Pset 5 solutions

P1: Let $U = \bigoplus_i U_i = \bigoplus U_j'$, where all U_i, U_j' are indecomposable. Let ι_i denote the inclusion $U_i \hookrightarrow U$, $\pi_i : U \twoheadrightarrow U_i$ be the projection, and ι_j', π_j' have the similar meaning.

Claim 1: $\forall i \exists j$ s.t. $\pi_j' \iota_i : U_i \to U_j'$ is an isomorphism.

Proof: Let us show that there is j s.t. $\pi_i \iota_j' \pi_j' \iota_i : U_i \to U_i$ is an isomorphism. Note that $\sum_j \iota_j' \pi_j' = id$. So $\sum_j \pi_i \iota_j' \pi_j' \iota_i = \pi_i \iota_i = id_{U_i}$. Since U_i is indecomposable, there is a maximal ideal $m \subset \operatorname{End}_A(U_i)$ consisting of nilpotent endomorphisms. We see that at least one of $\pi_i \iota_j' \pi_j' \iota_i$ doesn't lie in m hence is invertible. We further see that $\pi_j' \iota_i : U_i \to U_j'$ has ~~right~~ left inverse and hence U_i splits as a direct summand of U_j'. Since U_j' is indecomposable, this is impossible. $\square$

Claim 2: Let N, M, M' be finite dimensional A-modules ~~such that A~~. If $N \oplus M \simeq N \oplus M'$ then $M \simeq M'$ (here we assume that the base field is infinite).

Proof: Let $\varphi : M \oplus N \xrightarrow{\sim} M' \oplus N$ be an isomorphism and $\psi = \iota_N' \circ \pi_N : M \oplus N \to M' \oplus N$. Then $\varphi + t\psi : M \oplus N \to M' \oplus N$ is an isomorphism for all t but finitely many. Also $M' \oplus N = M' \oplus (\varphi + t\psi)(N)$ for infinitely many t. Replacing φ with $\varphi + t\psi$ we may assume that $\varphi(N) = N$. So φ induces $M = M \oplus N / N \xrightarrow{\sim} M' \oplus N / N = M'$. $\square$

To complete the proof of KS theorem we use Claims 1, 2 and easy induction.

P2: There is a source, say i, in Q. Then U given by $U_j = 0$, $j \neq i$, $U_i = V_i$ is a sub in a representation V. We conclude that either $V_i = 0$ or $V_j = 0$ for $j \neq i$. If $V_i = 0$ we can remove the vertex i and continue in this fashion. If $V_j = 0$ for $j \neq i$, then $\dim V_i = 1$ and we are done.

P3: a) We may assume that all $\alpha_i \neq 0$. What we need to prove is that ~~then,~~ the only indecomposable rep-n in this case has dimension $(1, \dots 1)$. Then all maps are bijections and we have one conjugacy class of representations.

$$\overset{V_1}{\circ} \xrightarrow{\varphi_1} \overset{V_2}{\circ} \xrightarrow{\varphi_2} \circ \longrightarrow \cdots \xrightarrow{\varphi_{m-1}} \overset{V_m}{\circ}$$

Let's prove that all φ_i are injective. The proof is by induction on i: assume that $\varphi_1 \ldots \varphi_{i-1}$ are injective, while φ_i is not. Let $U_i = \ker \varphi_i$, this is a proper subspace in V_i. Set $U_j = 0$ for $j > i$, $U_j = \varphi_j^{-1}(U_{j+1})$ for $j < i$, this is a sub. Now let U_i' be a complement to U_i in V_i. Set $U_j' = V_j$ for $j > i$ and $U_j' = \varphi_j^{-1}(U_{j+1}')$ for $j < i$. Then (U_j') is a sub and $(U_j) \oplus (U_j') = V_j$ (because φ_j w. $j < i$ are all injective) Contradiction.

We have proved that all φ_i are injective. Dualizing, we also get an indecomposable representation. We deduce that all φ_i are surjective. This completes the solution.

2) $\begin{tikzcd} V_1 \arrow[r, bend left, "A_1"] & V_2 \arrow[l, bend left, "A_2"] \end{tikzcd}$. Set $V = V_1 \oplus V_2$ and let $A = \begin{pmatrix} 0 & A_2 \\ A_1 & 0 \end{pmatrix}$. We view V as a $\mathbb{Z}/2\mathbb{Z}$-graded vector space and A as an endomorphism of deg 1. Clearly, the pair (A_1, A_2) is indecomposable $\Leftrightarrow$ V doesn't admit graded A-stable decomposition. Note that all spaces $\operatorname{Ker} A^k$, $\operatorname{Im} A^k$ are graded. Recall the Fitting decomp-n: $V = V_0 \oplus V_1$, where $V_0 = \bigcup \operatorname{Ker} A^k$, $V_1 = \bigcap \operatorname{Im} A^k$. It is graded & A-stable. So either $V_1 = V$, $V_0 = \{0\}$ or vice versa. In the first case, both A_1, A_2 are invertible. So we can identify V_1 & V_2 by means of A_1 and then our problem becomes to classify endomorphism $A_2 A_1^{-1}$ of V_2 — which is done by the Jordan normal form theorem. So suppose $V_1 = \{0\}$, $V_0 = V$. We can produce a Jordan basis in a way compatible w. decomp-n $V_1 \oplus V_2$. It follows that A has a single Jordan block. So when $|\dim V_1 - \dim V_2| \leq 1$ we have a single equivalence class of indecomposable representation, while for $|\dim V_1 - \dim V_2| > 1$, we have none. Therefore, $p(n,n) = 1$, $p(n, n\pm 1) = 0$ and there are no indecomposable representations for other dimension vectors.

