REPRESENTATION THEORY, LECTURE 0. BASICS

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Introduction

The aim of this lecture is to recall some standard basic things about the representation theory of finite dimensional algebras and finite groups. First, we recall restriction, induction and coinduction functors. Then we recall the Schur lemma and deduce consequences about the action of the center and the structure of completely reducible representations. Then we explain the structure and representation theory of simple finite dimensional algebras over algebraically closed fields. Next, we proceed to semisimple algebras. Finally, we use the latter to recall basics about the representation theory of finite groups.

1. Restriction, induction and coinduction

Let A, B be associative unital algebras over a field $\mathbb{F}$ with a homomorphism $B \to A$. Let M be an A-module. Of course, we can view M as a B-module. On the other hand, A is a left A-module and a right B-module. These two operations commute, one says in this case that A is an A-B-bimodule. It follows that, for a B-module N, the space $A \otimes_B N$ carries a natural structure of a left A-module (induced module). Also A is a B-A-bimodule. So $\operatorname{Hom}_B(A, N)$ is a left A-module via $(a\varphi)(a') := \varphi(a'a), \varphi \in \operatorname{Hom}_B(A, N), a, a' \in A$. This is a *coinduced* module.

Lemma 1.1. *For an A-module M and a B-module N we have natural isomorphisms*

$$\operatorname{Hom}_B(N, M) \cong \operatorname{Hom}_A(A \otimes_B N, M), \quad \operatorname{Hom}_B(M, N) \cong \operatorname{Hom}_A(M, \operatorname{Hom}_B(A, N)).$$

Proof. Consider the map $\operatorname{Hom}_B(N, M) \to \operatorname{Hom}_A(A \otimes_B N, M)$ that sends η to ψ_η given $\psi_\eta(a \otimes n) = a\eta(n)$ and the map in the opposite direction that sends ψ to η_ψ given by $\eta_\psi(n) := \psi(1 \otimes n)$. It is left as an exercise to check that the maps are well-defined (i.e., land in the required Hom spaces) and are mutually inverse. Establishing a natural isomorphism $\operatorname{Hom}_B(M, N) \cong \operatorname{Hom}_A(M, \operatorname{Hom}_B(A, N))$ is also left as an exercise. □

2. Schur lemma and its consequences

2.1. **Schur lemma.** The following important result is known as the Schur lemma.

Proposition 2.1. *Let $\mathbb{F}$ be algebraically closed, A be an associative unital $\mathbb{F}$-algebra and let U, V be finite dimensional irreducible A-modules. Then the following is true.*

(1) *If U, V are non-isomorphic, then* $\operatorname{Hom}_A(U, V) = 0$.
(2) $\operatorname{End}_A(U)$ *consists of constant maps. In particular,* $\dim \operatorname{End}_A(U) = 1$.

Under some assumptions, this can be generalized to infinite dimensional irreducible modules.

2.2. **The action of the center.** Consider the center of A, $Z(A) := \{z \in A | za = az, \forall a \in A\}$. This is a commutative algebra. The following claim is a corollary of the Schur lemma.

Corollary 2.2. *Let $z \in Z(A)$, and let U be a finite dimensional irreducible A-module. Then z acts on U by a scalar.*

It follows that there is an algebra homomorphism $\chi_U : Z(A) \to \mathbb{F}$ (called the central character of U) such that z acts on U as the multiplication by $\chi_U(z)$.

2.3. **Multiplicities in completely reducible modules.** Let A be an associative unital algebra over an algebraically closed field $\mathbb{F}$. Let $V_i, i \in I$, be the finite dimensional irreducible A-modules, where I is an indexing set. Now let V be a completely reducible finite dimensional A-module. Since V is completely reducible, there are non-negative integers $m_i, i \in I$, with only finitely many nonzero such that $V \cong \bigoplus_{i\in I} V_i^{\oplus m_i}$.

