Hard Lefschetz for Soergel bimodules

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Abstract

These are notes for the joint MIT/Northeastern Graduate seminar on category $\mathcal{O}$ and Soergel bimodules, Fall 2017. We prove the hard Lefschetz theorem for Soergel bimodules, closely following [EW12]. For the sake of continuity, we are mostly citing the previous talks of this seminar instead of loc. cit.

1 Setup

To fix a notation, lets recall the setup and some results described in the previous talks [Tsv17], [Kim17], [SA17], [Ven17].

Fix a Coxeter system (W, S). Let $\mathfrak{h}$ be a reflection faithful representation of W, $\alpha_s \in \mathfrak{h}^*, s \in S$ be a collection of simple roots, $\rho \in \mathfrak{h}^*$ be a fixed strictly dominant weight (see [SA17] for details).

We have a category of Soergel bimodules $\mathcal{B}$, which is a subcategory of the category of graded bimodules over the polynomial ring $R = \mathbb{R}[\mathfrak{h}]$. We have a collection of indecomposable bimodules $B_x(i), x \in W, i \in \mathbb{Z}$, where (i) denotes a grading shift. Category $\mathcal{B}$ is monoidal with respect to the product $\cdot \otimes_R \cdot$. Product of objects $B_1, B_2 \in \mathcal{B}$ is denoted simply as $B_1 B_2$. We also have a duality functor $\mathbb{D} : \mathcal{B} \to \mathcal{B}$.

For any sequence $\underline{x} = (s_1, \ldots, s_r), s_i \in S$, we have a Bott-Samelson bimodule $BS(\underline{x}) = B_{s_1} B_{s_2} \ldots B_{s_r}$. B_x is an indecomposable summand of $BS(\underline{x})$ if $\underline{x}$ is a reduced expression for x. For $B \in \mathcal{B}$, denote $\overline{B} = B \otimes_R \mathbb{R}$, a left R-module. In [SA17] an invariant form was defined on $BS(\underline{x})$ for any reduced expression $\underline{x}$, called an intersection form. It descends to a bilinear form on $\overline{BS(\underline{x})}$. For any embedding $B_x \to BS(\underline{x})$ we get a form on B_x by restriction. For any invariant form on B_x there is such a form on $B_x B_s$, called an ***induced form***, see loc.cit.

Fix $x \in W$. Recall that, for $\zeta \geq 0$, $L_\zeta : B_x B_s \to B_x B_s(2)$ is given by $L_\zeta = (\rho \cdot) \operatorname{id}_{B_s} + \zeta \operatorname{id}_{B_x} (\rho \cdot)$, where $(\rho \cdot)$ denotes the ***left*** multiplication by ρ on the corresponding factor.

Recall that if we have a finite-dimensional graded vector space V, such that either $H^{2i+1} = 0$ for all i(in which case H is called even) or $H^{2i} = 0$ for all i (in which case H is called odd), with a homogeneous bilinear pairing $(\cdot, \cdot) : V \times V \to V$, and a linear operator $L : V \to V(2)$ satisfying $(Lx, y) = (x, Ly)$, we can talk of a Hard Lefschetz theorem and Hodge-Riemann bilinear relations with standard sign for V, L, see [SA17].

Let $\mathcal{H} = \mathcal{H}(W, S)$ be a Hecke algebra associated to (W, S). It is an algebra over the ring $\mathbb{Z}[v, v^{-1}]$. $\mathcal{H}$ has a standard basis $H_x, x \in W$, and a bilinear form $(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{Z}[v, v^{-1}]$ satisfying $(H_x, H_y) = \delta_{x,y}$. There is a $\mathbb{Z}$-linear involution $x \to \overline{x}$ on $\mathcal{H}$, satisfying $\overline{v} = v^{-1}$, $H_x = H_{x^{-1}}^{-1}$. $\mathcal{H}$ has a basis $\underline{H}_x, x \in W$, called a Kazhdan-Lusztig basis. It is characterised by the properties $\overline{\underline{H}_x} = \underline{H}_x, \underline{H}_x \in H_x + \sum_{y<x} v\mathbb{Z}[v] H_y$. We have $(\underline{H}_x, \underline{H}_y) \in \delta_{x,y} + v\mathbb{Z}[v]$.

We have a map $ch : \mathrm{Ob}(\mathcal{B}) \to \mathcal{H}$, which descends to a homomorphism from a split K-ring of $\mathcal{B}$, see [SA17].

Recall that we have a following collection of statements for $x \in W$, $s \in S$:

a) $S(x)$ — Soergel's conjecture holds for x, that is $ch(B_x) = \underline{H}_x$.

b) $hL(x,s)$ — Hard Lefschetz theorem holds for $\overline{B_xB_s}, \rho$.

c) $hL(x,s)_\zeta$ — Hard Lefschetz holds for $\overline{B_xB_s}$, L_ζ. We will sometives write $hL(x,s)_0$ for $hL(x,s)$.

d) $HR(x)$ — Hodge-Riemann bilinear relations with standard sign hold for $\overline{B_x}, \rho$, with respect to a form restricted from any embedding $B(x) \to BS(\underline{x})$ for any reduced expression $\underline{x}$ for x.

e) $HR(x,s)_\zeta$ — Hodge-Riemann bilenear relations with standard sign hold for $\overline{B_xB_s}$, L_ζ, with respect to an induced form on $\overline{B_xB_s}$.