3) $1 \underset{B}{\overset{A}{\rightrightarrows}} 2$: i) Classification result.

If B is invertible, we again reduce the problem to classifying indecomposable linear operators. We get an indecomposable representation to be denoted by $R_\alpha(n)$, where α is the eigenvalue of $B^{-1}A$. When A is invertible, we get an indecomposable representation $R_{\alpha^{-1}}(n)$, where α is the eigenvalue of $A^{-1}B$. So we get a family of indecomposables $R_\alpha(n)$ w. $\alpha \in \mathbb{P}^1$.

Let's construct rep-s to be denoted by $R(n,n+1)$, $R(n+1,n)$. Let $\dim V_1 = n$, $\dim V_2 = n+1$. Let v_j^i, $i=1..n$ be a basis in V_j, $j=1,2$. Define A, B by $A(v_1^i) = v_2^i$, $B(v_1^i) = v_2^{i+1}$. This gives an indecomposable rep to be denoted by $R(n,n+1)$. A rep-n $R(n+1,n)$ is obtained from $R(n,n+1)$ by passing to duals.

Thm: $R_\alpha(n)$, $R(n,n+1)$, $R(n+1,n)$ are the indecomposable reps of Q.

ii) We may assume $\ker A \neq \{0\}$ (if $\dim V_1 = \dim V_2$, this means A isn't invertible, if $\dim V_1 < \dim V_2$ we can dualize). If $\ker A \cap \ker B \neq \{0\}$, then the rep is decomposable. So our representation, R, contains a subrepresentation $R_0(1)$. Let's describe extensions $0 \to R_0(1) \to R \to R' \to 0$.

iii) Lem: if $A': V_1' \to V_2'$ is ~~invertible~~ injective, then R' splits.

Proof: $R_0(1)$ is given by $Au = 0$, $Bu = v$. Complete u to a basis in V_1: $u_1 .. u_k$. Set $v_i = Au_i$ and complete $v_1 .. v_k, v$ to a basis in V_2: $v_{k+1} .. v_\ell$. Since $Bv = v$, we can modify $u_1 .. u_k$ w/o changing $v_1 .. v_k$ such that $B\,\mathrm{Span}(u_1 .. u_k) \subset \mathrm{Span}(v_1 .. v_\ell)$. So $(\mathrm{Span}(u_i), \mathrm{Span}(v_i))$ splits □

iv) So if R is indecomposable, then the only summands of R' are $R_0(?)$, $R(?+1, ?)$ (here we use an induction to prove Thm). Note that we have inclusions $R_0(n) \hookrightarrow R(n+1,n)$ and projections $R(n+1,n+1) \twoheadrightarrow R(n+1,n)$ that restrict to isomorphisms of $\ker A$, these kernels are 1-dimensional. Thm reduces to the claim that R' is indecomposable (the only non-split extension of $R(n,n-1)$ (resp

$R(n,n)$) by $R_0(1)$ is $R(n+1,n)$ (resp. $R(n+1,n)$)

v) Lem: Let R be a representation of Q containing $R_0(1)$ as a sub. Assume $R' = R^1 \oplus R^2$, where $\ker A^1, \ker A^2$ are both 1-dim-l and there is a homomorphism $\iota: R^1 \to R^2$ that induces an isomorphism $\ker A^1 \xrightarrow{\sim} \ker A^2$. Then R is not indecomposable

Proof: $\mathrm{id} \oplus z\iota$ gives a one-parameter family of direct summands R^1 in R'. Pick some lifts u^1, u^2 of generating elements of $\ker A^1, \ker A^2$ to R. Note that we can replace u^1 with $u^1 + zu^2$ by varying an embedding $R^1 \hookrightarrow R'$. If $Au^1 = 0$, then the argument of the proof of Lemma iii shows that R' splits. We can achieve $Au^1 = 0$ if $Au^2 \neq 0$. But if $Au^2 = 0$, then R^2 splits $\square$

4) $$\begin{tikzcd} V_1 \arrow[dr, "A_1"] & & \\ & V & V_3 \arrow[l, "A_3"'] \\ V_2 \arrow[ur, "A_2"] & & \end{tikzcd}$$

If $\ker A_i \neq \{0\}$, then the rep is decomposable. So we may assume that all A_i's are embeddings. So we can view V_1, V_2, V_3 as subspaces in V and we need to classify indecomposable triples of subspaces

The case when one of subspaces is 0 is easy: here $\dim V = 1$ and we get dimensions $(1,0,0,0)$, $(1,1,0,0)$ (up to permuting the last 3 entries), $(1,1,1,0)$ (again, up to permutation)