The following lemma is a consequence of the Schur lemma and the additivity of Hom's: $\operatorname{Hom}_A(V_i, V' \oplus V'') \cong \operatorname{Hom}_A(V_i, V') \oplus \operatorname{Hom}_A(V_i, V'')$.

Lemma 2.3. *The number m_i coincides with* $\dim \operatorname{Hom}_A(V_i, V)$.

We call $\operatorname{Hom}_A(V_i, V)$ the *multiplicity space* for V_i in V. The name is justified by the observation that the natural homomorphism

$$\bigoplus_{i\in I} V_i \otimes \operatorname{Hom}_A(V_i, V) \to V, \quad \sum_{i\in I} v_i \otimes \varphi_i \mapsto \sum_{i\in I} \varphi_i(v_i) \tag{2.1}$$

is an isomorphism of A-modules.

2.4. **Endomorphisms of completely reducible modules.** The isomorphism (2.1) together with the Schur lemma imply the following description of the endomorphism algebra $\operatorname{End}_A(V)$:

$$\operatorname{End}_A(V) = \bigoplus_{i\in I} \operatorname{End}(\operatorname{Hom}_A(V_i, V)).$$

Here we assume that the endomorphisms of the zero space are zero (and so we sum over all i such that $\operatorname{Hom}_A(V_i, V) \neq 0$, in particular, the sum is finite). The $\operatorname{End}_A(V)$-module structure on $\operatorname{Hom}_A(V_i, V)$ is given by the composition:

$$\varphi \cdot \psi := \varphi \circ \psi, \varphi \in \operatorname{End}_A(V), \psi \in \operatorname{Hom}_A(V_i, V).$$

3. Simple algebras

3.1. **Burnside theorem.** Let A be an associative algebra over an algebraically closed field $\mathbb{F}$ and let V be a finite dimensional A-module. So we have an algebra homomorphism $A \to \operatorname{End}(V)$.

Proposition 3.1. *If V is irreducible, then the homomorphism $A \to \operatorname{End}(V)$ is surjective.*

Proof. The proof is in several steps.

Step 1. Consider the A-module $V \otimes M$, where M is a finite dimensional vector space (and A acts on the first factor). We claim that every A-submodule $U \subset V \otimes M$ has the form $V \otimes M_0$, where M_0 is a subspace in M. Indeed, $\operatorname{Hom}_A(V, U) \hookrightarrow \operatorname{Hom}_A(V, V \otimes M)$. By the Schur lemma, the target space is naturally identified with M. The subspace $\operatorname{Hom}_A(V, U) \subset M$ is M_0 we need: by complete reducibility, if $U \neq V \otimes M_0$, there is a homomorphism $\varphi : V \to U$ that does not lie in M_0.

Step 2. The space V^* is a right A-module via $(\varphi \cdot a)(v) := \varphi(a \cdot v), \varphi \in V^*, v \in V, a \in A$. Note that V^* is irreducible (if $U' \subset V^*$ is a proper submodule, then the annihilator of U' is a proper A-submodule in V).

Step 3. Recall that $\mathrm{End}(V)$ is naturally identified with $V \otimes V^*$. Both $\mathrm{End}(V)$ and $V \otimes V^*$ are A-bimodules and the isomorphism $\mathrm{End}(V) \cong V \otimes V^*$ is that of A-bimodules. Replacing A with its image in $\mathrm{End}(V)$, we may assume that $A \subset \mathrm{End}(V)$. Clearly, $A \subset V \otimes V^*$ is a subbimodule. Apply Step 1 to A viewed as a left A-module. We get that $A = V \otimes M'$, where $M' \subset V^*$. Applying (the obvious analog of) Step 1 to the right A-module $V \otimes V^*$, we get $A = M \otimes V^*$, where $M \subset V$. Since $V \otimes M' = M \otimes V^*$, we see that $M = V, M' = V^*$. □

3.2. **Simple algebras over algebraically closed fields.** Let A be an associative unital algebra over $\mathbb{F}$. We say that A is simple if it has no proper two-sided ideals. For example, $\mathrm{Mat}_n(\mathbb{F})$ is a simple algebra, this can be deduced similarly to Step 3 of the proof of the Burnside theorem.

