Let $V$ be a finite dimensional vector space over $\mathbb{C}$. Let $\Gamma$ be a finite subgroup of $Sp(V)$. We can form the semidirect product $H_0 = SV \# \Gamma$, which is the tensor product $SV \otimes C\Gamma$, with multiplication rule $(f_1 \otimes \xi_1)(f_2 \otimes \xi_2) = f_1 \cdot \xi_1(f_2) \otimes \xi_1 \xi_2$. $H_0$ is a $\mathbb{Z}_+\text{-graded}$ algebra, with $\Gamma$ sitting in degree 0 and $V$ sitting in degree 1. We would like to study algebras $H$ which are filtered, and $\text{gr } H \cong H_0$. To this end, let $\varepsilon : \Lambda^2 V \rightarrow C\Gamma$ be a linear map, and define $H_\varepsilon$ to be the quotient of $TV \# \Gamma$ by the relation
\begin{equation}
[x, y] = \varepsilon(x, y)
\end{equation}
for $x, y \in V$ (note that LHS has degree 2 and RHS has degree 0). The algebra $H_\varepsilon$ has a natural filtration with $\deg(V) = 1$, $\deg(\Gamma) = 0$, and we have a natural surjective homomorphism $\varphi : H_0 \rightarrow \text{gr}(H_\varepsilon)$. It is not always an isomorphism. Indeed, let $x, y, z \in V$, and let us write down the Jacobi identity:
\[
0 = \left[ [x, y], z \right] + \left[ [y, z], x \right] + \left[ [z, x], y \right] \quad (2)
\]

Substituting (1) into (2), we get
\[
0 = \left[ x(x, y), z \right] + \left[ x(y, z), x \right] + \left[ x(z, x), y \right]
\]
Whiting \( x = \sum_{g \in \Gamma} x_g \cdot g \), we get
\[
0 = \sum_{g \in \Gamma} \left( x_g(x, y)(z^g - z) + x_g(y, z)(x^g - x) + x_g(z, x)(y^g - y) \right) g
\]
Thus, for any \( g \in \Gamma \) we must have
\[
x_g(x, y)(z^g - z) + x_g(y, z)(x^g - x) + x_g(z, x)(y^g - y)
\]
in order for \( \varphi \) to be an isomorphism (it's a necessary condition).
We claim that this implies \( x_g(x, y) = 0 \) for any \( g \) with \( \text{rk}(g-1)|_V \geq 2 \). Indeed,
\[
x^g - x \in \text{Im}(g-1), \text{ so if } x_g(x, y) \neq 0 \text{ then }
\]
\[
z^g - z = \frac{x_g(y, z)(x^g - x) + x_g(z, x)(y^g - y)}{x_g(x, y)}, \text{ hence }
\]
\[
z^g - z \text{ runs over a space of dimension } \leq 2 \text{ as } z \text{ varies. So } \dim \text{Im}(g-1) \leq 2, \text{ as claimed.}
\]
Note that if \( g \neq 1 \) then \( \dim \text{Im}(g-1) \geq 2 \), since \( \text{Im}(g-1) \) is a symplectic vector space.

Definition. A semisimple element \( g \in \mathfrak{sp}(V) \) is a symplectic reflection if \( \text{rk}(g-1) = 2 \).
Prop 1.1. If $\varphi$ is an isomorphism, then $x(x,y) = 0$ unless $g=1$ or $g$ is a symplectic reflection.

So denoting the set of symplectic reflections in $\Gamma$ by $S$, we get, when $\varphi$ is an isomorphism:

$$x(x,y) = x_1(x,y) + \sum_{g \in S} x_g(x,y)g.$$  \hfill (4)

Moreover, we can get information on the properties of $x_g(x,y)$ if $g=1$ or $g \in S$.

Indeed, first of all, conjugating (1) by $g \in \Gamma$, we see that $\varphi_1: \Lambda^2 V \to C$ is an $\Gamma$-invariant bilinear form. Also, if $g \in S$, $x_g(x,y)$ is $g$-invariant (for the same reason), so $x_g = x_g^{(1)} \oplus x_g^{(2)}$, where $x_g^{(1)}$ is a form on $\text{Im}(g^{-1})$ and $x_g^{(2)}$ is a form on $\text{Ker}(g^{-1}) = \text{Im}(g^{-1})^\perp$. But if $x,y \in \text{Ker}(g^{-1})$ then by (3), taking $z \in \text{Im}(g^{-1})$, we get that $x_g(x,y) = 0$. So $x_g(x,y) = g_w(P_gx, P_gy)$, where $g \in S$, $w \in \Lambda^2 V^*$ is the symplectic form, and $P_g: V \to \text{Im}(g^{-1})$ is the projection along $\text{Ker}(g^{-1})$. Note also that $x_g$ are invariant with respect to conjugation.
These are necessary conditions for \( \psi \) being an isomorphism, but they are also sufficient.

