Joel, lecture 1

0) Outline: motivation

$D^b Coh(T^*G(k,n)) \xrightarrow{?} DCoh(T^*G(n-k,n))$

1) Categorical sl_2-actions

2) Application to 0)

3) D-modules on Grassmanians & rel-n to Coh

4) Appl-n to categorification of knot invariants

- joint w. Cautis, Licata, Dodd

0) $T^*\mathbb{P}^1 = \{(X,V):\ 0 \subset V \subset \mathbb{C}^2\ X \in End(\mathbb{C}^2) \mid X\mathbb{C}^2 \subset V,\ XV \subset 0\}$

$= \{ 0 \underset{X}{\leftrightarrows} V \underset{X}{\leftrightarrows} \mathbb{C}^2 \}$

$\mathcal{O}_{\mathbb{P}^1}$ is spherical object in $D(Coh\, T^*\mathbb{P}^1)$, i.e.

1) $Ext^\bullet(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) \simeq H^\bullet(S^2) = \mathbb{C}[x]/(x^2)$.

2) $\mathcal{O}_{\mathbb{P}^1} \otimes \omega_{T^*\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}$

$\rightsquigarrow$ Seidel-Thomas twist $T: DCoh\, T^*\mathbb{P}^1 \longrightarrow DCoh(T^*\mathbb{P}^1)$

$T(A) = Cone(Ext^\bullet(\mathcal{O}_{\mathbb{P}^1}, A) \otimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow A)$

- categorification $a \mapsto a - \langle v, a \rangle v$

Thm (Seidel-Thomas) T is a category equivalence

$$T_{\mathcal{O}_{\mathbb{P}^1}}(A) = \begin{cases} A, & Ext^\bullet(\mathcal{O}_{\mathbb{P}^1}, A) = 0 \\ A[1], & A = \mathcal{O}_{\mathbb{P}^1} \otimes V \end{cases}$$

FM kernel: X, Y - smooth varieties $K \in DCoh(X \times Y)$

$\rightsquigarrow \Phi_K: DCoh(X) \longrightarrow DCoh(X)$

$A \longrightarrow \pi_{2*}(\pi_1^* A \otimes K)$ need $\pi_2|_{Supp(K)}$ - proper

The kernel for $T_{\mathcal{O}_{\mathbb{P}^1}}$

$A^\vee := RHom(A, \mathcal{O}_X)$

Kernel $= Cone(\mathcal{O}_{\mathbb{P}^1}^\vee \boxtimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_\Delta)$

$\mathcal{O}^{\vee}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1} \otimes L[-1]$, $L_{(V,X)} = Hom(\mathbb{C}^2/V, V)$ - line bundle

$Z = \mathbb{P}^1 \times \mathbb{P}^1 \cup \Delta \subset T^*\mathbb{P}^1 \times T^*\mathbb{P}^1$ $\quad [T^*\mathbb{P}^1 \to \mathcal{N} \subset \mathfrak{sl}_2]$

$= \{(X, V, X, W)\} = T^*\mathbb{P}^1 \times_{\mathcal{N}} T^*\mathbb{P}^1$

Get exact sequence $0 \to \mathcal{O}_\Delta(-\mathbb{P}^1 \times \mathbb{P}^1) \to \mathcal{O}_Z \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \to 0$

$$\mathcal{O}_\Delta(-\mathbb{P}^1 \times \mathbb{P}^1) = \mathcal{O}_\Delta \otimes L^*$$

$$\rightsquigarrow \underbrace{\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes L[1] \to \mathcal{O}_\Delta}_{\text{our previous cone}} \to \mathcal{O}_Z \otimes L \xrightarrow{+1} \cancel{\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}}$$

So $T_{\mathcal{O}_{\mathbb{P}^1}} = \Phi_{\mathcal{O}_Z \otimes L}$ (here $L = Hom(\mathbb{C}^2/V, W)$)

Cor: $\Phi_{\mathcal{O}_Z}$ is an equivalence

$$\begin{tikzcd} & Z \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ T^*\mathbb{P}^1 & & T^*\mathbb{P}^1 \end{tikzcd}$$

and $\Phi_{\mathcal{O}_Z} = p_{2*} p_1^*$

Generalization: $T^*\mathbb{P}^1 \rightsquigarrow T^*\mathbb{P}^{n-1} = T^*G(1,n)$

$$\begin{tikzcd} 0 \arrow[r, shift left] & V \arrow[l, shift left, "X"] \arrow[r, shift left] & \mathbb{C}^n \arrow[l, shift left, "X"] \end{tikzcd}$$

resolution $\{X \in End(\mathbb{C}^n) \mid X^2 = 0, \ rk\, X \leq 1\}$

another resol-n $T^*G(n-1,n)$

$$\begin{tikzcd} & Z \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ T^*G(1,n) \arrow[rr, phantom, "\square"] \arrow[dr] & & T^*G(n-1,n) \arrow[dl] \\ & \{X \in End(\mathbb{C}^n) \mid X^2 = 0, \ rk\, X \leq 1\} & \end{tikzcd}$$