Proposition 3.2. *Let $\mathbb{F}$ be algebraically closed and A be a finite dimensional simple A-algebra. Then $A \cong \mathrm{Mat}_n(\mathbb{F})$ for some n.*

Proof. Consider a minimal (w.r.t. inclusion) left ideal $I \subset A$ (just take a left ideal of minimal dimension). It is an irreducible left A-module. So we get a homomorphism $A \to \mathrm{End}(I)$. Its injective because A is simple. It is surjective by the Burnside theorem. So $A \xrightarrow{\sim} \mathrm{End}(I)$. □

3.3. **Representations of the matrix algebra.** Let V be a finite dimensional vector space over $\mathbb{F}$. We are going to understand the representation theory of the algebra $A = \mathrm{End}(V)$.

Proposition 3.3. *Every finite dimensional A-module U is completely reducible and the only irreducible module is V itself.*

Proof. There are $u_1, \ldots, u_k \in U$ such that $U = Au_1 + \ldots + Au_k$. This gives an A-module epimorphism $A^{\oplus k} \twoheadrightarrow U, (a_1, \ldots, a_k) \mapsto a_1u_1 + \ldots + a_ku_k$. Moreover, $A \cong V^{\oplus n}$, where $n = \dim V$ and so $V^{\oplus nk} \twoheadrightarrow U$. Being a quotient of a completely reducible module, U is completely reducible itself. If U is irreducible, $\mathrm{Hom}_A(V^{\oplus nk}, U) \neq 0$. Therefore $\mathrm{Hom}_A(V, U) \neq 0$. By the Schur lemma, $V \cong U$. □

4. Semisimple algebras

Let A be a finite dimensional algebra over $\mathbb{F}$ (still assumed to be algebraically closed). We say that A is *semisimple*, if it is a direct sum of simple algebras.

4.1. **Criteria for semisimplicity.** We are going to explain some criteria for semisimplicity.

Lemma 4.1. *Let A be a finite dimensional associative unital algebra. Let I, J be two-sided ideals in A consisting of nilpotent elements (we say that $a \in A$ is nilpotent if $a^n = 0$ for some $n > 0$). Then $I + J$ is a two-sided ideal consisting of nilpotent elements.*

The proof is left as an exercise.

So A has the unique maximal two-sided ideal consisting of nilpotent elements, it is called the *radical* of A and is denoted by $\mathrm{Rad}(A)$.

Also we can define a distinguished element $\mathrm{tr}_A \in A^*$. It sends $a \in A$ to the the trace of m_a, the operator $A \to A$ given by $m_a(b) = ab$. Note that $\mathrm{tr}_A(ab) = \mathrm{tr}_A(ba)$. So $(a, b)_A := \mathrm{tr}_A(ab)$ is a symmetric bilinear form.

Proposition 4.2. *Let A be a finite dimensional algebra over an algebraically closed field $\mathbb{F}$. The following conditions are equivalent.*

(i) *The algebra A is semisimple.*
(ii) $\mathrm{Rad}(A) = \{0\}$.
(iii) *A is completely reducible as a left A-module.*
(iv) *Every finite dimensional representation of A is completely reducible.*

If the characteristic of $\mathbb{F}$ is zero, then (i)-(iv) are equivalent to the following condition.

(v) *The form $(\cdot,\cdot)_A$ is non-degenerate.*

Proof. Proof of (i)⇒(ii). Note that the radical of a simple algebra is zero. Also the radical of the direct sum is the direct sum of radicals. This proves the required implication.