For a statement $I \in \{S(\cdot), hL(\cdot, s), \dots\}$ we write $I(\leq x), I(< x), \dots$ if $I(y)$ holds for all $y \leq x, y < x, \dots$.

The only missing piece for the induction machine is $hL(x,s)_\zeta, \zeta \geq 0$, assuming "everything else" for $y \leq x$ known. This is what we will prove in these notes.

2 Strategy of the proof

Recall the following Lemma from [Ven17].

Lemma 2.1. *Suppose that we have a map of graded* $\mathbb{R}[L]$*-modules (*$\deg L = 2$*)*

$$\phi : V \to W(1)$$

such that

a) ϕ *is injective in degrees* ≤ -1*,*

b) V *and* W *are equipped with graded bilinear forms* $(\cdot,\cdot)_V$ *and* $(\cdot,\cdot)_W$ *such that* $(\phi(\alpha), \phi(\beta))_W = (\alpha, L\beta)_V$ *for all* $\alpha, \beta \in V$*,*

c) W *satisfies the Hodge-Riemann bilinear relations.*

Then $L^i : V^{-i} \to V^i$ *is injective for* $i \geq 0$*.*

We are going to apply this Lemma, very roughly, for $V = \overline{B_xB_s}$ and ϕ given by the first differential in the Rouquier complex F_{xs}, see [Kim17] and below for the definition. In order to do this, we will have to equip the terms of F_{xs} with bilinear forms, which was almost done already in the previous talks, and prove that they satisfy Hodge-Riemann bilinear relations. In order to do this, we will need to borrow some more tools from geometry. Namely, we will mimic the definition of the perverse t-structure, and will prove, inspired by the corresponding fact from the geometry of the flag variety, that the Rouquier complexes are perverse for this definition. This will then allow us to prove Hodge-Riemann bilinear relations for Rouquier complexes and, applying the Lemma above, $hL(x,s)_\zeta$ for $\zeta \geq 0$.

We proceed to the definition of the perverse filtration on Soergel bimodules.

3 Perverse filtrations

3.1 Perverse filtration on bimodules

Soergel bimodule $B \in \mathcal{B}$ is called perverse if $ch(B) = \sum_z a_z \underline{H}_z, a_z \in \mathbb{Z}_{\geq 0}$. Soergel bimodule B is called p-split, if each of its indecomposable summands is of the form $B'(i), i \in \mathbb{Z}, B'$ is perverse. Note that it follows from the Soergel's conjecture that any Soergel bimodule is p-split. We need these definitions only while inductively proving it. Recall the following proposition from [SA17]:

Proposition 1 (Soergel's Hom formula). rk $Hom^\bullet(B_1, B_2) = \overline{(ch(B_1), \overline{ch(B_2)})}$ *for* $B_1, B_2 \in \mathcal{B}$.

Applying this proposition, we see that

$$Hom(B, B'(-i)) = 0 \tag{1}$$

if B, B' are perverse and $i > 0$ (note that $ch(B(k)) = v^k ch(B)$).

For $B \in \mathcal{B}$ p-split, choose a decomposition

$$B = \bigoplus_{i,x} B_x^{\oplus m_{x,i}}(i)$$

and define a perverse filtration on B by

$$\tau_{\leq j} B = \bigoplus_{i \geq -j} B_x^{\oplus m_{x,i}}(i).$$

By (1), this filtration does not depend on a choice of decomposition, and any morphism in $\mathcal{B}$ respects this filtration.

Define $\tau_{<j}, \tau_{\geq j}, \tau_{>j}$ in an obvious way. Finally, define

$$\mathcal{H}^j(B) = \tau_{\leq j} B / \tau_{<j} B(j).$$

Remark. This mimics the definitions from the theory of mixed equivariant perverse sheaves on flag varieties associated to (W, S). There, functors $\tau, \mathcal{H}$ are usually denoted ${}^p\tau$, ${}^p\mathcal{H}$. In fact, in the next subsection we define an analogue of the perverse t-structure on the homotopy category of Soergel bimodules.

3.2 Perverse filtration on complexes

Let $K^b(\mathcal{B})$ be a bounded homotopy category of $\mathcal{B}$. For a complex

$$F = \cdots \to {}^iF \to {}^{i+1}F \to \ldots$$

we say that it $F \in {}^pK^b(\mathcal{B})^{\geq 0}$ if, up to an isomorphism in $K^b(\mathcal{B})$, it satisfies the following properties:

- ${}^iF$ is p-split,
- $\tau_{<-i}\, {}^iF = 0$

for all $i \in \mathbb{Z}$.

We say that it $F \in {}^pK^b(\mathcal{B})^{\leq 0}$ if, up to an isomorphism in $K^b(\mathcal{B})$, it satisfies the following properties:

- ${}^{i}F$ is p-split,
- ${}^{i}F = \tau_{\leq -i}\, {}^{i}F$

for all $i \in \mathbb{Z}$.

