APPLICATIONS OF PROCESI BUNDLES TO CHEREDNIK ALGEBRAS

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1. Introduction

In this talk we describe some applications of Procesi bundles that appeared in Gufang's talk to type A Rational Cherednik algebras introduced in Jose's talk. We start by recalling Procesi bundles, quantum Hamiltonian reductions, and Cherednik algebras. Then we apply Procesi bundles to relating the spherical Rational Cherednik algebras to quantum Hamiltonian reductions. Finally, we study the deformed derived McKay equivalence (that will be a derived equivalence between the categories of modules over a quantum Hamiltonian reduction and over a Rational Cherednik algebra).

2. Reminder

2.1. **Procesi bundles.** Let $\mathfrak{h}$ denote the reflection representation of $\mathfrak{S}_n$, this is a $\mathbb{C}$-vector space of dimension $n-1$ embedded into $\mathbb{C}^n$ as $\{(x_1,\ldots,x_n)|x_1+\ldots+x_n=0\}$. We consider the "normalized" Hilbert scheme X – the preimage of $\mathfrak{h}^2/\mathfrak{S}_n\subset\mathbb{C}^{2n}/\mathfrak{S}_n$ under the Hilbert-Chow morphism. Of course, $\operatorname{Hilb}_n(\mathbb{C}^2)=X\times\mathbb{C}^2$. The variety X is a conical symplectic resolution of $X_0:=\mathfrak{h}^2/\mathfrak{S}_n$, where the $\mathbb{C}^\times$-action is induced from the dilation action on $\mathbb{C}^2$.

On X, we have a $\mathbb{C}^\times$-equivariant bundle, $\mathcal{P}$, called the Procesi bundle constructed in Gufang's lecture. This bundle satisfies the following conditions:

1. As a graded $\mathbb{C}[X]=\mathbb{C}[\mathfrak{h}^2]^{\mathfrak{S}_n}$-algebra, the endomorphism algebra $\operatorname{End}(\mathcal{P})$ is identified with the skew-group algebra $\mathbb{C}[\mathfrak{h}^2]\#\mathfrak{S}_n$.
2. $\operatorname{Ext}^i(\mathcal{P},\mathcal{P})=0$ for $i>0$.
3. $\mathcal{P}^{\mathfrak{S}_n}=\mathcal{O}_X$.

This bundle gives rise to a derived equivalence

$$R\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{P},\bullet):D^b(\operatorname{Coh}X)\xrightarrow{\sim}D^b(\text{mod-}\mathbb{C}[\mathfrak{h}^{2n}]\#\mathfrak{S}_n)$$

(the target category is of right modules, but we can do left modules as well – using an antiautomorphism of $\mathbb{C}[\mathfrak{h}^{2n}]\#\mathfrak{S}_n$ that is the identity on $\mathfrak{h}^{2n}$ and is the inversion of $\mathfrak{S}_n$).

2.2. **Hamiltonian reductions.** In this seminar, we have seen multiple occurrences of Hamiltonian reduction. Namely, we considered the vector space $V=\mathfrak{sl}_n\oplus\mathbb{C}^n$ (we had $\mathfrak{gl}_n$ but now we adopt to our normalized setting). On this space we have a natural action of the group $G=\operatorname{GL}_n(\mathbb{C})$.

1) As Barbara checked, $X_0=T^*V{/\!/\!/}_0G(=\mu^{-1}(0)/\!/G)$ and $X=T^*V{/\!/\!/}^\theta G=\mu^{-1}(0)^{\theta-ss}/\!/G$, where $\theta=\det^{-1}$. The Hilbert-Chow morphism becomes a natural projective morphism $\pi:X\to X_0$.

