is given by $(A, B) \mapsto BA$.*

We will need the following corollary of Theorem 1.3 combined with (4) of Theorem 1.2.

Corollary 1.4. *Let $X_0 \subset \operatorname{Hom}(V, V') \times \operatorname{Hom}(V', V)$ be a $\operatorname{GL}(V')$-stable subvariety. Then $X_0// \operatorname{GL}(V') = \{BA | (A, B) \in X_0\}$.*

Note that $X_0// \operatorname{GL}(V')$ is a closed $\operatorname{GL}(V)$-stable subvariety of $\operatorname{End}(V)$.

2. Orbit closures

2.1. **Induction lemma.** We are going to investigate the following question. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a Young diagram with n boxes (we assume that $\lambda_k > 0$). To this diagram we can assign the nilpotent orbit $O_\lambda \subset \operatorname{Mat}_n(\mathbb{C})$ consisting of all matrices whose Jordan normal form has blocks of sizes $\lambda_1, \ldots, \lambda_k$. Now set $\lambda' = (\lambda_1 - 1, \ldots, \lambda_k - 1)$, this is the Young diagram obtained from λ by removing the first column. Let $n' := |\lambda'|$, the number of boxes in λ'. Further, let $\lambda^{(i)}$ denote the diagram obtained from λ by removing the first i columns. Note that $O_\lambda = \{Y \in \operatorname{Mat}_n(\mathbb{C}) | \operatorname{rk} Y^i = |\lambda^{(i)}|\}$, while $\overline{O}_\lambda = \{Y \in \operatorname{Mat}_n(\mathbb{C}) | \operatorname{rk} Y^i \leqslant |\lambda^{(i)}|\}$.

Set $V := \mathbb{C}^n, V' := \mathbb{C}^{n'}$.

Lemma 2.1. *The set $\{BA | A \in \operatorname{Hom}(V, V'), B \in \operatorname{Hom}(V', V), AB \in \overline{O}_{\lambda'}\}$ coincides with $\overline{O}_\lambda$. Moreover, we have $BA \in O_\lambda$ if and only if A is surjective, B is injective, and $AB \in O_{\lambda'}$.*

Proof. Note that $(BA)^{i+1} = B(AB)^i A$. We have $\operatorname{rk}((BA)^{i+1}) \leqslant \operatorname{rk}((AB)^i) \leqslant |\lambda^{(i+1)}|$. It follows that $BA \in \overline{O}_\lambda$. The condition that $BA \in O_\lambda$ is equivalent to $\operatorname{rk}((BA)^{i+1}) = |\lambda^{(i+1)}|$ for all i. The condition that $\operatorname{rk}(BA) = |\lambda^{(1)}| = \dim V'$ is equivalent to A being surjective and B being injective. If that holds, then $\operatorname{rk}((BA)^{i+1}) = \operatorname{rk}((AB)^i)$. This completes the proof. □

2.2. **$\overline{O}_\lambda$ as a quiver variety.** Let $m := \lambda_1$. Consider the Dynkin quiver of type A_m, we number vertices by numbers from 0 to $m-1$. We orient it left to right. Set $v_i := |\lambda^{(i)}|$ and let $V_0 = V, V_1, \ldots, V_{m-1}$ be the spaces of these dimensions. The corresponding representation space $\operatorname{Rep}(\overline{Q}, v)$ consists of the $2m - 2$-tuples $(A_1, B_1, A_2, B_2, \ldots, A_{m-1}, B_{m-1})$, where $A_i \in \operatorname{Hom}(V_{i-1}, V_i)$ and $B_i \in \operatorname{Hom}(V_i, V_{i-1})$:

$$V_0 \underset{B_1}{\overset{A_1}{\rightleftarrows}} V_1 \underset{B_2}{\overset{A_2}{\rightleftarrows}} V_2 \underset{B_3}{\overset{A_3}{\rightleftarrows}} \cdots \underset{B_{m-1}}{\overset{A_{m-1}}{\rightleftarrows}} V_{m-1}$$

Consider the group $G = \operatorname{GL}(V_1) \times \ldots \times \operatorname{GL}(V_{m-1})$. The moment map $\mu : \operatorname{Rep}(\overline{Q}, v) \to \mathfrak{g}$ is given by $\mu = (\mu_1, \ldots, \mu_{m-1})$, where μ_i sends $(A_j, B_j)_{j=1}^m$ to $A_i B_i - B_{i+1} A_{i+1}$ (where we set $A_m = B_m = 0$).