Lemma 3.1 (exercise). *If $F' \to F \to F'' \xrightarrow{[1]}$ is a distinguished triangle in $K^b(\mathcal{B})$, and $F', F'' \in {}^{p}K^b(\mathcal{B})^{\geq 0}$ (respectively, ${}^{p}K^b(\mathcal{B})^{\leq 0}$) then $F \in {}^{p}K^b(\mathcal{B})^{\geq 0}$ (respectively, ${}^{p}K^b(\mathcal{B})^{\geq 0}$).*

The complex F is called perverse, if it is isomorphic to an object in ${}^{p}K^b(\mathcal{B})^{\geq 0} \cap {}^{p}K^b(\mathcal{B})^{\leq 0}$.

Remark. After Soergel's conjectures are known, one can show that $({}^{p}K^b(\mathcal{B})^{\geq 0}, {}^{p}K^b(\mathcal{B})^{\leq 0})$ give a non-degenerate t-structure on $K^b(\mathcal{B})$. Its heart can be thought of as a generalization of a category of mixed equivariant perverse sheaves on a flag variety to the case of the general Coxeter group.

4 Rouqier complexes

Recall from [Kim17] that we have complexes $F_s = B_s \to R(1)$ for every $s \in S$, and for $x \in W$ and a reduced expression $\underline{x}, x = s_1 \ldots s_r$, we have a complex $F_{\underline{x}} = F_{s_1} \ldots F_{s_r}$. For two different reduced expressions $\underline{x}, \underline{x}'$ of x, complexes $F_{\underline{x}}, F_{\underline{x}'}$ are canonically isomorphic in $K^b(\mathcal{B})$. We will replace $F_{\underline{x}}$ with its ***minimal subcomplex*** F_x, the notion we introduce below, and prove that F_x is perverse.

Remark. This generalizes the geometric fact that the standard and costandard sheaves on flag varieties are perverse.

4.1 Semisimplifaction and minimal subcomplexes

Let $\mathcal{A}$ be an additive category. There is an ideal $\mathrm{rad}_{\mathcal{A}}$ in $\mathcal{A}$, called a radical, defined by

$$\mathrm{rad}_{\mathcal{A}}(X, Y) = \{\phi \in Hom_{\mathcal{A}}(X, Y) : \phi \circ \psi \in J(End_{\mathcal{A}}(Y)), \forall \psi \in Hom_{\mathcal{A}}(Y, X)\}$$

where $J(A)$ stands for a Jacobson radical of an algebra A.

The quotient $\mathcal{A}/\operatorname{Rad}_{\mathcal{A}}$ is called a semisimplification of $\mathcal{A}$, denoted $\mathcal{A}^{ss}$. Denote $q : \mathcal{A} \to \mathcal{A}^{ss}$ the quotient functor.

Remark. Note that in case when $\mathcal{A}$ is a category of finitely generated projective modules over a finite-dimensional algebra A, $\mathcal{A}^{ss}$ is a category of finitely-generated modules over $A/J(A)$. q maps indecomposable projectives to simples.

Recall from [Tsv17], that $\mathcal{B}$ is a Krull-Schmidt category, and $\{B_x(i)\}$ is a full set of indecomposable objects. We have that $End(B_x)$ are finite-dimensional local $\mathbb{R}$-algebras. Note that we also have a surjection $End(B_x) \twoheadrightarrow End(\Gamma_{\leq x}/\Gamma_{<x} B_x) = \mathbb{R}$, so $End(q(B_x)) = \mathbb{R}$.

We have the following facts:

- $\{q(B_x(i))\}$ is a full set of pairwise non-isomorphic simple objects in $\mathcal{B}^{ss}$.
- Morphism $f : B \to B'$ in $\mathcal{B}$ is an isomorphism if and only if $q(f)$ is an isomorphism.
- More generally, $f : B \to B'$ is a split injection if and only if $q(f)$ is a split injection.

Complex $F \in C^b(\mathcal{B})$ — category of bounded complexes in $\mathcal{B}$ — is called minimal, if $q(F)$ has zero differentials. It is equivalent to $q(F)$ having no contractible (i.e. homotopic to 0) direct summands, which is equivalent to F having no contractible direct summands. For any $F \in C^b(\mathcal{B})$, we have a summand $F^{min} \Subset F, F^{min} \simeq F$ in $K^b(\mathcal{B}), F^{min}$ — minimal. F^{min} is called a minimal subcomplex of F. It is easy to see that any two minimal subcomplexes are isomorphic in $C^b(\mathcal{B})$.

For a reduced expression $\underline{x} = s_1 \dots s_m$, choose a minimal subcomplex $F_x \subset F_{\underline{x}}$. It does not depend on a choice of a reduced expression, up to an isomorphism in $C^b(\mathcal{B})$.

The following Lemma is easy:

Lemma 4.1 (exercise).

$$H^i(\overline{F_x}) = \begin{cases} \mathbb{R}(-l(x)), \textit{ if } i = 0, \\ 0, \textit{ otherwise.} \end{cases}$$

Proposition 2. *Assume $S(y)$ for all $y < x$. Then $F_x \in {}^pK^b(\mathcal{B})^{\geq 0}$.*

Proof. We will need the following

Lemma 4.2. *Assume $S(x)$ and pick $s \in S$.*

a) If $xs < x$, then $B_x F_s \simeq B_x(-1)$ in $K^b(\mathcal{B})$.

b) If $xs > x$, then $B_x F_s \in {}^pK^b(\mathcal{B})^{\geq 0}$.