2) Jose has considered Calogero-Moser spaces: $X_\lambda=T^*V{/\!/\!/}_\lambda G=\mu^{-1}(\lambda\cdot\operatorname{id})/\!/G$. We remark that all points in $\mu^{-1}(\lambda\cdot\operatorname{id})$ are θ-semistable and so, in particular, X_λ is

smooth and symplectic. We can also consider two families over $\mathbb{C}$ (with coordinate z): $\tilde{X}_0 := \mu^{-1}(\{z\cdot \mathrm{id}\})//G, \tilde{X} = \mu^{-1}(\{z\cdot \mathrm{id}\})^{\theta-ss}//G$.

3) Yi has constructed quantum Hamiltonian reductions on the level of algebras. Let us state the universal version of his construction. Consider the homogenized algebra $D_\hbar(V)$ of differential operators on V (where V, V^* have degree 1, and $\hbar$ has degree 2). We have a quantum comoment map $\Phi : \mathfrak{g} \to D_\hbar(V), x \mapsto x_V$. Consider the algebra $\mathcal{A}_{un} := [D_\hbar(V)/D_\hbar(V)\Phi([\mathfrak{g},\mathfrak{g}])]^G$. This is a free graded algebra over $\mathbb{C}[z,\hbar]$ deforming $\mathbb{C}[X_0]$. Its specialization to $\hbar = 0, z = \lambda$ gives $\mathbb{C}[X_\lambda]$. On the other hand, the specialization to $\hbar = 1, z = \lambda$ gives a usual quantum Hamiltonian reduction of $D(V)$. We remark that the algebra $\mathcal{A}_{un}$ is graded with $z, \hbar$ sitting in degree 2.

4) Finally, I have introduced the quantum Hamiltonian reduction on the sheaf level. I have considered the specialized version, but one has a universal version as well. Namely, we have a sheaf $\mathfrak{A}_{un} := D_\hbar(V)///^{\theta-ss}G := [D_\hbar(V)/D_\hbar(V)\Phi([\mathfrak{g},\mathfrak{g}])|_{T^*V^{\theta-ss}}]^G$. This is a sheaf of $\mathbb{C}[z,\hbar]$-algebras in conical topology on $\tilde{X}$ that quantizes the Poisson sheaf $\mathcal{O}_{\tilde{X}}$. The sheaf $\mathfrak{A}_{un}$ is $\mathbb{C}^\times$-equivariant. The sheaf $\mathfrak{A}_\lambda$ I have constructed is obtained as follows: we first specialize z to $\lambda = \hbar$ getting a sheaf on X and then specialize $\hbar = 1$. The global sections of $\mathfrak{A}_{un}$ coincide with $\mathcal{A}_{un}$.

2.3. **Cherednik algebras.** Jose have introduced the Rational Cherednik algebra H_{un} that is the quotient of $T(\mathfrak{h}\oplus\mathfrak{h}^*)\#\mathfrak{S}_n[t,c]$ by the following relations:

$$[x,x'] = 0 = [y,y'], [y,x] = \langle y,x\rangle t - c\sum_{i<j}(x_i - x_j)(y_i - y_j)(ij), \quad x,x' \in \mathfrak{h}^*, y,y' \in \mathfrak{h}.$$

In the last relation, we write $x = (x_1,\ldots,x_n), y = (y_1,\ldots,y_n)$ and (ij) denotes the transposition interchanging i and j in $\mathfrak{S}_n$. This algebra is graded with $\deg\mathfrak{S}_n = 0, \deg\mathfrak{h} = \deg\mathfrak{h}^* = 1, \deg c, t = 2$. Clearly, $H_{un}/(c,t) = S(\mathfrak{h}\oplus\mathfrak{h}^*)\#\mathfrak{S}_n$ (we remark that $S(\mathfrak{h}\oplus\mathfrak{h}^*)$ is naturally identified with $\mathbb{C}[\mathfrak{h}^2]$ because $\mathfrak{h}$ is self-dual).

