# Rational Cherednik Algebras of type A

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## 1 Rational Cherednik algebras

### 1.1 Smash-product algebras.

We are interested in filtered deformations of the algebra $\mathbb{C}[\mathrm{Sym}^n(\mathbb{C}^2)] = \mathbb{C}[(\mathbb{C}^2)^{\oplus n}]^{\mathfrak{S}_n} = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$. Here and for the rest of these notes we denote $\mathfrak{h} := \mathbb{C}^n$. The deformations of $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{\mathfrak{S}_n}$ we are going to produce arise from deformations of a closely related noncommutative algebra, the smash-product $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]\#\mathfrak{S}_n$. Let us define this algebra in a greater generality.

Let $A$ be an associative algebra with an action of a finite group $G$ by algebra automorphisms. The *smash product algebra* $A\#G$ is, as a vector space, simply $A \otimes \mathbb{C}G$. The product is given on pure tensors by $(f_1 \otimes g_1)(f_2 \otimes g_2) = f_1g_1(f_2)\otimes g_1g_2$ and is extended bilinearly. Note that the assignment $g \mapsto 1\otimes g$ (resp. $a \mapsto a\otimes 1$) identifies $\mathbb{C}G$ (resp. $A$) with a subalgebra of $A\#G$. Consider the trivial idempotent $e = \frac{1}{|G|}\sum_{g\in G} g \in A\#G$ and the corresponding *spherical subalgebra* $e(A\#G)e$. Note that this is not a unital subalgebra of $A\#G$, but $e$ is the unit in the spherical subalgebra. The connection of the smash product algebra with the algebra of invariants is given by the following result, whose proof is an exercise.

**Proposition 1.1** *The map $a \mapsto ae = ea = eae$ from $A^G$ to $e(A\#G)e$ is an isomorphism of algebras.*

Let us now explain how deformations of $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]\#\mathfrak{S}_n$ are related to deformations of $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$. First of all, note that the smash-product $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]\#\mathfrak{S}_n$ is graded, with $\mathfrak{S}_n$ on degree 0 and $\mathfrak{h}, \mathfrak{h}^*$ on degree 1. Let $\mathcal{A} = \bigcup_{n\geq 0}\mathcal{A}^{\leq n}$ be a filtered deformation of $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]\#\mathfrak{S}_n$, that is, $\mathrm{gr}\,\mathcal{A} = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$. Note that it follows that $\mathbb{C}\mathfrak{S}_n \subseteq \mathcal{A}^{\leq 0}$, so we can consider the spherical subalgebra $e\mathcal{A}e$. This algebra inherits a filtration from $\mathcal{A}$, $(e\mathcal{A}e)^{\leq n} = e\mathcal{A}^{\leq n}e$.

**Proposition 1.2** *We have* $\mathrm{gr}(e\mathcal{A}e) = e(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]\#\mathfrak{S}_n)e = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$.

So we can get filtered deformations of $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$ from those of the smash-product algebra. Even though this algebra is no longer commutative, a presentation by generators and relations is easier. Namely, we have:

$$\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]\#\mathfrak{S}_n = (T(\mathfrak{h} \oplus \mathfrak{h}^*)\#\mathfrak{S}_n)/(x\otimes y - y\otimes x, x, y \in \mathfrak{h} \oplus \mathfrak{h}^*).$$

Note that $T(\mathfrak{h} \oplus \mathfrak{h}^*)\#\mathfrak{S}_n$ inherits a filtration from $T(\mathfrak{h} \oplus \mathfrak{h}^*)$ (again, we put $\mathbb{C}\mathfrak{S}_n$ in degree 0). So, to get a deformation of $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]\#\mathfrak{S}_n$ we can correct the commutation relation on $T(\mathfrak{h} \oplus \mathfrak{h}^*)\#\mathfrak{S}_n$ by $[x, y] = \beta(x, y)$, where $\beta(x, y) \in (T(\mathfrak{h} \oplus \mathfrak{h}^*)\#\mathfrak{S}_n)^{\leq 1} = \mathbb{C}\mathfrak{S}_n \oplus (\mathfrak{h} \oplus \mathfrak{h}^*) \otimes \mathbb{C}\mathfrak{S}_n$. This is what we're going to do to define rational Cherednik algebras. But before, let us look at another motivation via Dunkl operators.