The following result is due to Kraft and Procesi.

Proposition 2.2. *The image of the map $\mu^{-1}(0) \to \operatorname{End}(V)$ sending $(A_i, B_i)_{i=1}^{m-1}$ to $B_1 A_1$ has image $\overline{O}_\lambda$ and induces an isomorphism $\mu^{-1}(0)//G \xrightarrow{\sim} \overline{O}_\lambda$.*

Proof. Our proof is by induction on i. The base is the empty quiver. To do the induction step, assume that the analog of our claim is established for the subquiver Q' with vertices $1, \ldots, m-1$ and the group $G' = \mathrm{GL}(V_2) \times \ldots \mathrm{GL}(V_{m-1})$. Let $\mu' : \mathrm{Rep}(\overline{Q}', v') \to \mathfrak{g}'$ be the corresponding moment map so that $\mathrm{Rep}(\overline{Q}, v) = \mathrm{Hom}(V, V_1) \times \mathrm{Hom}(V_1, V) \times \mathrm{Rep}(\overline{Q}', v'), G = \mathrm{GL}(V_1) \times G', \mu((A_i, B_i)) = (A_1 B_1 - B_2 A_2, \mu')$. We can produce the quotient $\mu^{-1}(0)//G$ in two steps: $(\mu^{-1}(0)//G')//\mathrm{GL}(V_1)$. Note that A_1, B_1 are G'-invariant. By the inductive step, $\mu'^{-1}(0)//G' \xrightarrow{\sim} \overline{O}_{\lambda'}$ via $(A_2, B_2, \ldots, A_{m-1}, B_{m-1}) \to B_2 A_2$. It follows that $\mu^{-1}(0)//G' = \{(A_1, B_1) | A_1 B_1 \in \overline{O}_{\lambda'}\}$. Combining Corollary 1.4 with Lemma 2.1, we see that the map $(A_1, B_1) \mapsto B_1 A_1$ gives rise to an isomorphism of $\mu^{-1}(0)//G$ and $\overline{O}_\lambda$. □

Remark 2.3. Let us explain how this helps to prove normality. First, if an affine variety X is normal, then so is $X//G$, this is an exercise. Now to prove that $\mu^{-1}(0)$ is normal we need to do two things: to show that it is a complete intersection (its codimension equals $\dim G$) and apply the Serre normality criterium saying that $\mu^{-1}(0)$ is normal if the complement of the locus where μ is not submersive has codimension bigger than 1. In order to do this one observes that μ is a submersion at a point v precisely when G_v is discrete (in our case, this is equivalent for $G_v = \{1\}$). Both claims are proved inductively based on Lemma 2.1.

2.3. **Closure of an arbitrary orbit.** Lemma 2.1 generalizes to arbitrary orbits as follows. Take an orbit $C \subset \mathrm{Mat}_n(\mathbb{C})$. Let λ be the Young diagram corresponding to the nilpotent component of C. Set $n' := n - |\lambda| + |\lambda'|$ and let C' be the orbit in $\mathrm{Mat}_{n'}(\mathbb{C})$ obtained from C by replacing the nilpotent part O_λ with $O_{\lambda'}$. The proof of the following lemma is left to the reader.

Lemma 2.4. *The set $\{BA | A \in \mathrm{Hom}(V, V'), B \in \mathrm{Hom}(V', V), AB \in \overline{C}'\}$ coincides with $\overline{C}$. Moreover, we have $BA \in C$ if and only if A is surjective, B is injective, and $AB \in C'$.*

When C consists of non-degenerate operators we can replace C with $C - \chi$, where χ is an eigenvalue of C (and we write $C - \chi = \{Y - \chi \,\mathrm{id}_{\mathbb{C}^n} | Y \in C\}$) and apply Lemma 2.4 to that orbit. This motivates the following construction.