What is more difficult is that H_{un} is graded free over $\mathbb{C}[c,t]$. Moreover, it is universal in the following sense: if P is a vector space and H' is a free graded $S(P)$-algebra (with P sitting in degree 2) such that $H'/(P) = S(\mathfrak{h}\oplus\mathfrak{h}^*)\#\mathfrak{S}_n$, then there is a unique linear map $\mathrm{Span}(c,t) \to P$ and a unique graded $S(P)$-linear isomorphism $S(P)\otimes_{\mathbb{C}[c,t]} H_{un} = H'$ that is the identity modulo P.

We can specialize t, c to numerical parameters and we will get filtered deformations $H_{t,c}$ of $S(\mathfrak{h}\oplus\mathfrak{h}^*)\#\mathfrak{S}_n$.

Inside H_{un} we can consider the spherical subalgebra $eH_{un}e$, where $e = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sigma$. This is an associative subalgebra with unit e that is a free graded deformation of $S(\mathfrak{h}\oplus\mathfrak{h}^*)^{\mathfrak{S}_n}$ over $\mathbb{C}[c,t]$. In fact (and this is important) it is not universal in the sense above but it should have some semi-universality property: for example, all filtered quantizations of $S(\mathfrak{h}\oplus\mathfrak{h}^*)^{\mathfrak{S}_n}$ are expected to be of the form $eH_{1,c}e$.

3. Spherical subalgebras vs Hamiltonian reductions

3.1. **Main result.** Now we have two free graded deformations of $S(\mathfrak{h}\oplus\mathfrak{h}^*)^{\mathfrak{S}_n}$: the quantum Hamiltonian reduction $\mathcal{A}_{un}$ (an algebra over $\mathbb{C}[z,\hbar]$) and the spherical Cherednik algebra $eH_{un}e$ (over $\mathbb{C}[c,t]$). In fact, these two algebras are isomorphic (we have seen partial results in this direction in Jose's talk: after specialization to $\hbar = 0$ and $t = 0$).

Theorem 3.1. *There is an isomorphism $eH_{un}e \xrightarrow{\sim} \mathcal{A}_{un}$ with the following properties:*

- *it preserves gradings;*

- *it maps t to $\hbar$ and c to $-z$;*
- *the induced endomorphism of $eH_{un}e/(c,t) = S(\mathfrak{h}\oplus\mathfrak{h}^*)^{\mathfrak{S}_n} = \mathcal{A}_{un}/(z,t)$ is the identity.*

Such an isomorphism is unique.

This theorem is proved in [L1]. This result was essentially known before, due to Gan and Ginzburg, [GG]. An advantage of an approach in [L1] that it generalizes easily to the case of so called wreath-product groups.

3.2. **Deformation from Procesi bundle.** We want to show that there is a linear map $\nu : \mathrm{Span}(c,t) \to \mathrm{Span}(z,\hbar)$ such that

$$\mathcal{A}_{un} = \mathbb{C}[z,\hbar] \otimes_{\mathbb{C}[c,t]} eH_{un}e. \tag{3.1}$$

The problem is that $eH_{un}e$ does not have a universality property, it is H_{un} that does. So we need to produce an algebra $\tilde{\mathcal{A}}_{un}$ that deforms $S(\mathfrak{h}\oplus\mathfrak{h}^*)\#\mathfrak{S}_n$ and such that $e\tilde{\mathcal{A}}_{un}e = \mathcal{A}_{un}$.