Proof of (ii)⇒(iii). We have an A-module filtration $A = A_0 \supsetneq A_1 \ldots \supsetneq A_n \supsetneq A_{n+1} = 0$ such that A_i/A_{i+1} is irreducible for all $i = 0,\ldots,n$. Consider the corresponding algebra homomorphism $\varphi : A \to \bigoplus_{i=0}^n \mathrm{End}(A_i/A_{i+1})$. The inclusion $a \in \ker\varphi$ is equivalent to $aA_i \subset A_{i+1}$ for all i. So for any $a \in \ker\varphi$ we have $a^{n+1} = 0$ and hence $\ker\varphi \subset \mathrm{Rad}(A)$. Therefore $\ker\varphi = \{0\}$ and we have an embedding $A \hookrightarrow \bigoplus_{i=0}^n \mathrm{End}(A_i/A_{i+1})$ of algebras, so, in particular, of left A-modules. The A-module $\bigoplus_{i=0}^n \mathrm{End}(A_i/A_{i+1}) = \bigoplus_{i=0}^n (A_i/A_{i+1}) \otimes (A_i/A_{i+1})^*$ is completely reducible. Being a submodule in a completely reducible module, A is completely reducible.

Proof of (iii)⇒(iv) repeats (a part of) the proof of Proposition 3.3.

Proof of (iv)⇒(i). We just need that A is completely reducible. Let $A = \bigoplus_{i=1}^k V_i^{\oplus m_i}$, where all V_i are irreducible and all m_i are positive. Since A is a faithful A-module (only zero acts by zero), the same is true for $\bigoplus_{i=1}^k V_i$. So we get an algebra embedding $A \hookrightarrow \bigoplus_{i=1}^k \mathrm{End}(V_i)$. In particular, this is a left A-module embedding. It follows that $m_i \leqslant \dim V_i$, as the right hand side is the multiplicity of V_i in the left A-module $\bigoplus_{i=1}^k \mathrm{End}(V_i)$.

On the other hand, by the Burnside theorem, the composition of the embedding $A \hookrightarrow \bigoplus_{i=1}^k \mathrm{End}(V_i)$ with the projection to $\mathrm{End}(V_i)$ is surjective. It follows that $m_i \geqslant \dim V_i$. We conclude that $m_i = \dim V_i$ and $A \xrightarrow{\sim} \bigoplus_{i=1}^k \mathrm{End}(V_i)$.

Proof of (ii)⇔(v). It is enough to show that the radical of A coincides with $\ker(\cdot,\cdot)_A$ when $\mathrm{char}\,\mathbb{F} = 0$. Indeed, if a is in the radical, then ab is nilpotent for all $b \in A$, and $(a,b)_A = 0$. So $\mathrm{Rad}(A) = \ker(\cdot,\cdot)_A$. On the other hand, $\ker(\cdot,\cdot)_A$ is a two-sided ideal because $(ab,c)_A = (a,bc)_A$ for all $a,b,c \in A$. Also if $a \in \ker(\cdot,\cdot)_A$, then $\mathrm{tr}_A(a^n) = 0$ for all $n > 0$. It follows that a is nilpotent (here we use that $\mathrm{char}\,\mathbb{F} = 0$). We see that $\ker(\cdot,\cdot)_A$ is contained in the radical. This finishes the proof of (ii)⇔(v). □

4.2. **Representations of semisimple algebras.**

Lemma 4.3. *Let $A = \bigoplus_{i=1}^k \mathrm{End}(V_i)$. Then the set of irreducible A-modules, to be denoted by $\mathrm{Irr}(A)$, coincides with $\{V_1,\ldots,V_k\}$.*

Proof. Let A_1, A_2 be associative algebras. Then any $A_1 \oplus A_2$-module V canonically decomposes as $V_1 \oplus V_2$, where V_i is an A_i-module. Namely, if e_i is the unit in A_i, then $V_i = e_iV$. In particular, $\mathrm{Irr}(A_1 \oplus A_2) = \mathrm{Irr}(A_1) \sqcup \mathrm{Irr}(A_2)$. The claim of the lemma follows from here (and trivial induction). □

Often one wants to compute the number of irreducible representations of A without referring to the decomposition $A = \bigoplus_{i=1}^k \mathrm{End}(V_i)$.

