In this talk we compute the quantum connection of T^*G/B, identify its monodromy with the action of the affine braid group from Mitya Vaintrob's talk, and briefly discuss the heuristics for this identification. The material is almost entirely from [1], but of course all errors are mine.

1 Review of the geometry of T^*G/B

Let G be a complex semisimple simply-connected Lie group. We write T for a maximal torus, and g, t for the respective lie algebras.

1.1 Symplectic Resolutions

Let X be a smooth algebraic variety equipped with a holomorphic symplectic form Ω. Let X_0 be the affinization of X, i.e. the spectrum of the ring of algebraic functions on X. There is a natural map $\pi : X \to X_0$.

Definition: We call X a symplectic resolution if π is birational and proper, and there is an action of $\mathbb{C}^*$ which dilates the symplectic form by a nonzero character $\hbar$.

Proposition: T^*G/B, equipped with the canonical holomorphic symplectic form and the $\mathbb{C}^*$ action dilating the cotangent fibers, is a symplectic resolution. Its affinization is $X_0 = \mathcal{N}$, the cone of nilpotent elements in g.

For more details, see [2]. Many of the techniques we use below apply to a general symplectic resolution: other examples include T^*G/P for a general parabolic P, resolutions of slices to nilpotent orbits, $Hilb_n(\mathbb{C}^2)$ and more generally Nakajima quiver varieties, and hypertoric varieties.

1.2 Cohomology and Curve Classes

G acts on G/B, hence acts symplectically on T^*G/B. Let $\mathbf{G} = G \times \mathbb{C}^*$, where $\mathbb{C}^*$ acts on T^*G/B by dilating the cotangent fibers by a character $\hbar$. We have

$$\begin{aligned} H^*_{\mathbf{G}}(T^*G/B, \mathbb{Z}) &= H^*_{\mathbf{G}}(G/B) \\ &= H^*_G(G/B) \otimes H^*_{\mathbb{C}^*}(pt) \\ &= H^*_B(pt) \otimes H^*_{\mathbb{C}^*}(pt) \\ &= H^*_T(pt) \otimes H^*_{\mathbb{C}^*}(pt) \\ &= Sym(P) \otimes \mathbb{Z}[\hbar] \\ &= \mathbb{Z}[u_1, ..., u_n] \otimes \mathbb{Z}[\hbar] \end{aligned}$$

where T is a maximal torus for G, n is the rank of G and P is the weight lattice. In particular, we have $H^2(X, \mathbb{Z}) = P$ and $H^2(X, \mathbb{C}) = t^*$, where $t = Lie(T)$.

Every positive coroot $\alpha^\vee \in Hom(P, \mathbb{Z}) = H_2(X, \mathbb{Z})$ corresponds an SL_2 subgroup $G_{\alpha^\vee} \subset G$. Its orbits in G/B are rational curves of class $\alpha^\vee$. These generate the effective cone of T^*G/B.

1.3 Action of the graded affine Hecke Algebra on cohomology

In Yi Sun's talk, we saw an action of the Weyl group W on the cohomology of Springer fibers. Here we describe the equivariant analogue of this action on the Springer fiber G/B; we will use it later to describe quantum multiplication.

Recall that $Z = T^*G/B \times_{\mathcal{N}} T^*G/B$ is a union of lagrangians in $T^*G/B \times T^*G/B$ indexed by the Weyl group. Any class $\gamma \in H^*_{\mathbf{G}}(Z)$ (thought of as equivariant Borel-Moore homology) defines an endomorphism of $H^*_{\mathbf{G}}(T^*G/B)$ by

$$\gamma(\theta) = (\pi_2)_*\gamma \cap \pi_1^*\theta.$$

As before, these endomorphisms form an algebra under convolution. Recall that the degenerate affine Hecke algebra $\mathcal{H}'$ is generated by the symmetric algebra $Sym(t^*) = \mathbb{C}[u_1, ..., u_n]$ and the group algebra $\mathbb{C}W$, subject to the relation

$$s_i u - s_i(u) s_i = \hbar(\alpha_i, u)$$

for any simple reflection s_i and linear generator $u \in t^*$. The definition we use in this chapter is slightly narrower than the one previously introduced.

Theorem: [3] There is an isomorphism $\phi : \mathcal{H}' \xrightarrow{\sim} H^*_{\mathbf{G}}(Z)$, where the RHS is viewed as a convolution algebra. We have

$$\phi(u_\lambda) = c_1(L_\lambda), \lambda \in P$$

where L_λ is the equivariant line bundle associated to λ, supported along the diagonal component of Z. We also have

$$\phi(s_i - 1) = [Z_i]$$

where s_i is a simple reflection, and Z_i is defined as follows. Let $P_i \subset G$ be the parabolic corresponding to s_i. Let $Y_i = G/B \times_{G/P_i} G/B$. Then $Z_i = N^*_Y i$, the conormal bundle of Y in $T^*G/B \times T^*G/B$.