This is where we use the Procesi bundle. Recall that axiom (2) implies that $\mathcal{P}$ uniquely lifts to a left module over any formal deformation of $\mathcal{O}_X$, commutative or non-commutative, does not matter. If the deformation of $\mathcal{O}_X$ is $\mathbb{C}^\times$-equivariant, then the deformation of $\mathcal{P}$ is also $\mathbb{C}^\times$-equivariant (and in a unique way). For example, we can take the $(z,\hbar)$-adic completion of $\mathfrak{A}_{un}$ for a deformation of $\mathcal{O}_X$. Then, thanks to the contracting $\mathbb{C}^\times$-action, we can extend the lifting to $\mathfrak{A}_{un}$ itself. Let $\tilde{\mathcal{P}}_\hbar$ be that extension, this is a $\mathbb{C}^\times$-equivariant left $\mathfrak{A}_{un}$-module with $\tilde{\mathcal{P}}_\hbar/(z,\hbar) = \mathcal{P}$. The condition $\mathrm{Ext}^1(\mathcal{P},\mathcal{P}) = 0$ further implies that $H_{\mathcal{P}} := \mathrm{End}_{\mathfrak{A}_{un}}(\tilde{\mathcal{P}}_\hbar)^{opp}$ is a flat deformation of $\mathrm{End}(\mathcal{P})$ that by axiom (1) coincides with $S(\mathfrak{h}\oplus\mathfrak{h}^*)\#\mathfrak{S}_n$. It follows that there is a unique map $\nu_{\mathcal{P}} : \mathrm{Span}(c,t) \to \mathrm{Span}(z,\hbar)$ such that

$$H_{\mathcal{P}} = \mathbb{C}[z,\hbar] \otimes_{\mathbb{C}[c,t]} H_{un}.$$

Further, by axiom (3), $\mathcal{O}_X$ is a direct summand of $\mathcal{P}$. Because of the uniqueness of the deformation, there is a direct summand of $\tilde{\mathcal{P}}_\hbar$ that deforms $\mathcal{O}_X$. But $\mathfrak{A}_{un}$ is a deformation of $\mathcal{O}_X$. So $\mathfrak{A}_{un}$ is a direct summand of $\tilde{\mathcal{P}}_\hbar$, let e be the projector. Then e is a degree 0 element in $H_{\mathcal{P}}$ and $eH_{\mathcal{P}}e = \mathrm{End}_{\mathfrak{A}_{un}}(\tilde{\mathcal{P}}_\hbar e) = \Gamma(\mathfrak{A}_{un}) = \mathcal{A}_{un}$.

So we get $\mathcal{A}_{un} = \mathbb{C}[z,\hbar] \otimes_{\mathbb{C}[c,t]} eH_{un}e$, as required.

3.3. **Correspondence between parameters.** What remains to prove is that one can take the linear map $\nu : \mathrm{Span}(c,t) \to \mathrm{Span}(z,\hbar)$ in (3.1) as in the theorem. The uniqueness of the isomorphism there is a nice problem that is left to the audience.

First thing to notice is that $\nu(t) = \hbar$. This is because the bracket on $S(\mathfrak{h}\oplus\mathfrak{h}^*)^{\mathfrak{S}_n}$ induced by the deformation $\mathcal{A}_{un}$ is $\hbar\{\cdot,\cdot\}$, while $eH_{un}e$ induces the bracket $t\{\cdot,\cdot\}$.

The second idea is to reduce the proof to the first interesting case: $n = 2$. This works to some extent. For this we take a point x in the stratum of $\mathfrak{h}^2/\mathfrak{S}_n$ corresponding to just two points equal and take the completions of $\mathcal{A}_{un}(n)$ and $eH_{un}e$ at that point x (the definition of the completion as an inverse limit still makes perfect sense). The algebra $\mathcal{A}_{un}(n)^{\wedge_x}$ splits into the completed tensor product of the formal Weyl algebra $W_\hbar^{\wedge_0}$ (quantization of the symplectic leaf through x) and of $\mathcal{A}_{un}(2)^{\wedge_0}$ (quantization of the slice that is $\mathbb{C}^2/\{\pm 1\}$): $\mathcal{A}_{un}(n)^{\wedge_x} = \mathrm{W}_\hbar^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\mathcal{A}_{un}(2)^{\wedge_0}$. Similarly, $(eH_{un}(n)e)^{\wedge_x} = \mathrm{W}_\hbar^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[t]]}(eH_{un}(2)e)^{\wedge_0}$. One can show that (3.1) gives rise to an isomorphism with $n = 2$ having the same linear map ν.