Proof of a). By $S(x)$, $ch(B_x B_s) = \underline{H}_x \underline{H}_s = (v + v^{-1})\underline{H}_s$, so $B_x B_s \simeq B_x(-1) \oplus B_x(1)$. Then $B_x F_s$ if of the form

$$0 \to B_x(1) \oplus B_x(-1) \to B_x(1) \to 0.$$

Multiplication by F_s is an invertible endofunctor of $K^b(\mathcal{B})$, so the complex above is indecomposable (meaning that any summand of this complex is contractible) since B_x is indecomposable. By $S(x)$ and Soergel's Hom formula, $End(B_x) = \mathbb{R}$, so the differential must take the summand $B_x(1)$ isomorphically to $B_x(1)$. Contracting the subcomplex $B_x(1) \to B_x(1)$, we get that the complex above is homotopic to $B_x(-1)$. □

Proof of b). By $S(x)$, $ch(B_x B_s) = \underline{H}_x \underline{H}_s \in \sum_{x'} \mathbb{Z}_{\geq 0} \underline{H}_{x'}$, so, by definition, $B_x B_s$ is perverse. So

$$B_x F_s = 0 \to B_x B_s \to B_x(1) \to 0 \in {}^pK^b(\mathcal{B})^{\geq 0}.$$

□

Corollary 4.1. *If $F \in {}^pK^b(\mathcal{B})^{\geq 0}$ and $S(y)$ holds for all indecomposable summands B_y of ${}^iF$ for all i, then $FF_s \in {}^pK^b(\mathcal{B})^{\geq 0}$.*

Proof. We can assume that F is a minimal complex. Consider the "stupid" filtration

$$w_{\geq k} F = 0 \to {}^kF \to {}^{k+1}F \to \dots .$$

For all k, we have triangles

$$w_{\geq k+1} F \to w_{\geq k} F \to {}^kF[-k] \xrightarrow{[1]} .$$

By Lemma 3.1, we know that if $w_{\geq k+1}FF_s \in {}^pK^b(\mathcal{B})^{\geq 0}$ and ${}^kF[-k]F_s \in {}^pK^b(\mathcal{B})^{\geq 0}$, so is $w_{\geq k}FF_s$. But the latter holds by Lemma 4.2, so we are done by induction. □

Now proposition follows by induction: $F_s \in {}^pK^b(\mathcal{B})^{\geq 0}$ for all $s \in S$ by definition, so for any reduced expression $F_{\underline{x}} \in {}^pK^b(\mathcal{B})^{\geq 0}$, and so is F_x.

□

We are now ready to prove that F_x are perverse, as promised.

Theorem 1. *Assume $S(y)$ for all $y \leq x$. Then*

a) ${}^0F_x = B_x$.

b) For $i \geq 1, {}^iF_x = \oplus_z B_z(i)^{\oplus m_{z,i}}, z < x, m_{z,i} \in \mathbb{Z}_{\geq 0}$.

In particular, $F_x \in {}^pK^b(\mathcal{B})^{\geq 0} \cap {}^pK^b(\mathcal{B})^{\leq 0}$.

Proof. We will need the following Lemma from [LW14]. Recall from [SA17], that we have a canonical support filtration Γ on any object in $\mathcal{B}$, and for an element $y \in W$ we have functors $\Gamma_{\geq y}, \Gamma_{>y}$. Also recall that for $y \in W$, R_y is an R-bimodule which is a free rank 1 left R-module with the right R-action twisted by x. Denote $\Delta_y = R_y(-l(y))$.

Lemma 4.3 ([LW14]).

$$\Gamma_{\geq y}/\Gamma_{>y}F_x \simeq \begin{cases} \Delta_y, y = x, \\ 0, \textit{otherwise.} \end{cases}$$

in the homotopy category of R-bimodules.

Knowing this Lemma, we proceed as follows. Consider a summand $B_z(j)$ of ${}^iF_x$. By assumption, $z < x$, and $S(z)$ holds. Consider the image of $B_z(j)$ under the differential. Since $S(y)$ holds for any y such that $B_y(k)$ is a summand of ${}^{i+1}F_x$, and by Soergel's Hom formula, $B_z(j)$ can map only to $B_y(k) \Subset {}^{i+1}F_x$ with $k \geq j$. Again, by Soergel's Hom formula, any non-zero map in $Hom(B_z(j), B_y(j))$ is an isomorphism, and such an isomoprphism can't appear in a minimal complex. So we must have $k > j$. Similarly, only summands $B_y(k'), k' < j$ of ${}^{i-1}F$ can map to $B_z(j)$ non-trivially.

Consider $\Gamma_{\geq z}/\Gamma_{>z}F_x$. The summand $B_z(j)$ of ${}^iF_x$ contributes $\Delta_z(j)$ to this sub-quotient, and it must cancel, by Lemma 4.3, with such a summand from ${}^{i-1}F_x$ or ${}^{i+1}F_x$. Note that such a summand can come only from $B_y(k)$ with $y > z$. By $S(y)$,

$$ch(B_y(l)) \in v^l(H_y + \sum_{y'<y} v\mathbb{Z}[v]H_{y'}),$$

so, by the discussion above, $\Delta_z(j)$ can only cancel with a contribution of the summand $B_y(k')$ with $y > z, k' < j$, from ${}^{i-1}F_x$.