We therefore have an action of $\mathcal{H}'$ on $H^*_{\mathbf{G}}(T^*G/B) = \mathbb{C}[u_1, ..., u_n] \otimes \mathbb{C}[\hbar]$. Lusztig also describes the representation explicitly:

Theorem: [3] Under ϕ, u_λ acts by multiplication, while s_i acts by the following 'discrete derivative':

$$(1 - s_i)f(u) = (f(u) - f(s_i(u)))\left(1 - \frac{\hbar}{\alpha_i}\right)$$

1.4 Poisson deformations of a symplectic resolution

The poisson deformations of X are classified by the image of Ω in $H^2(X, \mathbb{C})$. In our case, $H^2(T^*G/B, \mathbb{C}) = t^*$, and the space of poisson deformations coincides with the Grothendieck simultaneous resolution. Non-affine deformations live over certain 'root hyperplanes' $H_\alpha \subset H^2(X, \mathbb{C})$; for T^*G/B, these are the usual root hyperplanes.

The fiber over a generic point of t^* is the affine space G/T, whereas T^*G/B is the fiber over zero. The fiber X_α over a generic point of a root hyperplane H_α is described as follows:

Let $T_\alpha \subset T$ be the kernel of α, and let $L_\alpha \subset G$ be the centralizer of T_α We have an exact sequence

$$1 \to T \to L_\alpha \to PGL(2, \mathbb{C}) \to 1$$

which defines an action of L_α on $T^*\mathbb{P}^1$, via the action of $PGL(2, \mathbb{C})$.

Proposition: The generic fiber over the hyperplane H_α is given by

$$X_\alpha = G \times_{L_\alpha} T^*\mathbb{P}^1.$$

2 Computing the quantum cohomology of a symplectic resolution

2.1 Triviality of non-equivariant quantum cohomology

The ordinary quantum cohomology of a symplectic resolution is equal to the classical cohomology ring. In our case, this can be seen directly, since T^*G/B deforms to the affine space G/T, whose Gromov-Witten invariants vanish (recall that Gromov-Witten invariants are invariant under deformations of the complex structure).

However, there is no $\mathbb{C}^*$-equivariant deformation to an affine space, and in fact the $\mathbb{C}^*$-equivariant quantum cohomology is non-trivial, as we will see.

2.1.1 Quantum product from the deformation

For a divisor u and $\beta \neq 0$, we have

$$\langle \gamma_1, u, \gamma_2 \rangle_{0,3,\beta}^{X,\mathbf{G}} = (u, \beta) \langle \gamma_1, \gamma_2 \rangle_{0,2,\beta}^{X,\mathbf{G}}$$

by the divisor equation (the superscript $\mathbf{G}$ indicates equivariant invariants). If cohomology is generated by divisors, as is the case for T^*G/B, the quantum cohomology is thus determined by the two-point invariants.

One can rewrite them as follows. We have

$$\langle \gamma_1, \gamma_2 \rangle_{0,2,\beta}^{X,\mathbf{G}} = \langle L_\beta(\gamma_1), \gamma_2 \rangle^{\mathbf{G}}.$$

where

$$L_\beta = (\mathrm{ev}_1 \times \mathrm{ev}_2)_* [\mathcal{M}_{0,2}(X, \beta)]^{vir} \in H_{2\dim X - 1}^{BM,\mathbf{G}}(X \times X, \mathbb{C}).$$

Our task is therefore to characterize L_β. We know that L_β vanishes in ordinary cohomology for $\beta \neq 0$, i.e. it should be divisible by $\hbar$. To understand the quotient by $\hbar$, we use the deformations of T^*G/B.

Choose a line $l = \mathbb{C} \subset t^*$ through the origin, not contained in any root hyperplane. Consider the total space $\pi : X(l) \to l$ of the deformation of T^*G/B over this line; π has fiber $T^*G/B = X$ over 0, and G/T everywhere else. Since there is only one non-affine fiber, we have

$$\mathcal{M}_{0,2}(X(l), \beta) = \mathcal{M}_{0,2}(X, \beta).$$

However, the virtual fundamental classes differ:

Theorem: [1]

$$[\mathcal{M}_{0,2}(X, \beta)]^{vir} = \hbar[\mathcal{M}_{0,2}(X(l), \beta)]^{vir}$$