The case $n = 2$ is easy enough to be analyzed by hand. What we get is two possibilities $\nu(c) = -z, \nu(c) = z - \hbar$. To complete the proof it is enough to produce an automorphism, φ, of $eH_{un}(n)e$ with the following properties:

(i) It preserves the grading and the subspace $\operatorname{Span}(c, t)$.
(ii) It induces the identity map on $S(\mathfrak{h} \oplus \mathfrak{h}^*)^{\mathfrak{S}_n}$.
(iii) It preserves t and maps c to $-c - t$.

It is possible to do this but all constructions I know are sort of implicit. Also one can show that the group of automorphisms satisfying (i) and (ii) is just $\mathbb{Z}/2\mathbb{Z}$.

One pleasant consequence of the last fact is that there are two bundles on X satisfying (1)-(3): $\mathcal{P}$ and $\mathcal{P}^*$, see [L2] for details.

4. Derived equivalences

4.1. **Main result.** Recall that we have a derived equivalence $R\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{P}, \bullet) : D^b(\operatorname{Coh} X) \to D^b(\text{mod-} S(\mathfrak{h} \oplus \mathfrak{h}^*)\#W)$. We are going to produce its deformed analog. Let us observe that since $\mathcal{P}$ is $\mathbb{C}^\times$-equivariant, this equivalence upgrades to an equivalence of the derived categories of $\mathbb{C}^\times$-equivariant modules.

The target category will be of finitely generated graded H_{un}-modules. For the source category, we take that of coherent $\mathbb{C}^\times$-equivariant $\mathfrak{A}_{un}$-modules. Those are sheaves (in conical topology) of graded $\mathfrak{A}_{un}$-modules that are locally finitely generated. In particular, for such a module M, the quotient $M/(\hbar, z)M$ is a coherent sheaf.

Theorem 4.1 ([GL]). *The functors $R\operatorname{Hom}_{\mathfrak{A}_{un}}(\tilde{\mathcal{P}}_\hbar, \bullet)$ and $\tilde{\mathcal{P}}_\hbar \otimes^L_{H_{un}} \bullet$ define mutually inverse equivalences between $D^b(\operatorname{Coh}^{\mathbb{C}^\times} \mathfrak{A}_{un})$ and $D^b(H_{un}\text{-grmod})$.*

This is, more or less, or less a formal consequence of the equivalence on the undeformed level, see [GL, Section 5] for technical details.

4.2. **Ramifications.**

4.2.1. *Specialization.* First, we can specialize $\hbar, z$. The specialization $\hbar = 0$ will be useful in Kostya's talk. We can also plug numerical parameters: in this way we get an equivalence $D^b(\operatorname{Coh} \mathfrak{A}_\lambda) \to D^b(H_{1,-\lambda}\text{-mod})$ (if we use $\mathcal{P}$) and $D^b(\operatorname{Coh} \mathfrak{A}_\lambda) \to D^b(H_{1,\lambda-1}\text{-mod})$ (if we use $\mathcal{P}^*$) – there are two choices of Procesi bundles and they give rise to the two different maps between the spaces of parameters. The category $\operatorname{Coh} \mathfrak{A}_\lambda$ consists of all coherent $\mathfrak{A}_\lambda$-modules, where "coherent" means: having a global (complete and separated) filtration such that the associated graded module is coherent.

4.2.2. *Supports.* A nice property of this equivalence is that it preserves supports. Namely, for an $H_{1,c}$-module M we can define its support as follows. Take a filtration on M such that $\operatorname{gr} M$ is a finitely generated $S(\mathfrak{h} \oplus \mathfrak{h}^*)\#\mathfrak{S}_n$-module. Then view $\operatorname{gr} M$ as a $\mathbb{C}[X_0]$-module (via the inclusion $\mathbb{C}[X_0] \hookrightarrow S(\mathfrak{h} \oplus \mathfrak{h}^*)\#\mathfrak{S}_n$) and we can consider its support (=the variety defined by the annihilator) in X_0. The same thing is declared to be the support of M, it is an exercise to check that this is independent of the choice of the filtration. The support of a $\mathfrak{A}_\lambda$-module M is defined in a similar fashion, this is a subvariety in X.