We conclude: *if ${}^iF_x$ contains a summand $B_z(j), z < x$, ${}^{i-1}F_x$ must contain a summand $B_y(k'), y > z, k' < j$.*

Since ${}^{-1}F_x = 0$, we get ${}^0F_x = B_x$. Using Proposition 2, we get the rest by induction. □

4.2 Rouquier complexes are Hodge-Riemann

In this subsection we will prove that terms ${}^iF_x$ of a (minimal) Rouquier complex satisfy Hodge-Riemann bilinear relations, after an appropriate graded shift. To do this, we must first define an invariant form on these terms.

Fix some reduced expression $\underline{x} = s_1 \dots s_m$ of x. We have

$$ {}^jF_x \stackrel{\oplus}{\subset} {}^j(F_{s_1} \dots F_{s_m}) = \bigoplus_{\underline{x}' \in X} BS(\underline{x}')(j), $$

for some index set X. For a tuple $\lambda = (\lambda_{\underline{x}'})_{\underline{x}' \in X}$ of positive numbers, define a form $(\cdot,\cdot)^\lambda$ on

$$ \bigoplus_{\underline{x}' \in X} BS(\underline{x}') $$

as $(\cdot,\cdot)^\lambda = \sum_{\underline{x}'} \lambda_{\underline{x}'} (\cdot,\cdot)_{BS(\underline{x}')}$.

We say that F_x satisfies Hodge-Riemann bilinear relations if, for any choice of $\underline{x} = s_1 \dots s_m$ we can choose an embedding

$$ F_x \stackrel{\oplus}{\subset} F_{s_1} \dots F_{s_m} $$

such that for any choice of a tuple λ, $\overline{{}^jF_x(-j)}$ satisfies Hodge-Riemann bilinear relations with respect to $(\cdot,\cdot)^\lambda$ and left multiplication by ρ, where sign is determined so that the form is positive definite in degrees congruent to $-m+j$ modulo 4, for all j.

Theorem 2. *Assume $S(\leq x)$. Assume $HR(y,s)$ for $y < x, s \in S, ys > y$. Then F_x satisfies Hodge-Riemann bilinear relations.*

Proof. We will use the following Lemma repeatedly in inductive arguments:

Lemma 4.4. *Fix $\zeta \geq 0, s \in S$ and $B = \oplus_{z \in W} B_z^{\oplus m_z}, m_z \in \mathbb{Z}_{\geq 0}$.*

Assume that, if $m_z \neq 0$, we have $S(z), HR(z,s)_\zeta$. If $\zeta = 0$, assume in addition $m_z = 0$ for $zs < z$ (we need this assumption since $HR(z,s)$ fails if $zs < z$).

Assume that $\overline{B}$ is even or odd, and B is equipped with an invariant non-degenerate form such that $\overline{B}$ satisfies Hodge-Riemann bilinear relations with standard signs, with respect to $(\rho\cdot)$.

Then $\overline{BB_s}$ satisfies Hodge-Riemann bilinear relations with standard signs with respect to L_ζ and an induced form.

Proof. The proof is straightforward, we only give a plan. One first shows, using Soergel's Hom formula, that one can choose a decomposition $B = \oplus_{z \in W} B_z^{\oplus m_z}$ to be orthogonal. By inductive assumption, all summands of the decomposition $\overline{BB_s} = \oplus_{z \in W} \overline{B_z B_s}^{\oplus m_z}$ satisfy Hodge-Riemann bilinear relations with standard signs, so the only thing left to check is that the signs are the same in every degree in each summand, using Hodge-Rimeann bilinear relations for $\overline{B}$. $\square$

We proceed to the proof of the Theorem. Pick $y, s, ys = x > y$ and assume that the statement of the Theorem is already known for F_y. We have ${}^jF_x(-j) \stackrel{\oplus}{\subset} {}^jF_y B_s(-j) \oplus {}^{j-1}F_y(-j+1)$. Note that these two summands come from different Bott-Samelson bimodules, as in the definition of the form $(\cdot,\cdot)^\lambda$, so they are orthogonal with respect to this form.

By $S(\leq y), {}^jF_y(-j) = \oplus_{z\in W} V_z \otimes_{\mathbb{R}} B_z$, where all multiplicity spaces V_z have degree 0. Let

$$B^\uparrow = \bigoplus_{z\in W, zs>z} V_z \otimes_{\mathbb{R}} B_z, B^\downarrow = \bigoplus_{z\in W, zs<z} V_z \otimes_{\mathbb{R}} B_z.$$

Decomposition ${}^jF_y(-j) = B^\uparrow \oplus B^\downarrow$ is orthogonal: $Hom(B^\uparrow, \mathbb{D}B^\downarrow) = Hom(B^\downarrow, \mathbb{D}B^\uparrow) = 0$, so there are no non-zero invariant pairings between the summands.