**Theorem 1.2.** If \( x(x, y) = x, (x, y) + \sum g \omega (P_g x, P_g y), \)

where \( x, \) is invariant and \( c_g \) are invariant under conjugation, then \( \psi \) is an isomorphism.

**Definition.** In the situation of Thm 1.2, the algebra \( H_x \) is called the symplectic reflection algebra.

**Theorem 1.2.** is the PBW theorem for symplectic reflection algebras.

There are two proofs of Thm 1.2. One proof is based on the Koszul deformation principle of Drinfeld (also Braverman–Gaitsgory, Beilinson–Ginzburg–Soergel).

Namely, the algebra \( H_0 \) is Koszul, so to check flatness of a deformation of this algebra, it suffices to check flatness in degrees up to \( 3 \) inclusively, which gives exactly the conditions above.
I will explain another proof based on classical deformation theory and calculation of the Hochschild cohomology of $H_0$.

**Theorem 1.3.**

$$HH^i(H_0, H_0) = \bigoplus_{g \in \mathbb{G}_m} SV^g \otimes \Lambda^i(V^g)^*$$

**Proof.** $HH^i(H_0, H_0) = \text{Ext}^i_{H_0-\text{Gmod}}(H_0, H_0) = \text{Ext}^i_{(SV \# \Gamma) \otimes (SV \# \Gamma)}(SV \# \Gamma, SV \# \Gamma) = \text{Ext}^i_{SV \otimes SV} (SV, SV \# \Gamma)_{(SV \otimes SV) \# \Gamma} = \bigoplus_{g \in \mathbb{G}_m} \text{Ext}^i_{SV \otimes SV} (SV, SV \cdot g)$.  

Now, $\text{Ext}^*_{SV \otimes SV} (SV, SV \cdot g) = \bigotimes_{i=1}^{\text{dim} V} \text{Ext}^*_{C[x] \cdot \text{Gmod}} C[x]^i$.

Where $C[x]^i$ is the $C[x]$-bimodule with action $f_0 h f_1(x) = f_1(x) h(x) f_2(\lambda x)$, and $\lambda$ are the eigenvalues of $g$ on $V$.

We have a resolution of $C[x]$ as a $C[x]$-bimodule:

$$0 \rightarrow C[x_1, x_2] \xrightarrow{(x_1 - x_2)} C[x_1, x_2] \rightarrow C[x] \rightarrow 0.$$  

Taking Hom from this to $C[x]^*$, we get
\[ 0 \to \mathbb{C}[x]^7 \left( \frac{1-\lambda}{\lambda} \right)^{-6} \mathbb{C}[x] \to 0. \]

Its cohomology is:

1) \( \mathbb{C} \) in degree 1, 0 in degree 0 if \( \lambda \neq 1 \)

2) \( \mathbb{C}[x] \) in degrees 0 and 1 if \( \lambda = 1 \).

So altogether we get

\[ \text{HH}^i(H_0, H_0) = \left( \bigoplus_{g \in G} \text{SV}^g \otimes \text{V}^g(\text{V}^g)^* \right)^{\mathbb{Z}_g}, \]

as desired. 

This answer can be written as

\[ \text{HH}^i(H_0, H_0) = \bigoplus_{\text{Conj}} \left( \bigoplus_{g \in \mathbb{Z}_g} \text{SV}^g \otimes \text{V}^g(\text{V}^g)^* \right)^{\mathbb{Z}_g}, \]

where \( \mathbb{Z}_g \) is the centralizer of \( g \).

The grading on \( \text{HH}^i \) is as follows:

\( \text{deg}(\text{V}^g) = 1, \quad \text{deg}(\text{V}^g)^* = -1 \), and also terms corresponding to \( g \) have overall degree \(-rk(g-1)\).

We are interested in \( \text{HH}^2 \) and \( \text{HH}^3 \) (since we want to study deformations).
We get
\[ HH^2 = (SV \otimes \Lambda^2 V^*)^{\Gamma} \oplus \bigoplus_{g \in S} SV^g \bigg] \]

In particular, we'll be interested in
\[ HH^2[-2] = (\Lambda^2 V^*)^{\Gamma} \oplus S \bigg[ S \bigg] \Gamma = E \]

We have a first order deformation of \( H_0 \) parametrized by this space.
Moreover, obstructions to this deformation lie in \( HH^3[\leq -4] = 0 \), according to our computation. Thus, the deformations defined by \( E \) are unobstructed, and we have a universal graded deformation \( H \) of \( H_0 \) over \( O(E) \), such that \( \text{deg}(E) = 2 \).

Hence, given \( x \in E \), we have specialization \( H_x \) of \( H \) at \( x \), a filtered algebra with \( \text{gr}(H_x) = H_0 \).