Thm (Kawamata, Namikawa)

$$p_{2*} p_1^* : DCoh(T^*G(1,n)) \xrightarrow{\sim} DCoh(T^*G(n-1,n))$$

More general:

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ T^*G(k,n) \arrow[rr, phantom, "\square"] \arrow[dr] & & T^*G(n-k,n) \arrow[dl] \\ & \{X \mid X^2 = 0, \ rk\, X \leq k\} = B_k & \end{tikzcd}$$

$k \leq n-k$

$$Z = \left\{ \begin{tikzcd} & V \arrow[dl, "X"'] & \\ 0 & & \mathbb{C}^n \arrow[ul, "X"'] \arrow[dl, "X"] \\ & W \arrow[ul, "X"] & \end{tikzcd} \right\}$$

Irreducible components:

$Z_s = \{(X,V,W) \mid \dim V\cap W \geq s, \operatorname{rk} X \leq s\}$, $s=0,..k$.

~~Reason: $n = \operatorname{rk} X + \dim \ker X \leq \dim \ker X$~~

e.g. $Z_0 = \{(0,V,W)\} = Gr(k,n)\times Gr(n-k,n)$

$Z_k = \{(X,V,W) \mid V\subset W\}$ (for $n=2k$, get diagonal)

$B_s^o = \{X \mid X^2=0, \operatorname{rk} X = s\}$, then $Z_s = q^{-1}(B_s^o)$, $q: Z \to B_k$

$\cap B_k$

Thm (Namikawa) $n=4, k=2$ $p_{2*}p_1^*$ is not equivalence $DCoh(T^*G(2,4))\circlearrowleft$

Kawamata constructed sheaf supported on Z which gives equivalence

Problem: produce equivalence$^{(*)}$ $DCoh(T^*G(k,n)) \to DCoh(T^*G(n-k,n))$

to be solved using categorical $\mathfrak{sl}_2$-actions

(*) commuting w. $GL_n(\mathbb{C})$-action

1) Categorical $\mathfrak{sl}_2$-action

$\mathfrak{sl}_2(\mathbb{C})$ has basis e,f,h. $e=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $f=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

V-fin. dim. rep. of $\mathfrak{sl}_2$ h acts on V diagonalizably w. integral e-values

$\rightsquigarrow V = \bigoplus_r V_r$ (e-decomp for h)

$e: V_r \to V_{r+2}$, $f: V_r \to V_{r-2}$ so that $ef-fe|_{V_r} = r\cdot id_{V_r}$

Eg. $P_r := \{$subsets of $\{1,..n\}$ of size k, $r = n-2k\}$

$V_r = \mathbb{C}P_r$

$e: [S] \mapsto \sum_{\substack{T\subset S\\ |T|=|S|-1}} [T]$, $f: [S] \mapsto \sum_{\substack{T\supset S\\ |T|=|S|+1}} [T]$

Preliminary def-n:

A categorical $\mathfrak{sl}_2$-action is a sequence of additive categories $(\mathcal{D}_r)\big|_{r=-n}^{n}$

functors $E: \mathcal{D}_r \to \mathcal{D}_{r+2}$, $F: \mathcal{D}_r \to \mathcal{D}_{r-2}$ s.t. $\begin{cases} EF \simeq FE \oplus I_{\mathcal{D}_r}^{\oplus r} & r\geq 0 \\ FE \simeq EF \oplus I_{\mathcal{D}_r}^{\oplus -r}, & r\leq 0\end{cases}$

$\rightsquigarrow \mathfrak{sl}_2 \curvearrowright \bigoplus_r K(\mathcal{D}_r) \leftarrow$ split Grothendieck group.