By $S(\leq x)$, $B^\uparrow B_s$ is perverse, $B^\downarrow B_s = B^\downarrow(1) \oplus B^\downarrow(-1)$. We get an orthogonal decomposition

$${}^jF_x(-j) \Subset B^\downarrow B_s \oplus B^\uparrow B_s \oplus {}^{j-1}F_y(-j+1). \tag{2}$$

$\overline{{}^{j-1}F_y(-j+1)}$ satisfies Hodge-Riemann bilinear relations by the inductive assumption. $\overline{B^\uparrow}$ satisfies them too, by inductive assumption, so, by Lemma 4.4, $\overline{B^\uparrow B_s}$ satisfies Hodge-Riemann bilinear relations.

Since ${}^jF_x(-j)$ is perverse, its projection to $B^\downarrow B_s$ in the inclusion (2) lands in $B^\downarrow(1)$, by Soergel's Hom formula. We will now prove that its image in $B^\downarrow(1)$ does not contribute to the pairing, using the following linear-algebraic Lemma:

Lemma 4.5 (exercise). *Let H be a graded vector space, $(\cdot,\cdot)$ — graded pairing on H, not necessary non-degenerate, $L : H^\bullet \to H^{\bullet+2}$ — Lefschetz operator. Assume that $L^i : H^{-i-1} \to H^{-1+i}$ is an isomorphism for all i. Then, for $i \geq 0$, the form $(h,h')^{-i} = (h, L^i h')$ on H^{-i} is zero.*

Hint. Use the primitive decomposition. □

Applying this Lemma to $H = \overline{B^\downarrow}(1)$, which we can do since $\overline{B^\downarrow}$ satisfies hard Lefschetz by assumption, we see that the image of ${}^jF_x(-j)$ in $B^\downarrow(1)$ does not contribute to the pairing.

We now need to prove that

$$\iota : \overline{{}^jF_x(-j)} \to \overline{B^\uparrow B_s} \oplus \overline{{}^{j-1}F_y(-j+1)}$$

is injective. To do this, recall the quotient functor $q : \mathcal{B} \to \mathcal{B}^{ss}$. Apply it to the split injection (2). Recall, that $\{q(B_x)(i)\}$ give a full set of pairwise non-isomorphic simple objects in $\mathcal{B}^{ss}$. It follows that

$$q({}^jF_x(-j) \to B^\downarrow(1)) = 0,$$

since ${}^jF_x(-j)$ is a sum of objects of the form B_z, and $B^\downarrow(1)$ is the sum of objects of the form $B_{z'}(1)$. We conclude that

$$q({}^jF_x(-j) \to B^\uparrow \oplus {}^{j-1}F_y(-j+1))$$

is a split injection. Hence ${}^jF_x(-j) \to B^\uparrow \oplus {}^{j-1}F_y(-j+1)$ is a split injection.

We got an injection (respecting the forms) of $\overline{{}^jF_x(-j)}$ to a sum of orthogonal spaces, each satisfying Hodge-Riemann bilinear relations. The only thing left to do is to confirm that the signs of the form on these orthogonal spaces coincide in each degree, which we omit. □

5 Hard Lefschetz

Recall our plan from Section 2. Fix a reduced expression $\underline{x}$ of x and $s \in S$ with $xs > x$. We have a map

$$\phi : BS(\underline{x}s) \to \oplus BS(\underline{xs}_{\hat{i}}),$$

which restricts to the first differential

$$\phi : {}^0(F_xF_s) \to {}^1(F_xF_s)$$

in the Rouquier complex F_xF_s. After killing the right R-action, we have

$$(b, L_\zeta b')_{\overline{BS(\underline{xs})}} = (\phi(b), \phi(b'))^\gamma_{\mathbb{R}},$$

where $(\cdot,\cdot)^\gamma_{\mathbb{R}}$ denotes the form on $\oplus BS(\underline{xs}_{\hat{i}})$ rescaled by the certain tuple γ of positive numbers. Lets first prove the easier case of the hard Lefschetz theorem:

Theorem 3. *Suppose* $\zeta > 0, x \in W, s \in S$*,* $xs > x$*. Assume:*

a) $S(\leq x)$.

b) $HR(z,t), z < x, t \in S, zt > z$.

c) $HR(< x, s)_\zeta$.

d) $HR(x)$.

Then $hL(x,s)_\zeta$ *holds.*

Proof. By a), b) we know that F_x satisfies Hodge-Riemann bilinear relations.

Write

$$\phi : B_xB_s \to {}^1(F_xF_s) = {}^1F_xB_s \oplus B_x(1)$$

and let $d_1 : B_xB_s \to {}^1F_xB_s, d_2 : B_xB_s \to B_x(1)$ be the components of ϕ. Recall that $L_\zeta = (\rho\cdot)\,\mathrm{id}_{B_s} + \mathrm{id}_X(\zeta\rho\cdot)$ on XB_s. It is easy to see that L_ζ commutes with d_1 and $d_2(L_\zeta b) = \rho d_2(b) + d_2(b)(\cdot\zeta\rho)$. Denote by L the operator on $\overline{{}^1F_xB_s \oplus B_x(1)}$ given by $\overline{L_\zeta}$ on the first summand and ρ on the second. Since summands of ${}^1F_xB_s \oplus B_x(1)$ come from the different Bott-Samelson bimodules, as in the definition of the form $(\cdot,\cdot)^\gamma$, this decompostion is orthogonal with respect to this form.