We claim that \( H_x \) coincides with \( H_x \) defined above. Indeed, let \( x, y \in V \), and consider the commutator \([x, y]\) in \( H_x \). It has degree \(-2\), so it must have the form \([x, y] = x(x, y)\), as in the beginning of the lecture (by looking at the first order terms w.r.t. parameters). This gives...
Remark. We can assume, essentially without loss of generality, that $\Gamma$ is generated by symplectic reflections. Indeed, if $\Gamma = \langle S \rangle \subset \Gamma$, then $H = \overline{H} \otimes \mathbb{C} \Gamma$, where $\overline{H}$ is the SRA associated to $\Gamma$. Also, it suffices to assume that $\Gamma$ is symplectically irreducible, i.e. $\langle x^2, y^2 \rangle = \mathbb{C}$.

Note that groups generated by symplectic reflections can be classified. This was done by A. Cohen in 1980. Here are some examples.

1. Complex reflection groups, i.e. $\Gamma \subset \text{GL}(V)$ generated by $y_n(z_i)$ such that $\Gamma \subset \text{Sp}(V)$, $V = \mathbb{R} \oplus \mathbb{R}^*$, and any complex reflection in $\Gamma$ is a symplectic reflection in this repr, so $\Gamma$ is generated by symplectic reflections inside $\text{Sp}(V)$.

This includes

1a) Coxeter groups, in particular Weyl groups. Notably the symmetric group $S_n$, $\mathbb{C}^n - 1$ (or $\mathbb{C}^n$).

1b) Cyclotomic groups.
\[ \Gamma = S_n \times \mathbb{Z}_e^n \] acting naturally on \( V = \mathbb{C}^n \).

2. Finite subgroups of \( SL_2(\mathbb{C}) \):

- **ADE classification**: \( \mathbb{Z}_e \) (type \( A_{e-1} \))
- Dihedral gp \( D_e \) (type \( D_{e+2} \)), \( e \geq 1 \)
- Tetrahedral gp \( T_{24} \) (\( E_6 \)), Cube group \( C_{48} \)
- Icosahedral gp \( I_{120} \) (\( E_8 \))

Higher rank version:

\[ \Gamma = S_n \times G^n, \quad G \subset SL_2(\mathbb{C}), \text{acting naturally on } (\mathbb{C}^2)^n. \]

If \( G = \mathbb{Z}_e \), get cyclotomic groups from previous example.

**Examples of SRA**:

\[ \Gamma < SL_2(\mathbb{C}), \quad H = \mathcal{C}(x, y) \# \Gamma / [x, y] = \sum_{\gamma \in \Gamma} c_{\gamma} y \]

E.g. \( \Gamma = 1 \Rightarrow \) Weyl algebra \( [x, y] = t \)

\( \Gamma = \mathbb{Z}_2 \Rightarrow \) Cherednik algebra for \( \mathbb{Z}_2 \):

\[ [x, y] = t + C s, \quad s^2 = 1, \quad sx = -sx, \quad sy = -sy. \]
Spherical SRA: let $E = \frac{1}{G} \sum_{x \in R} x$

be the symmetrizer.

**Definition.** The spherical SRA is the algebra $eH^e$ (universal) or $eH^e e$ (specialized).

We have $gr(eH^e e) = e(sV \# 1)e = (sV)^T$, a commutative algebra.

So $eH^e e$ is a filtered deformation (quantization) of the comm. algebra $(sV)^T$.

**Theorem 1.4.** The Poisson bracket on $(sV)^T$ of degree $-2$ defined by $eH^e e$ is given by $x_1 e(K^2 V^*)^T$.

**Proof.** Direct calculation.

**Corollary 1.5.** If $x_1 = 0$ then $eH^e e$ is commutative.

**Proof.** Lemma 1.6. Any Poisson bracket on $(sV)^T$ of degree $\leq -3$ is zero.
Proof: We have $(SV)^{-1} = O(V^{*})$, let $V^{*}$ be the set of pts of $V^{*}$ with trivial stabilizer. A Poisson bracket on $V^{*}$ defines a bivector field $\Pi$ on $V^{*}$, hence a $\Gamma$-invariant bivector field $\Pi$ on $V^{*}$. Since $V^{*} \setminus V^{*\circ}$ has codim $\Pi$ extends to a bivector field on $\mathcal{O}$.

If $\deg(\Pi) \leq -3$ then $\Pi = 0$.

Now, assume $\Pi \neq 0$, and let $d$ be the smallest integer such that

$\exists f_1 \in F(H^2) \cap F(H^2)$ such that $[f_1, f_2]$ has degree exactly $i + j - d$. Consider the degree $-d$ Poisson bracket induced by $\varepsilon H^2 \varepsilon$ on $(SV)^{-1}$.

Since $d \geq 3$, this bracket is zero. So $d$ does not exist, i.e. $\varepsilon H^2 \varepsilon$ is commutative.