Now let's consider the case when $V_1, V_2, V_3 \neq \{0\}$. We claim that $V_i \cap V_j = \{0\}$ for $i \neq j$. Assume the converse: let $v \in V_i \cap V_j$. If $v \in V_k$ then we pick a complement U to $\mathbb{C}v$ in V and get a decomposition $(V_1, V_2, V_3) = (\mathbb{C}v, \mathbb{C}v, \mathbb{C}v) \oplus (V_1 \cap U, V_2 \cap U, V_3 \cap U)$. So we can assume

proper if $\dim V > 1$.

that $V_i \cap V_j \cap V_k = \{0\}$. Then pick U complementing $\mathbb{C}v$ and containing V_k. Then we get decomposition $(V_1, V_2, V_3) = (\mathbb{C}v, \mathbb{C}v, 0) \oplus (V_1 \cap U, V_2 \cap U, V_3)$

So we only need to consider the case when $V_i \cap V_j = \{0\}$ for $i \neq j$

Dually, we can assume $V_i + V_j = V$ for $i \neq j$. (we can pass to V^* and the annihilators of V_i's).

So $V_i \oplus V_j = V$ $\forall i \neq j$. It follows that $\dim V_i = m$ $\forall i$, $\dim V = 2m$. Having fixed V_1, V_2, to give V_3 is to specify an isomorphism $A: V_1 \to V_2$ (so that $V_3 = \{v + A(v) \mid v \in V_1\}$). The condition that (V_1, V_2, V_3) is indecomposable is equivalent to A being indecomposable. This shows that $\dim V_i = 1$. All such triples are equivalent.

So the possible dimensions w. $\dim V_i \neq 0$ are $(2,1,1,1)$, $(1,1,1,1)$. We recover the positive root system for D_4.

P4 We start by reminding the structure of $\mu^{-1}(0) \subset V \oplus V^*$ (that appeared in Lec 17)

Recall that μ is given as follows: $\langle \mu(v, \alpha), x \rangle = \langle \alpha, xv \rangle$ $v \in V$, $\alpha \in V^*$, $x \in \mathfrak{g}$. Recall that we write V_k for $\{v \in V \mid \dim Gv = k\}$. Now consider $\pi: \mu^{-1}(0) \to V$, $(v, \alpha) \mapsto v$. For $v \in V_k$, $\pi^{-1}(v) \subset V^*$ is a subspace of dimension $\dim V - k$. It follows that

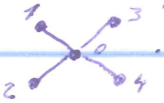
(1) $\dim \mu^{-1}(0) = \max\limits_k \{\dim V_k + \dim V - k\}$

(2) There is a bijection between the irreducible components of $\mu^{-1}(0)$ and the irreducible components of all V_k's

(1) implies the following:

(1') $\dim \mu^{-1}(0) = \dim V \iff G$ has finitely many orbits in V

To finish the proof, we note that if μ^* is the moment map $\mu^*: V^* \oplus V \to \mathfrak{g}$, $\mu^*(\alpha, v) = \langle v, \alpha x \rangle = -\langle \alpha, xv \rangle = -\mu(v, \alpha)$. So $\mu^{-1}(0) = \mu^{*-1}(0)$. We are done by (1') and (2)

P5. All matrices have quadratic minimal polynomial. So the quiver we get is $\widetilde{D}_4$: . We $v_0 = n$ and $v_i = \mathrm{rk}(Y_i - \lambda_i)$, where λ_i is one of eigenvalues of Y_i. The parameters are given by $\xi_0 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$, $\xi_i = \lambda_i' - \lambda_i$, where λ_i' is another root of the minimal polynomial of Y_i.

a) Here $v = \delta$ - the imaginary root. We have $p(\delta) = 1$ and $p(v) \le 0$ for any summand of v, $v \neq \delta$ (recall that $p(v) = 1 - \frac{1}{2}(v,v) = 1 - \sum_{i=0}^{4} v_i^2 + v_0(v_1 + v_2 + v_3 + v_4)$). We conclude that $\mathrm{Irr}(\Pi^\xi(\delta))$ is a non-empty 2-dim-l variety.

b) Here $v = 2\delta$ and $p(v) > \sum_{i=1}^{k} p(v^i)$ fails for $k=2$, $v^1 = v^2 = \delta$.

c) Take $\lambda_1 = 2a$, $\lambda_2 = b$, $\lambda_3 = c$, $\lambda_4 = d$. Then $v = 2\delta + \epsilon_1$ is a real root. But ξ is generic with $v \cdot \xi = 0$. Because of this, we have no proper decomposition $v = \sum_i v^i$ w. $v^i \cdot \xi = 0$. We see that $\mathrm{Irr}(\Pi^\xi(v))$ is a single point.

d) Here v is as in c) but $\xi = 0$. Decompose v into the sum of simple roots v^i. We have $p(v) = p(v^i) = 0$. So $p(v) > \sum_i p(v^i)$ fails and $\mathrm{Irr}(\Pi^\xi(v)) = \emptyset$.

A: a) infinitely many solutions

c) one solution

b,d) no solutions