Let $\chi_1, \ldots, \chi_m$ be all roots of the minimal polynomial of C counted with multiplicities so that, for $Y \in C$, we have $\prod_{i=1}^m (Y - \chi_i) = 0$. Check the Dynkin diagram of type A_m oriented as before. Set $v_i := \mathrm{rk} \prod_{j=1}^i (Y - \chi_i)$ and $\xi_i = \chi_{i+1} - \chi_i, i = 1, \ldots, m-1$. Consider G, μ as before and set $\xi := \sum_{i=1}^m \xi_i \,\mathrm{id}_{V_i}$.

The proof of the following proposition repeats that of Theorem 1.3

Proposition 2.5. *We have an isomorphism $\mu^{-1}(\xi)//G \xrightarrow{\sim} \overline{C}$ induced by $(A_i, B_i)_{i=1}^{m-1} \mapsto B_1 A_1 + \chi_1$.*

Example 2.6. Let us consider an example of this. Let C consist of the projectors in $\mathrm{Mat}_n(\mathbb{C})$ of rank $r < n$. The quiver Q will have type A_2. Then we can set $\chi_1 = 0$ and $\chi_2 = 1$. We have $\xi_2 = 1, v_1 = r$, the moment map equation is $AB = 1$ and the map $\mu^{-1}(0) \to \overline{C} = C$ is $(A, B) \mapsto BA$. Or we can set $\chi_1 = 1, \chi_2 = 0$. In this case, $v_1 = n - r, \xi_1 = -1$. The moment map equation is $AB = -1$ and the map $\mu^{-1}(0) \to C$ is $(A, B) \mapsto BA + 1$.

Remark 2.7. Let $(A_i, B_i)_{i=1}^{m-1} \in \mu^{-1}(\xi)$ be such that $Y := B_1 A_1 + \chi_1 \in C$. We can recover $(A_i, B_i)_{i=1}^{m-1}$ up to G-conjugacy from Y as follows. Since $Y \in C$, we see that all A_i are surjective and all B_i are injective. We can assume that $V = V_0 \supset V_1 \supset V_2 \supset \ldots$ and all B_i's are inclusions. Then $A_1 = Y - \chi_1 : V_0 \to V_1 = \mathrm{im}(Y - \chi_1)$. We have $A_2|_{V_1} + \chi_2 = B_2 A_2 + \chi_2 = A_1 B_1 + \chi_1 = (Y - \chi_1)|_{V_1} + \chi_1 = Y|_{V_1}$. So $A_2 = (Y - \chi_2)|_{V_1}$. Continuing in this way we see that $V_i = \mathrm{im} \prod_{j=1}^i (Y - \chi_j)$ and $A_i = (Y - \chi_i)|_{V_{i-1}}$ for all i.

3. Solution to Deligne-Simpson problem

Recall that we are interested in solutions of the DS problem: describe $\mathrm{GL}_n(\mathbb{C})$-orbits $C_1,\ldots,C_k$ in $\mathrm{Mat}_n(\mathbb{C})$ such that there are $Y_i\in C_i$ with

(1) $\sum_{i=1}^k Y_i=0$,
(2) and there are no proper subspaces in $\mathbb{C}^n$ stable under all Y_i.

We are going to produce a quiver Q, a dimension vector v and a parameter λ out of $C_1,\ldots,C_k$.

The quiver Q and the dimension vector v are as follows. Produce the type A quivers from $C_1,\ldots,C_k$, let $[0,i],[1,i],\ldots,[m_i,i]$ denote the vertices of the quiver corresponding to C_i. Identify the vertices $[0,i]$ with a single vertex 0 getting a star-shaped quiver. Denote the dimension vector v so that $(v_{[0,i]},\ldots,v_{[m_i,i]})$ is the dimension vector produced from C_i (so that $v_0=n$). Define $\xi_{[j,i]}$ as above for $j>0$, set $\xi_0=\sum_{i=1}^k\chi_{[1,i]}$. Note that ξ_0 is chosen in such a way that the equalities $\sum_{i=1}^k\mathrm{tr}(Y_i)=0$ (a necessary condition for the DS problem to have a solution) and $v_0\xi_0+\sum_{i=1}^k\sum_{j=1}^{m_i}v_{[j,i]}\xi_{[j,i]}=0$ (a necessary condition for $\Pi^\xi(Q)$ to have a representation of dimension v) are equivalent.