Preliminary: because we need to pick some data producing isomorphisms

~~Def~~ Data: E, F - biadj-t

Actually, $\mathcal{D}_r$ will have a shift [1]. And we want

$\mathcal{D}_{r-1} \underset{F}{\overset{E}{\rightleftarrows}} \mathcal{D}_{r+1}$: $E_R \cong F[r]$, $E_L \cong F[-r]$

adj-s

Now $EF \to I \leftrightsquigarrow E \to E$ (ignoring shifts)

To give $EF \to FE \leftrightsquigarrow E^2 \to E^2$

So to give $EF \to FE \oplus I[r-1] \oplus I[r-3] \oplus \dots \oplus I[1-r]$

is the same thing as to give $E^2 \to E^2_{[shift]}$ & r homomorphisms $E \to E[shift]$.

$t: E^2 \to E^2[-2]$, $id: E \to E$, $x: E \to E[2]$ & $x^k: E \to E[2k]$

So our additional data are:

$t: E^2 \to E^2[-2]$ & $x: E \to E[2]$

s.t. 1) t, x produce isomorphisms of functors

2) In $Hom(E^n, E^n)$ have endomorphisms $t_1, \dots t_{n-1}, x_1, \dots x_n$ that define an action of nil-affine Hecke algebra (defined later)

- complete def-n:

A cat-l $\mathfrak{sl}_2$-action: 1) Categories $\mathcal{D}_r$ (w. shift functors)

2) Functors E, F

3) transformations x, t (+ units & counits of adjunction)

$\exists$ 2-cat-y $\mathcal{U}(\mathfrak{sl}_2)$, and cat-l $\mathfrak{sl}_2$-action is a 2-functor

$\mathcal{U}(\mathfrak{sl}_2) \longrightarrow Cat.$

Objects of $\mathcal{U}(\mathfrak{sl}_2)$ = integers

1-morphisms: monomials of E's & F's

2-morphisms: gen-d by x, t subject to Hecke rel-ns + adjunctions

E.g.

$\mathcal{D}_{-2} \rightleftarrows \mathcal{D}_0 \underset{E}{\overset{F}{\rightleftarrows}} \mathcal{D}_2$ s.t. $EF|_{\mathcal{D}_2} = I[1] \oplus I[-1]$, $E = F_R[-1]$

So F is a spherical functor in the following sense

$F: \mathcal{C} \to \mathcal{D}$ is spherical if $\exists\ \varphi: \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ (often $\varphi = [-2]$) s.t.

$F_R = \varphi \circ F_L$ & disting. triangle $I \to F_R F = \varphi F_L F \to \varphi$

Often triangle splits & $F_R F = I \oplus I[-2] = H^*(S^2) \otimes I$

If $(\mathcal{D}_r)$ is a cat-l $\mathfrak{sl}_2$-action, $\mathcal{D}_r$ are triang-d, then we'll define equivalence $T: \mathcal{D}_r \to \mathcal{D}_{-r}$ (generalizing spherical twist)

De-categor. level:

$$SL_2 \curvearrowright V, \quad t = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Lem: (Lusztig) $t|_{V_r} = f^{(r)} - f^{(r+1)}e + f^{(r+2)}e^{(2)} - \dots$, where $f^{(n)} = \frac{f^n}{n!}$

If we have a cat-l $\mathfrak{sl}_2$-action, then F^n is acted on by nil-affine HA NH_n

$NH_n = \langle t_1, \dots, t_{n-1}, x_1, \dots, x_n \mid$ braid rel-ns on t_i's

$t_i^2 = 0$, $x_i x_j = x_j x_i$

$t_i x_i = x_{i+1} t_{i+1} + 1$, for every x's & t's commute $\}$

NH_n acts on $\mathbb{C}[x_1, \dots, x_n] / (\mathbb{C}[x_1, \dots, x_n]^{S_n}_+) = H^*(Fl_n)$: $t_i(p) = \frac{p - s_i(p)}{x_i - x_{i+1}}$

Result (Rouquier) $F^n = F^{(n)} \otimes H^*(Fl_n)$ - w. symmetrized grading

Let $\Theta_s : F^{(r+s)} E^{(s)} : \mathcal{D}_r \to \mathcal{D}_{-r}$

$$\begin{tikzcd}[column sep=small]
\to \Theta_3 \arrow[r] & \Theta_2 \arrow[rr] \arrow[d, phantom, "\shortparallel"] & & \Theta_1 \arrow[rr] \arrow[d, phantom, "\shortparallel"] & & \Theta_0 \arrow[d, phantom, "\shortparallel"] \\
& F^{(r+2)}E^{(2)} \arrow[dr] & & F^{(r+1)}E \arrow[dr] & & F^{(r)} \\
& & F^{(r+1)}FEE \arrow[ur] & & F^{(r)}FE \arrow[ur, "\text{adj-n}"'] &
\end{tikzcd}$$

- Rickard complex

We'll get complex of FM kernels $\rightsquigarrow$ get a functor defined by the complex of kernels