### 1.2 Dunkl operators.

For $i \neq j \in \{1, \ldots, n\}$, let $s_{ij} \in \mathfrak{S}_n$ denote the transposition $i \leftrightarrow j$. For each reflection $s_{ij} \in \mathfrak{S}_n$, let $P_{ij} \subseteq \mathfrak{h}$ denote the reflection hyperplane associated to $s_{ij}$, that is, $P_{ij} = \{x_i = x_j\}$, where $x_i \in \mathfrak{h}^*$ is the standard $i$-th coordinate function. Let $\mathfrak{h}^{reg} = \mathfrak{h} \setminus \bigcup_{i<j} P_{ij}$. Note that $\mathfrak{h}^{reg}$ is the locus where the action of $\mathfrak{S}_n$ is free, that it is Zariski open, and $\mathfrak{h}^{reg} = \mathfrak{h} \setminus \{\prod_{i<j}(x_i - x_j) = 0\}$. Let $\mathcal{D}(\mathfrak{h}^{reg})$ be the algebra of algebraic differential operators on $\mathfrak{h}^{reg}$. Note that $\mathfrak{S}_n$ acts on $\mathfrak{h}^{reg}$ and therefore it also acts on $\mathcal{D}(\mathfrak{h}^{reg})$.

**Definition 1.3** *For any $i = 1, \ldots, n$, $t, c \in \mathbb{C}$, the Dunkl operator is defined to be*

$$D_i = t\frac{\partial}{\partial x_i} - c\sum_{j\neq i}\frac{1}{x_i - x_j}(1 - s_{ij}) \in \mathcal{D}(\mathfrak{h}^{reg})\#\mathfrak{S}_n. \quad (1)$$

Note that for $\sigma \in \mathfrak{S}_n$ we have $\sigma D_i \sigma^{-1} = D_{\sigma(i)}$. Also, for $j \neq i$:

$$[D_i, x_j] = cs_{ij},$$

and

$$[D_i, x_i] = t - \sum_{j \neq i} cs_{ij}.$$

Finally, we have the following important technical lemma.

**Lemma 1.4** *Dunkl operators $D_i$, $D_j$ commute.*

*Proof.* The proof is left as an exercise for the reader.□

In our discussion, we will need a slightly modified version of the construction above. Namely, let $\hbar$ be a variable and consider $\mathrm{Rees}_\hbar(\mathcal{D}(\mathfrak{h}^{reg})\#\mathfrak{S}_n)$, the Rees algebra with respect to the usual filtration on $\mathcal{D}(\mathfrak{h}^{reg})$ by the order of a differential operator (and $\mathfrak{S}_n$ is in filtration degree 0). Recall that this is $\bigoplus_{n\geq 0}(\mathcal{D}(\mathfrak{h}^{reg})\#\mathfrak{S}_n)^{\leq n}\hbar^n \subseteq \mathcal{D}(\mathfrak{h}^{reg})\#\mathfrak{S}_n[\hbar]$. Then, let

$$D_i^\hbar = \hbar \frac{\partial}{\partial x_i} - c\sum_{i\neq j} \frac{1}{x_i - x_j}(1 - s_{ij}) \in \mathrm{Rees}_\hbar(\mathcal{D}(\mathfrak{h}^{reg})\#\mathfrak{S}_n).$$

Clearly, the relations above also hold in this setting if we replace $t$ by $\hbar$. The reason why we pass to the Rees algebra is the following: note that for $t \neq 0$, setting $\hbar = t$ we recover the notion of Dunkl operators above. However, this is no longer true for $t = 0$. Recall that $\mathrm{Rees}_\hbar(\mathcal{D}(\mathfrak{h}^{reg})\#\mathfrak{S}_n)/(\hbar) = \mathrm{gr}(\mathcal{D}(\mathfrak{h}^{reg})\#\mathfrak{S}_n) = \mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*]\#\mathfrak{S}_n$. So we have:

$$D_i^0 = y_i - c\sum_{i\neq j} \frac{1}{x_i - x_j}(1 - s_{ij}) \in \mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*]\#\mathfrak{S}_n,$$

where $y_i \in \mathfrak{h}$ in the right hand side is dual to $x_i \in \mathfrak{h}^*$.