We write

$$(ev_1 \times ev_2)_*[\mathcal{M}_{0,2}(X(l), \beta)]^{vir} = L_\beta^{red}.$$

The image of a rational curve C must lie in a single fiber of the affinization map, since all algebraic functions are constant on C. It follows that L_β and L_β^{red} are supported on the Steinberg variety. Moreover, recall that

$$dim[\mathcal{M}_{0,n}(X(l), \beta)]^{vir} = dim X(l) + c_1(TX(l), \beta) + n - 3.$$

Hence we have

$$dim L_\beta^{red} = dim X + 1 + 0 + 2 - 3 = dim X.$$

It follows that L_β^{red} must be a linear combination of components of Z with rational coefficients. Using the isomorphism ϕ, we schematically write

$$L_\beta^{red} = \sum_{w \in W} c_w w$$

where $c_w \in \mathbb{Q}$. We must now determine the coefficients c_w. Since we are looking for rational numbers, it is enough to work in non-equivariant cohomology. This allows us to perturb l, which we do in the next section.

2.2 Reduction to rank 1

Choose a generic shift $l + a$ of l, which intersects the hyperplanes H_α in distinct points. Let $\pi_a : X(l + a) \to l + a$ be the family above $l + a$.

The generic fiber is again G/T, while the fibers X_α over the intersections with H_α are $T^*\mathbb{P}^1$ fibrations over G/L_α, as described in **??**. Each such fiber X_α contains a unique primitive curve class $\alpha^\vee$ corresponding to a positive root.

Invariance of Gromov-Witten invariants with respect to deformations of the complex structure implies

$$\langle \gamma_1, \gamma_2 \rangle_{0,2,\beta}^{X(l)} = \langle \gamma_1, \gamma_2 \rangle_{0,2,\beta}^{X(l+a)}$$

Since the domain of a stable map is connected, its image in $X(l + a)$ must lie in a single fiber X_α, hence the only curve classes which contribute are multiples $m\alpha^\vee$ of the positive root classes.

One must hence compute the class $L_{m\alpha}$ where α is a positive root. The Steinberg variety of X_α has two components: the diagonal Δ and the fiber product of the natural $\mathbb{P}^1$ fibration over G/L_α, which we denote Z_α. We have

$$L_{m\alpha} = c_0\Delta + c_1 Z_\alpha \in H^{BM}_{dimX}(X_\alpha \times X_\alpha)$$

One can show [1] that the computation of c_0, c_1 reduces to that for a single fiber $T^*\mathbb{P}^1$ above G/L_α. We have already seen the answer:

$$c_0 = 0, c_1 = \frac{1}{m}.$$

We now describe the action of L_α on $H^*_{\mathbf{G}}(X, \mathbb{C})$. More precisely, we have an equality of non-equivariant cohomology

$$H^*(X_\alpha, \mathbb{C}) = H^*(X, \mathbb{C})$$

which allows us to identify Z_α with a class in $H^*(X \times X)$. This class is a unique rational linear combination of components of the Steinberg variety of X. Then L_α is the natural equivariant lift of this rational linear combination. One sees

$$L_\alpha = \phi(s_\alpha - 1) \tag{1}$$

We can now describe the operator of quantum multiplication by a divisor u. By the above, we have

$$u* = u + \hbar \sum_{\alpha\in R^+} \sum_m (u, m\alpha^\vee) q^{m\alpha^\vee} \frac{1}{m}(s_\alpha - 1) \tag{2}$$

Since the sum over m is a geometric series, we can write the analytic continuation of the quantum product as

$$u* = u + \hbar \sum_{\alpha\in R^+} (u, \alpha^\vee) \frac{q^{\alpha^\vee}}{1 - q^{\alpha^\vee}} (s_{\alpha^\vee} - 1) \tag{3}$$

3 The Monodromy of the Quantum Connection

Formula 3 shows that the quantum connection of T^*G/B is

$$\nabla_u = \frac{d}{du} - u - \hbar \sum_{\alpha\in R^+} (u, \alpha^\vee) \frac{q^{\alpha^\vee}}{1 - q^{\alpha^\vee}} (s_\alpha - 1) \tag{4}$$

It is a meromorphic connection on the trivial vector bundle E with fiber $H^*_{\mathbf{G}}(X, \mathbb{C})$ and base the torus $H^2(T^*G/B, \mathbb{C})/H^2(T^*G/B, \mathbb{Z}) = t^*/P = T^\vee$, i.e. the adjoint torus for the Langlands dual group G^L. It is nonsingular on

$$(T^\vee)^{reg} = T^\vee \setminus \ \{q^{\alpha^\vee} = 1\}$$

After a gauge transform given by the function $\delta^\hbar = \prod_{\alpha\in R^+}(q^{\alpha^\vee} - 1)^\hbar$, it becomes the affine KZ connection studied by Cherednik and Matsuo:

$$\nabla'_u = \frac{d}{du} - u - \hbar \sum_{\alpha\in R^+} (u, \alpha^\vee) \frac{s_\alpha}{q^{\alpha^\vee} - 1}.$$