Thm (Chuang-Rouquier, Cautis-Kamnitzer-Licata, Rouquier)

(Θ_s) has unique convolution (iterated cone), say T &

$T: \mathcal{D}_r \to \mathcal{D}_{-r}$ is an equivalence

$$\mathcal{D}_{-v-2p} \qquad \mathcal{D}_{-v} \qquad \mathcal{D}_{v} \overset{F^{(p)}}{\frown} \overset{A}{\mathcal{D}_{v+p}}$$

A is called highest wt object if $E(A)=0$

Property: $A \in \mathcal{D}_{v+p}$ is highest wt $\Rightarrow T(F^{(p)}(A)) = F^{(v+p)}(A)[p(v+p+1)]$

Note: $F^{(p)}A$'s form a spanning class

Get a required generalisation

2) $\mathcal{D}_v = D Coh(T^* G(k,n))$, $v = n-2k$

$G(k,n) \times G(k+p,n) \supset I^p(k,n)$

$$\overset{\shortparallel}{\{(V,W) \mid V \subset W\}}$$

$C^p(k,n) := T^*_{I^p(k,n)}(G(k,n) \times G(k+p,n)) = \{(X,V,W) \mid V \subset W\}$

Define $F^{(p)}: \underset{\mathcal{D}_v}{\underset{\shortparallel}{D Coh(T^* G(k,n))}} \rightarrow \underset{\mathcal{D}_{v-p}}{\underset{\shortparallel}{D Coh(T^* G(k+p,n))}}$

by using $\mathcal{O}_{C^p(k,n)} \otimes \det(W/V)^{n-2k}$

$E^{(p)}$ using $\mathcal{O}_{C^p(k,n)} \otimes \det(\mathbb{C}^n/W)^{-p} \det(V)^p$

as kernels

Thm (Cautis-Kamnitzer-Licata)

These functors give cat-l $\mathfrak{sl}_2$-actions (w. suitable t, x)

Consider $\widetilde{T^* G(k,n)} = \{(X, V, \underset{\mathbb{C}}{\underset{\Uparrow}{a}}),\ XV \subset V \mid X|_{\mathbb{C}^n/V} = a \cdot id,\ X|_V = -a \cdot id\}$

$\mathcal{F} \in D Coh(T^* G(k,n) \times T^* G(k+1,n))$

$(a,b) \rightsquigarrow \widetilde{T^* G(k,n)} \times \widetilde{T^* G(k+1,n)}$

$\exists$ morphism $\mathcal{F} \rightarrow \mathcal{F}[2]$ (depending a, b) given an obstruction to extending $\mathcal{F}$.

$\rightsquigarrow x: F \rightarrow F[2]$ which does the job

Can get t in a similar way

This gives categorical $\mathfrak{sl}_2$-action $\rightsquigarrow$ equiv. $T: DCoh(T^*G(k,n)) \to DCoh(T^*G(n-k,n))$

$\Theta_s = F^{(v+s)} E^{(s)}[-s]$, $s = 0, \ldots k$

$Z = Z_0 \cup Z_1 \cup \ldots \cup Z_k$

Steinberg

Θ_s w. $\Phi_{\Theta_s} = \Theta_s$

Lem: $\Theta_s = \mathcal{O}_{\tilde{Z}_s} \otimes \det(\mathbb{C}^n/W)^s \det V^s [-s]$

normalization

easy comp-n using composition of kernels

$\Rightarrow T$ is a Cohen-Macaulay sheaf (i.e $T^\vee$ is shifted sheaf)

Recall that for $T^*\mathbb{P}^{n-1}$ get $\Theta = \mathcal{O}_Z$

Let Z^0 - open dense subset

$= \{(V, W, X) : \dim \ker X + \dim V \cap W \le n+1\}$

i.e $\operatorname{im} X \subset V \cap W$ is codim 1 subspace

Fact (easy) $Z_s \cap Z_{s'}$ is a divisor $\Leftrightarrow |s-s'| = 1$

$Z^0 \cap Z_s \cap Z_{s'}$ is empty unless $|s-s'|=1$ (or $s=s'$)

Thm $\exists$ line bundle $\mathcal{L}$ on Z^0 s.t. $T = j_* \mathcal{L}$, $j: Z^0 \to Z$

Cautis.

Cor: $T^{-1}(A) = L' \otimes T(A \otimes L')$ for some line bundles

$\Rightarrow$ existence of affine braid group action

Cor: for $T^*Gr(2,4)$ CKL equivalence is same as Kawamata's