## 1.3 Rational Cherednik algebras of type A.

For $c \in \mathbb{C}$, let $H_{\hbar,c}$ be the $\mathbb{C}[\hbar]$-subalgebra inside $\mathrm{Rees}_\hbar(\mathcal{D}(\mathfrak{h}^{reg}))\#\mathfrak{S}_n$ generated by $\mathfrak{h}^*$, $\mathfrak{S}_n$, and Dunkl operators $D_i^\hbar$, $i = 1, \ldots, n$.

**Proposition 1.5** *The algebra $H_{\hbar,c}$ is the quotient of $(\mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle\#\mathfrak{S}_n)[\hbar]$ by the ideal generated by the following relations*

$$[x_i, x_j] = 0, \;\; [y_i, y_j] = 0, \;\; [y_i, x_j] = cs_{ij} \;\; [y_i, x_i] = \hbar - \sum_{j\neq i} cs_{ij}. \tag{2}$$

*For $i, j = 1, \ldots, n$.*

*Proof.* Denote the algebra defined in the proposition by $H'$. By the results of Subsection 1.2 it is clear that we have an epimorphism $H' \to H_{\hbar,c}$ defined via $x_i \mapsto x_i$, $y_i \mapsto D_i^\hbar$, $\mathfrak{S}_n \ni \sigma \mapsto \sigma$. Let us show that this is injective. It is clear by its definition that $H'$ is generated, over $\mathbb{C}[\hbar]$, by elements of the form $\left(\prod_{i=1}^n x_i^{a_i}\right)\left(\prod_{i=1}^n y_i^{b_i}\right)\sigma$. Note that the algebra $\mathrm{Rees}(\mathcal{D}(\mathfrak{h}^{reg})\#\mathfrak{S}_n)$ can be filtered by the ordered of a differential operator, and its associated graded is $(\mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*]\#\mathfrak{S}_n)[\hbar]$. The symbols of the elements $\left(\prod_{i=1}^n x_i^{a_i}\right)\left(\prod_{i=1}^n (D_i^\hbar)^{b_i}\right)\sigma$ in $(\mathbb{C}[T^*(\mathfrak{h}^{reg})]\#\mathfrak{S}_n)[\hbar] = (\mathbb{C}[\mathfrak{h}^{reg} \oplus \mathfrak{h}^*]\#\mathfrak{S}_n)[\hbar]$ are clearly linearly independent, so the result follows. □

Now we specialize $\hbar$ to be a complex number.

**Definition 1.6** *For $t, c \in \mathbb{C}$, the rational Cherednik algebra $H_{t,c}$ associated to $t, c$ is the quotient of the algebra $\mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle\#\mathfrak{S}_n$ by the following relations:*

$$[x_i, x_j] = 0, \;\; [y_i, y_j] = 0, \;\; [y_i, x_j] = cs_{ij} \;\; [y_i, x_i] = t - \sum_{j\neq i} cs_{ij}. \tag{3}$$

*For $i, j = 1, \ldots, n$. In other words, $H_{t,c} = H_{\hbar,c}/(\hbar - t)$.*

So, for example, $H_{0,0} = \mathbb{C}[x_1, \dots, x_n, y_1 \dots, y_n]\#\mathfrak{S}_n$, and $H_{1,0} = \mathcal{D}(\mathbb{C}^n)\#\mathfrak{S}_n$.

Note that, by definition, for every $t \in \mathbb{C}^\times$, $c \in \mathbb{C}$, using Dunkl operators we get an injective homomorphism

$$\Theta_{t,c} : H_{t,c} \to \mathcal{D}(\mathfrak{h}^{reg})\#\mathfrak{S}_n, \tag{4}$$

given by $\Theta_{t,c}(x_i) = x_i$, $\Theta_{t,c}(y_i) = D_i$. For $t = 0$, we get

$$\Theta_{0,c} : H_{0,c} \to \mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*]\#\mathfrak{S}_n, \tag{5}$$

given by similar formulae.