Example 3.1. Assume that $n=6$ and $k=3$. We take the following conjugacy classes C_1,C_2,C_3: C_1 contains $\mathrm{diag}(-2,-1,0,1,2,3)$, C_2 contains $\mathrm{diag}(-1,-1,-1,0,0,0)$, and $C_3=O_\lambda$, for $\lambda=(3,3)$. Then we get $m_1=5,m_2=1,m_3=2$ with $v_{[?,1]}=(5,4,3,2,1),v_{[1,2]}=3,v_{[?,3]}=(4,2)$. Further, $\xi_{[?,1]}=(1,1,1,1,1),\xi_{[1,2]}=1,\xi_{[?,3]}=(0,0),\xi_0=-3$. We note that the resulting quiver is of type $\tilde{E}_8$ and v is the indecomposable imaginary root δ.

Theorem 3.2. *There is a bijection between the solutions $Y_1,\ldots,Y_k$ of the DS problem (up to conjugacy) and the irreducible representations in $\mathrm{Rep}(\Pi^\xi(Q),v)$ (up to isomorphism).*

First, let us get a necessary and sufficient condition for a representation of $\Pi^\xi(Q)$ to be irreducible.

Lemma 3.3. *The following two conditions are equivalent:*

(1) *A representation $(A_{[j,i]},B_{[j,i]})\in\mathrm{Rep}(\Pi^\xi(Q),v)$ is irreducible.*
(2) *All maps $A_{[j,i]}$ are surjective, all maps $B_{[j,i]}$ are injective, and the space $\mathbb{C}^n$ is irreducible with respect to the operators $B_{[1,i]}A_{[1,i]}$.*

Proof. Suppose (1) holds and let us establish (2). Assume the map $A_{[j,i]}$ is not surjective. Set $U_{[j',i']}:=V_{[j',i']}$ if $i'\neq i$ or $j'<j$, $U_{[k,i]}:=A_{[k,i]}U_{[k-1,i]}$ for $k\geqslant j$. The moment map equation implies that $B_{[k,i]}U_{[k+1,i]}\subset U_{[k,j]}$. Indeed, we have $B_{[k,i]}U_{[k+1,i]}=B_{[k,j]}A_{[k,j]}U_{[k,j]}=(A_{[k-1,j]}B_{[k-1,j]}+\xi_{[k-1,i]})U_{[k,i]}\subset U_{[k,i]}$. We deduce that $(U_{[k,i]})\subset(V_{[k,i]})$ defines a proper subrepresentation, a contradiction. The remaining parts of (2) are left as exercises.

Conversely, suppose that (2) holds. Let $(U_{[j,i]})\subset(V_{[j,i]})$ be a proper subrepresentation. The condition that V_0 is irreducible w.r.t. $B_{[1,i]}A_{[1,i]}$ implies that $U_0=\{0\}$ or $U_0=V_0$. In the former case, $U_{[j,i]}=\{0\}$ because all $B_{[j,i]}$ are injective, in the latter case, $U_{[j,i]}=V_{[j,i]}$ because all $A_{[j,i]}$ are surjective. □

Proof of Theorem 3.2. Let $(A_{[j,i]},B_{[j,i]})$ be an irreducible representation in $\mathrm{Rep}(\Pi^\xi(Q),v)$. Then $Y_i:=B_{[1,i]}A_{[1,i]}+\chi_{[1,i]},i=1,\ldots,k$ is a solution to the Deligne-Simpson problem (the claim that $\sum_i Y_i=0$ is the moment map condition at 0, while the claim that $\mathbb{C}^n$ is irreducible w.r.t. the Y_i's follows from Lemma 3.3). Conversely, let $(Y_i)_{i=1}^k$ be the solution to the DS problem. We can form the maps $A_{[i,j]},B_{[i,j]}$ as in Remark 2.7, and this will give an irreducible representation in $\mathrm{Rep}(\Pi^{\xi}(Q), v)$. On the level of isomorphism classes, these two maps are inverse to each other. □

Example 3.4. In the example above, the DS problem has a solution. Moreover, the variety of conjugacy classes of (Y_1, Y_2, Y_3) can be shown to be 2-dimensional.