Now, for a $\mathbb{C}^\times$-stable subvariety $Y_0 \subset X_0$ consider the category $D^b_{Y_0}(H_{1,c}\text{-mod})$ of all complexes in $D^b(H_{1,c}\text{-mod})$ whose homology is supported on Y_0. Similarly, for a $\mathbb{C}^\times$-stable subvariety $Y \subset X$, we can define the category $D^b_Y(\operatorname{Coh} \mathfrak{A}_\lambda)$.

Then the equivalence $D^b(\operatorname{Coh} \mathfrak{A}_\lambda) \to D^b(H_{1,c}\text{-mod})$ restricts to $D^b_{\pi^{-1}(Y_0)}(\operatorname{Coh} \mathfrak{A}_\lambda) \to D^b_{Y_0}(H_{1,c}\text{-mod})$.

4.2.3. *One more torus.* So far, we have considered the contracting torus action, let us write T_c for that copy of $\mathbb{C}^\times$. We also have Hamiltonian torus actions on X, X_0 (and also on $\mathfrak{A}_\lambda, H_c$). The action on $\mathfrak{A}_\lambda$ is induced from the action on $D(V)$, which in turn comes from the dilation action on V. The action on H_c is given by $t.x = tx, t.y = t^{-1}y, t.w = w$ (the corresponding quantum comoment map sends $1 \in \mathbb{C}$ to the Euler element h constructed by Jose). Let us denote this Hamiltonian torus by T_h. The Procesi bundle is T_h-equivariant (for the same reasons as were explained in Gufang's talk) and one can achieve that the isomorphism $\operatorname{End}(\mathcal{P}) \cong S(\mathfrak{h} \oplus \mathfrak{h}^*)\#\mathfrak{S}_n$ is T_h-equivariant as well. Then all our constructions become additionally T_h-equivariant and we get an equivalence $D^b(\operatorname{Coh}^{T_h} \mathfrak{A}_\lambda) \cong D^b(\mathrm{H}_{1,c}\text{-mod}^{T_h})$.

4.2.4. *Categories $\mathcal{O}$.* The previous two ramifications allow to get equivalences between categories $\mathcal{O}$. Namely, let us point out that the category $\mathcal{O}$ for $H_{1,c}$ can be characterized as the category of all modules supported on $\mathfrak{h}/\mathfrak{S}_n \subset (\mathfrak{h} \oplus \mathfrak{h}^*)/\mathfrak{S}_n$ that can be made equivariant for the $\mathbb{C}^\times$-action. We can *define* the category $\mathcal{O}$ for $\mathfrak{A}_\lambda$ in a similar fashion, we require that the support is contained in $\pi^{-1}(\mathfrak{h}/\mathfrak{S}_n)$. We can further define the subcategories $D^b_{\mathcal{O}}(\operatorname{Coh}(\mathfrak{A}_\lambda)) \subset D^b(\operatorname{Coh}(\mathfrak{A}_\lambda)), D^b_{\mathcal{O}}(H_{1,c}\text{-mod}) \subset D^b(H_{1,c}\text{-mod})$ and we get equivalences between these categories. One can check that the natural functor $D^b(\mathcal{O}(H_{1,c})) \to D^b_{\mathcal{O}}(H_{1,c}\text{-mod})$ is an equivalence (and the same is true on the $\mathfrak{A}_\lambda$ side as well). So we get a derived equivalence between $D^b(\mathcal{O}(\mathfrak{A}_\lambda)), D^b(\mathcal{O}(H_{1,c}))$.

4.3. **Applications.** We want to get some applications of this construction: one about Cherednik algebras that does not mention quantum Hamiltonian reductions and the other about quantum Hamiltonian reductions that does not mention Cherednik algebras.