We recall a few facts from Yaping Yang's talk. There is an action of the Weyl group on the torus which lifts to an action on E, with respect to which ∇' is equivariant. Hence it descends to a connection on the quotient $(T^\vee)^{reg}/W$. We have

$$\pi_1(T^\vee)^{reg}/W) = \hat{B}_{g^L}$$

(since the action of W is not free, this is an orbifold π_1) and the monodromy of ∇' factors through the map

$$\hat{B}_{g^L} \to \mathbb{H}_{g^L}$$

to the affine Hecke algebra (note: this is the usual affine Hecke algebra, not the graded one).

Similarly, the action of $\hat{B}_{g^L}$ on $D^b Coh_{\mathbf{G}}(T^*G/B)$ described in chapter **??** descends to an action on K-theory which again factors through $\mathbb{H}_{g^L}$. Using an appropriate character map from K-theory to cohomology, one can show that the two actions coincide.

3.1 Commuting difference equation

As we have just seen, the monodromy of the quantum connection comes from an action on K-theory. In particular, the K-theoretic action is linear over $K_{\mathbf{G}}(pt) = Rep(\mathbf{G})$, the ring of finite dimensional representations of $\mathbf{G}$, and any monodromy operator can be written as a matrix with entires in $K_{\mathbf{G}}(pt)$. Under the character map to cohomology, an element $V \in K_{\mathbf{G}}(pt)$ is quite literally sent to its character. For instance, the basic representation of $\mathbb{C}^*$ is sent to $e^{2\pi i\hbar}$. In particular, the monodromy matrix is invariant under shifts $\hbar \to \hbar + 1$, and shifts of the equivariant parameters of G by elements of $P^\vee$.

This implies that for any $s \in P^\vee \oplus \mathbb{Z}$, we have an intertwiner $S(s,q) : E \to E$

$$S(s,q)\nabla(a) = \nabla(a+s)S(s,q)$$

where $\nabla(a)$ is the quantum connection with equivariant parameters a. The $S(s,q)$ form a commuting family of difference operators. Such operators were originally constructed by Seidel in a different setting. In the case of T^*G/B, they are the shift operators described by Opdam.

3.2 Heuristics

Why is the monodromy of a quantum connection related to automorphisms of $D^b Coh(X)$? We can only give a very schematic and conjectural answer here.

Briefly, under a phenomenon called homological mirror symmetry, pioneered by Kontsevich, one expects $D^b Coh(X)$ to be identified with a variant of the Fukaya category $D^\pi Fuk(Y)$ of some symplectic manifold Y, and vice-versa with X and Y interchanged. The objects of the Fukaya category are lagrangian submanifolds of Y (with some extra data), and the morphisms encode intersections of these lagrangians.

Just as X carries a family of (complexified) symplectic structures parametrized by an open set $U \in H^2(X, \mathbb{C})$, Y will carry a family of complex structures parametrized by the same set, in other words, one really has a 'mirror family' of complex manifolds $Y_b, b \in U$, all with the same symplectic structure.

The identification of $D^b Coh(X)$ and the Fukaya category induces an identification of $H^*(X, \mathbb{C})$ with $H^*(Y, \mathbb{C})$, such that the quantum connection of X is mapped to the Gauss-Manin connection of the family Y_b. The latter is the flat connection induced by the continuous family of lattices $H^*(Y,\mathbb{Z}) \subset H^*(Y,\mathbb{C})$.

Given a family of symplectictomorphic spaces such as Y_b, one can often produce a 'symplectic connection' which associates to a path in the base a symplectomorphism between the fibers. Up to a hamiltonian isotopy, this symplectomorphism depends only on the homotopy class of the path. Since two objects of the Fukaya category are isomorphic if they are related by a Hamiltonian isotopy, we obtain a 'flat connection' on the bundle of Fukaya categories associated to Y_b. Its monodromy produces automorphisms of $D^\pi Fuk(Y)$, lifting the monodromy of the Gauss-Manin connection.

Since mirror symmetry identifies $D^bCoh(X)$ and (a variant of) $D^\pi Fuk(Y)$, one obtains automorphisms of the former. Since the Gauss-Manin connection should match the quantum connection, these automorphisms should match the monodromy of the quantum connection.

Unfortunately (or fortunately), much of this remains to be proven.

References

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