**Remark 1.7** *From the relations (3) it's not hard to see that, if $\lambda \in \mathbb{C}^\times$, then we get a natural isomorphism $H_{t,c} \to H_{\lambda t, \lambda c}$. Hence, we have essentially two different cases: $t = 0$ and $t \neq 0$.*

The algebra $H_{t,c}$ clearly inherits a filtration from the grading in $\mathbb{C}\langle x_1, \dots, x_n, y_1 \dots, y_n\rangle\#\mathfrak{S}_n$, where in the latter algebra $\mathfrak{S}_n$ has degree 0. The next theorem tells us that $H_{t,c}$ is indeed a filtered deformation of the smash-product algebra $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]\#\mathfrak{S}_n$.

**Theorem 1.8 (PBW theorem for Rational Cherednik Algebras)** $\mathrm{gr}(H_{t,c}) = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]\#\mathfrak{S}_n$.

*Proof.* It is easy to see that we have a map $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]\#\mathfrak{S}_n \twoheadrightarrow \mathrm{gr}(H_{t,c})$. From the commutation relations it is clear that any element of $H_{t,c}$ can be written as a linear combination of monomials of the form $\left(\prod_{i=1}^n x_i^{a_i}\right)\left(\prod_{i=1}^n y_i^{b_i}\right)\sigma$, so the map is surjective. The PBW theorem, then, is equivalent to the claim that these monomials form a basis of $H_{t,c}$, and this follows from the proof of Proposition 1.5. □

Recall that $e = \frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \sigma$ is the trivial idempotent in $\mathbb{C}\mathfrak{S}_n$. Then, we get.

**Corollary 1.9** *The spherical subalgbera $eH_{t,c}e$ is a filtered deformation of $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$.*

## 1.4 Rational Cherednik algebra as universal deformation.

In this subsection, we explain how a closely related algebra to $H_{t,c}$ satisfies a certain universality property with respect to deformations. Take elements $\overline{x}_i = x_i - \frac{1}{n}\sum_{j=1}^n x_j$, $\overline{y}_i = y_i - \frac{1}{n}\sum_{j=1}^n y_j \in H_{t,c}$. Note that $\sum \overline{x}_i = 0 = \sum \overline{y}_i$. Let $H^+_{t,c}$ be the subalgebra of $H_{t,c}$ generated by $\overline{x}_i, \overline{y}_i$ and $\mathfrak{S}_n$. Note that we can present the algebra $H^+_{t,c}$ by generators and relations as the quotient of $\mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle\#\mathfrak{S}_n$ by the relations:

$$\sum_{i=1}^n x_i = 0, \;\; \sum_{y=1}^n y_i = 0, \;\; [x_i, x_j] = 0, \;\; [y_i, y_j] = 0, \;\; [y_i, x_j] = cs_{ij} - \frac{t}{n}, \;\; [y_i, x_i] = t\frac{n-1}{n} - \sum_{j \neq i} cs_{ij}. \tag{6}$$

It follows from the theory developed above that $H^+_{t,c}$ is a filtered deformation of the algebra $\mathbb{C}[\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*]\#\mathfrak{S}_n$, where $\mathfrak{h}^+ = \{(a_1, \dots, a_n) \in \mathbb{C}^n : \sum a_i = 0\}$ is the reflection representation of $\mathfrak{S}_n$. Take any deformation of the form $H_\kappa := T(\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*)\#\mathfrak{S}_n/([x, y] = \kappa(x, y))$, where $T(\bullet)$ denotes tensor algebra and $\kappa(x, y)$ is a map $\kappa : \bigwedge^2(\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*) \to \mathbb{C}\mathfrak{S}_n$, $\kappa(x, y) = \sum_{\sigma \in \mathfrak{S}_n} \kappa_\sigma(x, y)\sigma$. Let us see why $\kappa$ must be of the form (6). We will see that some $\kappa_\sigma$ are identically 0. First, it is an exercise to see that if $\mathrm{gr}\, H_\kappa = \mathbb{C}[\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*]\#\mathfrak{S}_n$, then $\kappa$ must be $\mathfrak{S}_n$-equivariant (otherwise the condition fails already in filtration degree 1). Note that