Lemma 4.4. *Let A be a semisimple algebra. Then $|\mathrm{Irr}(A)| = \dim Z(A)$.*

Proof. This follows from the observation that the center of $\mathrm{End}(V_i)$ consists of scalar operators combined with $Z(A_1 \oplus A_2) = Z(A_1) \oplus Z(A_2)$. □

4.3. **Irreducible representations of arbitrary finite dimensional algebras.** Now let A be a finite dimensional $\mathbb{F}$-algebra. By (ii) of Proposition 4.2, the algebra $A/\operatorname{Rad}(A)$ is semisimple. Besides $\operatorname{Rad}(A)$ acts by 0 on all irreducible representations. We conclude that pulling back a representation from $A/\operatorname{Rad}(A)$ to A gives rise to a bijection between $\operatorname{Irr}(A/\operatorname{Rad}(A))$ and $\operatorname{Irr}(A)$.

5. Finite groups

5.1. **Group algebra and its semisimplicity.** Let G be a finite group and $\mathbb{F}$ be a field. We can form the group algebra $\mathbb{F}G$ of G, a vector space with basis G, where the basis elements multiply as in G. A representation of G is the same thing as a representation of $\mathbb{F}G$.

Proposition 5.1. *Let $\mathbb{F}$ be algebraically closed and of characteristic* 0. *Then $\mathbb{F}G$ is a semisimple algebra. In particular, any finite dimensional representation of G over $\mathbb{F}$ is completely reducible.*

Proof. We will check (v) of Proposition 4.2. On the basis elements, we have $\operatorname{tr}_{\mathbb{F}G}(g) = \delta_{g1}|G|$, where δ_{g1} is the Kroneker symbol. So $(g,h)_{\mathbb{F}G} = \delta_{g,h^{-1}}|G|$. Clearly, this form is nondegenerate. □

We remark that this proposition is no longer true when the characteristic of $\mathbb{F}$ is positive.

5.2. **The number of irreducible representations.** By Lemma 4.4, the number of the irreducible representations of G coincides with the dimension of the center. So let us investigate the structure of $Z(\mathbb{F}G)$ as a vector space.

Proposition 5.2. *There is a basis $b_C \in Z(\mathbb{F}G)$, where C runs over the set of conjugacy classes in G. It is given by $b_C := \sum_{g\in C} g$.*

Proof. The inclusion $\sum_{g\in G} c_g g \in Z(\mathbb{F}G)$ is equivalent to $h\sum_{g\in G} c_g g = \sum_{g\in G} c_g gh$, which in its turn is equivalent to $\sum_{g\in G} c_g(hgh^{-1}) = \sum_{g\in G} c_g g$. In other words, $\sum_{g\in G} c_g g \in Z(\mathbb{F}G)$ if and only if the function $g \mapsto c_g$ is constant on conjugacy classes. This implies the claim of the proposition. □

6. What happens when $\mathbb{F}$ is not algebraically closed

First, we need to modify the second part of the Schur lemma: the endomorphism algebra $\mathrm{End}_A(V)$ is a skew-field. An analog of the Burnside theorem still works: the image of A in $\mathrm{End}(V)$ is $\mathrm{End}_{\mathbb{S}}(V)$, where $\mathbb{S}$ is the skew-field $\mathrm{End}_A(V)$. The proof is somewhat more involved. The simple algebras are precisely $\mathrm{Mat}_n(\mathbb{S})$, where $\mathbb{S}$ is a finite dimensional skew-field over $\mathbb{F}$, the proof repeats that of Proposition 3.2. An analog of Proposition 3.3 holds for $\mathrm{Mat}_n(\mathbb{S})$. An analog of Proposition 4.2 holds too. The details are left to the reader.