We have $\overline{\phi}(\overline{L_\zeta}b) = L\overline{\phi}(b)$, and by $HR(x)$, Hodge-Riemann bilinear relations for F_x, and $HR(< x, s)_\zeta$ with Lemma 4.4, $\overline{{}^1F_xB_s \oplus B_x(1)}$ satisfies Hodge-Riemann bilinear relations. Since $\overline{\phi}$ is the first differential in the complex $\overline{F_xF_s} \simeq \overline{F_{xs}}$, it is injective in degrees $< l(xs) = l(x) + 1$, by Lemma 4.1. By Lemma 2.1 we get that $\overline{L_\zeta}^k$ is injective on $(\overline{B_xB_s})^{-k}$, and since $\overline{B_xB_s}$ has symmetric Betti numbers, we are done. $\square$

We now turn to the harder case.

Theorem 4. *Suppose* $x \in W, s \in S$*,* $xs > x$*. Assume:*

a) $S(\leq x)$.

b) $HR(z,t), z < x, t \in S, zt > z$.

c) $HR(x)$.

d) $hL(z), z < xs$.

Then $hL(x,s)$ *holds.*

Proof. Again, write $\phi : B_xB_s \to B_x(1) \oplus {}^1F_xB_s$. Note that since $\zeta = 0$, we ***don't have Hodge-Riemann bilinear relations for*** ${}^1F_xB_s(-1)$, so we'll have to circumvent it.

As we did before, write

$$^1F_x(-1) = B^{\uparrow} \oplus B^{\downarrow},$$

with $B^{\uparrow}B_s$ perverse, $\mathcal{H}^0(B^{\downarrow}B_s) = 0$. As before, this decomposition is orthogonal, and the full decomposition

$$\phi : B_xB_s \to B_x(1) \oplus B^{\uparrow}B_s(1) \oplus B^{\downarrow}B_s(1)$$

is orthogonal.

By Corollary 4.1 $F_xF_s \in {}^pK^b(\mathcal{B})^{\geq 0}$. Recall that $B^{\downarrow}B_s = B^{\downarrow}(-1) \oplus B^{\downarrow}(1)$, so $B^{\downarrow}B_s(1) = B^{\downarrow} \oplus B^{\downarrow}(2)$. So the restriction of the second differential to $B^{\downarrow}(2)$ is a split injection. After contracting this summand, the first two terms of the complex become

$$d : B_xB_s \to B_x(1) \oplus B^{\uparrow}B_s(1) \oplus B^{\downarrow}.$$

Let $d_1 : B_xB_s \to B_x(1), d_2 : B_xB_s \to B^{\uparrow}B_s(1), d_3 : B_xB_s \to B^{\downarrow}$ be the components of this differential.

Again, by Lemma 4.1 $\overline{d}$ is injective in degrees $\leq l(x)$.

We want to prove that, for any $b \in (\overline{B_xB_s})^{-k}$, $\rho^k b \neq 0$. We consider two cases.

Case a). $\overline{d_3}(b) \neq 0$. $\overline{d_3}$ commutes with the left multiplication by ρ, and all summands of $B^{\downarrow}$ satisfy hL by our assumption, so

$$\overline{d_3}(\rho^k(b)) = \rho^k\overline{d_3}(b) \neq 0.$$

Case b). $\overline{d_3}(b) = 0$. Note that $B^{\downarrow}$ was the summand that prevented us form using Hodge-Riemann bilinear relations as in the previous Theorem. So on $\ker \overline{d_3}$ proof is identical to the previous case.

□

We are left with the last case of hard Lefschetz theorem. Its proof is completely different (and much easier) and does not use Rouquier complexes.

Theorem 5. *Suppose* $\zeta > 0, x \in W, s \in S, xs < x$. *If* $hL(x)$ *holds, then* $hL(x,s)_\zeta$ *holds.*

Proof. We will need the following Lemma from [Will10]. See Appendix for the proper context.

Lemma 5.1 ([Will10]). *If* $x \in W, s \in S, xs < s$, *then there exists* (R, R^s)*-bimodule* B' *such that* $B_x = B' \otimes_{R^s} R$.

We now construct an explicit isomorphism $R \simeq R^s \oplus R^s(-2)$: consider inclusions

$$\iota_1, \iota_2 : R^s \to R, \iota_1(r) = r, \iota_2(r) = \frac{1}{2}\alpha_s r,$$

and projections

$$\pi_1, \pi_2 : R \to R^s, \pi_1(r) = \frac{1}{2}(r + sr), \pi_2(r) = \partial_s r.$$

It is easy to check that ι_1, ι_2 split π_1, π_2. This fixes an isomorphism

$$B_x B_s \simeq B' \otimes_{R^s} R \otimes_{R^s} R \simeq B_x(1) \oplus B_x(-1).$$

With respect to this isomorphism, L_ζ is given by the matrix

$$\begin{pmatrix} (\rho\cdot) + \zeta(\cdot\pi_1(\rho)) & \frac{1}{2}\zeta(\cdot\pi_1(\alpha_s\rho)) \\ \zeta\rho(\alpha_s^\vee) & (\rho\cdot) + \frac{1}{2}\zeta(\cdot\partial_s(\alpha_s\rho)) \end{pmatrix}.$$

After tensoring with $\otimes_R \mathbb{R}$, we get

$$\begin{pmatrix} \rho & 0 \\ \zeta\rho(\alpha_s^\vee) & \rho \end{pmatrix}.$$

Completing the action of ρ to the action of $\mathfrak{sl}_2(\mathbb{R})$ on $\overline{B_x}$, which we can do by $hL(x)$, we see that this matrix describes an action of e on a representation $\overline{B_x} \otimes V_2$, where V_2 is a standard 2-dimensional representation of $\mathfrak{sl}_2(\mathbb{R})$, after rescaling. We get $hL(x, s)$. $\square$

Appendix. Singular Soergel bimodules

We describe the proper context for Lemma 5.1, following [Wil10].