$$[\kappa(u, v), w] = \sum_{\sigma \in \mathfrak{S}_n} \kappa_\sigma(u, v)(\sigma(w) - w)\sigma, \tag{7}$$

this follows easily from the definition of the smash-product algebra. Now, the Jacobi identity in $H_\kappa$ tells us that we must have:

$$[\kappa(u, v), w] + [\kappa(v, w), u] + [\kappa(w, u), v] = 0$$

for any $u, v, w \in \mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*$. Since $\mathrm{gr}\, H_\kappa = \mathbb{C}[\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*]\#\mathfrak{S}_n$, the map $(\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*) \otimes \mathbb{C}\mathfrak{S}_n \to H_\kappa$ is injective. Then, by (7), for every $\sigma \in \mathfrak{S}_n$, $u, v, w \in \mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*$ we have:

$$\kappa_\sigma(u, v)(\sigma(w) - w) + \kappa_\sigma(v, w)(\sigma(u) - u) + \kappa_\sigma(w, u)(\sigma(v) - v) = 0. \tag{8}$$

It follows that, if $\operatorname{rank}(\sigma - 1) > 2$, then $\kappa_\sigma$ must be identically 0. Since the action of $\mathfrak{S}_n$ on $\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*$ is lifted from an action on $\mathfrak{h}^+$, it follows that $\kappa_\sigma$ must be identically zero unless $\sigma = s_{ij}$ for some $i, j$ or $\sigma = 1$.

Consider the case $\sigma = s_{ij}$. Note that, if $k \neq i, j$ then $s_{ij}(x_k) = x_k$. It follows from (8) that $\kappa_{s_{ij}}(x_k, \bullet) = 0$. Similarly, $\kappa_{s_{ij}}(y_k, \bullet) = 0$. Now, $2x_j = x_j - x_i - \sum_{k \neq i,j} x_k$, and similarly for $x_i$. From here it follows that $\kappa_{s_{ij}}(x_j, x_i) = 0$. Using a similar formula for the $y's$, one has that $\kappa_{s_{ij}}(y_i, x_i) = -\kappa_{s_{ij}}(y_i, x_j)$.

Now, note that $\kappa_1$ is a $\mathfrak{S}_n$-invariant skew-symmetric form on $\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*$. By irreducibility of $\mathfrak{h}^+$ and Schur's lemma, there is, up to scaling, a unique such form, namely the canonical symplectic form on $\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^* = T^*(\mathfrak{h}^+)$. Finally, the $\mathfrak{S}_n$-equivariance of the form $\kappa$ forces us to have relations of the form (6).

The discussion above makes us see that the algebra $H^+_{\hbar,\overline{c}}$, where $\hbar, \overline{c}$ are formal variables in degree 2 (so that $H^+_{\hbar,\overline{c}}$ is graded), should be a 'universal' deformation of $\mathbb{C}[\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*]\#\mathfrak{S}_n$. It turns out that is the case. First, let us explain what we mean by a universal deformation. Recall that a graded deformation of a $\mathbb{Z}_{\geq 0}$-graded algebra $A$ of degree $n$ over a vector space $P$, is a free, graded $S(P)$-algebra $\mathcal{A}$ where $P$ sits in $\mathcal{A}$ in degree $n$, and such that $\mathcal{A}/\mathcal{A}P = A$.

**Definition 1.10** *A universal graded deformation of a $\mathbb{Z}_{\geq 0}$-graded algebra $A$ of degree 2 is a graded deformation $\mathcal{A}_{un}$ over a vector space $P_{un}$, where $P_{un}$ sits inside $\mathcal{A}_{un}$ in degree 2, such that, for any other deformation $\mathcal{A}'$ of $A$ over a vector space $P'$ where $P'$ sits in degree 2, there exists a unique linear map $P_{un} \to P'$ such that the deformations $\mathcal{A}'$ and $S(P') \otimes_{S(P_{un})} \mathcal{A}_{un}$ are equivalent (via a unique equivalence).*

In the next subsection, we'll see that $H^+_{\hbar,\overline{c}}$ is indeed a universal deformation of $\mathbb{C}[\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*]\#\mathfrak{S}_n$.