4.3.1. *Translation equivalences.* One is based on translation equivalences between $\operatorname{Coh} \mathfrak{A}_\lambda$ and $\operatorname{Coh} \mathfrak{A}_{\lambda+m}$ for integral m. Those are based on quantizations of line bundles. Namely, let χ be a character of G. Let us write ρ for the quotient morphism $\mu^{-1}(0)^{ss} \to X$. For a character χ of G we can define the line bundle

$$\mathcal{O}(\chi) = (\rho_* \mathcal{O}_{\mu^{-1}(0)^{ss}})^{G,\chi}$$

where the superscript G, χ means taking (G, χ)-semiinvariants, sections s satisfying $gs = \chi(g)s$. The line bundle $\mathcal{O}_\chi$ quantizes to the $\mathfrak{A}_{\lambda+\chi}$-$\mathfrak{A}_\lambda$-bimodule

$$\mathfrak{A}_\lambda(\chi) = (\rho_*[D(R)/D(R)\{x - \langle \lambda, x\rangle\}|_{T^*R^{\theta-ss}}])^{G,\chi}.$$

The multiplication by $\mathfrak{A}_\lambda$ from the right should be clear. To check that we can multiply by elements from $\mathfrak{A}_{\lambda+\chi}$ from the left we need to check that, for a local section s of $\mathcal{A}_\lambda(\chi)$, we have $(x_R - \langle \lambda + \chi, x\rangle)s = 0$. Since $x \mapsto x_R$ is a quantum comoment map, we have $[x_R, s] = \langle \chi, x\rangle s$. Also $s(x_R - \langle \lambda, x\rangle) = 0$. Our claim follows.

It is an exercise to check that the associated graded of $\mathfrak{A}_\lambda(\chi)$ is $\mathcal{O}(\chi)$. Further, we have a natural homomorphism

$$\mathfrak{A}_{\lambda+\chi}(-\chi) \otimes_{\mathfrak{A}_{\lambda+\chi}} \mathfrak{A}_\lambda(\chi) \to \mathfrak{A}_\lambda \tag{4.1}$$

whose associated graded is an isomorphism $\mathcal{O}(-\chi) \otimes \mathcal{O}(\chi) \xrightarrow{\sim} \mathcal{O}$ so (4.1) is an isomorphism as well.

It follows that $\mathfrak{A}_\lambda(\chi) \otimes_{\mathfrak{A}_\lambda} \bullet$ is an equivalence $\operatorname{Coh} \mathfrak{A}_\lambda \xrightarrow{\sim} \operatorname{Coh} \mathfrak{A}_{\lambda+\chi}$. It gives rise to a derived equivalence $D^b(H_{1,c}\text{-mod}) \xrightarrow{\sim} D^b(H_{1,c+\chi}\text{-mod})$ descending to the derived equivalences between categories $\mathcal{O}$.

4.3.2. *Derived localization.* Another application does not mention Cherednik algebras. Consider the derived global section functor $R\Gamma : D^b(\operatorname{Coh}\mathfrak{A}_\lambda) \to D^b(\mathcal{A}_\lambda\text{-mod})$. Let us notice that this functor equivalent to $eR\operatorname{Hom}(\tilde{\mathcal{P}}_{1,\lambda}, \bullet)$. The $R\operatorname{Hom}$-functor is a derived equivalence independently of the parameter. The functor $M \mapsto eM$ is always exact and is an equivalence if $-\lambda = c$ is not a rational number with denominator $\leqslant n$ lying in $(-1, 0)$. Our conclusion is the derived localization theorem: $R\Gamma$ is a derived equivalence if λ is not a rational number in $(0, 1)$ with denominator $\leqslant n$.

4.3.3. *Images of Verma modules.* Perhaps the most interesting application: to Macdonald positivity, will be discussed in Kostya's talk. This application is based on computing the images of Verma modules for H_{un} in $D^b(\operatorname{Coh}\mathfrak{A}_{un})$ (for example, we will see that these images are concentrated in homological degree 0). This description can also be used to establish the abelian localization theorem that will be a subject of my talk.

References

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