$I \subset S$ is called finitary, if W_I — parabolic, corresponding to I — is finite. In what follows, all $I, J, K \dots \subset S$ are finitary (but S may not be).

Let w_I be the longest element of W_I,

$$\pi(I) = v^{l(w_I)} \sum_{w \in W_I} v^{-l(w)}.$$

We have

$$\underline{H}_I := \underline{H}_{w_I} = \sum_{w \in W_I} v^{l(w_I) - l(w)} H_w,$$

$\underline{H}_I^2 = \pi(I)\underline{H}_I, \underline{H}_K\underline{H}_I = \pi(K)\underline{H}_I, K \subset I.$

Write $\mathcal{H}_I$ for the Hecke algebra of $(W_I, I), \mathcal{H} = \mathcal{H}_S$.

Let ${}^I\mathcal{H} = \underline{H}_I\mathcal{H}, \mathcal{H}^I = \mathcal{H}\underline{H}_I, {}^I\mathcal{H}^J = {}^I\mathcal{H} \cap \mathcal{H}^J$.

We can define a multiplication:

$$*_J : {}^I\mathcal{H}^J \times {}^J\mathcal{H}^K \to {}^I\mathcal{H}^K, h_1 *_J h_2 = \frac{1}{\pi(J)} h_1 h_2.$$

Remark. If we regard ${}^I\mathcal{H}$ as a right $\mathcal{H}$-module, then ${}^I\mathcal{H} \simeq \mathrm{Ind}_{\mathcal{H}_I}^{\mathcal{H}}(\mathrm{triv})$. $Hom_{\mathcal{H}}({}^J\mathcal{H}, {}^I\mathcal{H}) \simeq {}^I\mathcal{H}^J$ and $*_J$ becomes a composition. Note that $\underline{H}_I$ is a projector to ${}^I\mathcal{H}$, up to a factor of $\pi(I)$.

${}^I\mathcal{H}^J$ is a free $\mathbb{Z}[v, v^{-1}]$-module of rank $\#(W_I\backslash W/W_J)$. For $p \in W_I\backslash W/W_J$ let $p_+ \in W$ be its maximal length representative.

Define ${}^I\underline{H}_p^J = \underline{H}_{p_+}$ — the Kazhdan-Lusztig basis of ${}^I\mathcal{H}^J$. Define ${}^I H_p^J = \sum_{x\in p} v^{l(p_+)-l(x)} H_x$ — standard basis in ${}^I\mathcal{H}^J$.

Structure

$$({}^I\mathcal{H}^J, *_J)_{I,J\subset S,\text{finitary}}$$

is called a Schur algebroid (cf. Remark above).

${}^I\mathcal{B}^J$ be a subcategory of graded (R^I, R^J)-bimodules, where $R^I = R^{W_I}$, that is generated by graded shifts of summands of bimodules

$$R^{I_1} \otimes_{R^{J_1}} R^{I_2} \otimes_{R^{J_2}} \cdots \otimes_{R^{J_{n-1}}} R^{I_n},$$

$I = I_1 \subset J_1 \supset I_2 \subset J_2 \supset I_3 \ldots$.

Theorem 6 ([Will10]). *a) Isomorphism classes of indecomposable objects in ${}^I\mathcal{B}^J$ up to grading shifts are in one-to-one correspondence with $W_I\backslash W/W_J$.*

b) We have a commutative diagram

$$\begin{tikzcd} {}^I\mathcal{B}^J \times {}^J\mathcal{B}^K \arrow[r, "\otimes_{R^J}"] \arrow[d, "ch\times ch"] & {}^I\mathcal{B}^K \arrow[d, "ch"] \\ {}^I\mathcal{H}^J \times {}^J\mathcal{H}^K \arrow[r, "*_J"] & {}^I\mathcal{H}^K \end{tikzcd}$$

where ch is a certain analogue of ch defined before for $\mathcal{H} = {}^\emptyset\mathcal{H}^\emptyset$. Moreover, for any $p \in W_I\backslash W/W_J$ there is a unique indecomposable ${}^I B_p^J$ with

$$ch({}^I B_p^J) = {}^I H_p^J + \sum_{q<p} g_{q,p}\, {}^I H_q^J.$$

c) $R \otimes_{R^I} {}^I B_p^J \otimes_{R^J} R \simeq B_{p_+}$.

Remarks. Lemma 5.1 follows from c) immediately by setting $J = \{s\}, I = \emptyset$. Also note that Soergel's conjecture we just finished proving implies

$$ch({}^I B_p^J) = {}^I\underline{H}_p^J,$$

see discussion in loc.cit.

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