## 1.5 Hochschild cohomology.

Let $A$ be an associative algebra, and let $M$ be an $A$-bimodule. The space of Hochschild $n$-cochains on $M$, $C^n(A, M)$, is the space of $\mathbb{C}$-linear maps $A^{\otimes n} \to M$. We have a map $d : C^n(A, M) \to C^{n+1}(A, M)$ given by the formula:

$$\begin{aligned} df(a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}) = \quad & f(a_1 \otimes \cdots \otimes a_n)a_{n+1} - f(a_1 \otimes \cdots \otimes a_n a_{n+1}) + f(a_1 \otimes \cdots \otimes a_{n-1}a_n \otimes a_{n+1}) + \cdots \\ & +(-1)^n f(a_1 a_2 \otimes \cdots \otimes a_n) + (-1)^{n+1} a_1 f(a_2 \otimes \cdots \otimes a_{n+1}). \end{aligned}$$

It is an exercise to see that $d^2 = 0$, so that we have a complex $C^0(A, M) \to C^1(A, M) \to \cdots$. The cohomology of this complex is called the Hochschild cohomology and is denoted by $\overline{\mathrm{HH}}^i(A, M)$. We denote $\overline{\mathrm{HH}}^i(A) := \overline{\mathrm{HH}}^i(A, A)$. Note that, using the standard resolution of the $A$-bimodule $A$, we can see that, actually, $\overline{\mathrm{HH}}^i(A, M) = \mathrm{Ext}^i_{A\text{-}Bimod}(A, M)$.

Now assume that $A$ is $\mathbb{Z}$-graded, and that $M$ is a $\mathbb{Z}$-graded $A$-bimodule. We can modify the construction above to get a notion of graded Hochschild cohomology as follows. Define $C^n(A, M)^m = \{f \in C^n(A, M) : f((A^{\otimes n})^i) \subseteq M^{i+m}\}$. Note that $d(C^n(A, M)^m) \subseteq C^{n+1}(A, M)^m$. So we can define $\mathrm{HH}^\bullet(A, M)^m$ to be the cohomology of the complex $d : C^n(A, M)^m \to C^{n+1}(A, M)^m$, and define $\mathrm{HH}^\bullet(A, M) := \bigoplus_m \mathrm{HH}^\bullet(A, M)^m$. Note, however,that in general $\overline{\mathrm{HH}}^i(A, M) \neq \mathrm{HH}^i(A, M)$. The following result is well-known in deformation theory.

**Theorem 1.11** *Assume that*

- $\mathrm{HH}^2(A)^{-2}$ *is finite dimensional.*
- $\mathrm{HH}^2(A)^i = 0$ *for* $i < -2$.
- $\mathrm{HH}^3(A)^i = 0$ *for* $i < -3$.

*Then, modulo uniqueness of the map in the universal property, a universal graded deformation of $A$ exists, with $P_{un} = (\mathrm{HH}^2(A)^{-2})^*$.*

For a proof of Theorem 1.11 see, for example, [7]. A condition that guarantees uniqueness of the map in the universal property for a universal graded deformation is that $\mathrm{HH}^1(A)^i = 0$ for $i \leq -2$. We will not check this. Instead, we will see uniqueness in the case of interest for us by more elementary methods.

So we need to compute $\mathrm{HH}^2(\overline{A})^i$, $i \leq -2$ and $\mathrm{HH}^3(\overline{A})^j$, $j < -3$ for $\overline{A} = \mathbb{C}[\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*]\#\mathfrak{S}_n$. Let us sketch how to do this. Let $A = \mathbb{C}[\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*]$. For $\sigma \in \mathfrak{S}_n$, define the $A$-bimodule $A\sigma$ as follows: as a left $A$-module, $A\sigma$ is just $A$. The right multiplication is twisted by the action of $\sigma$: $(a_1\sigma)a_2 = (a_1\sigma(a_2))\sigma$. Note that $\mathfrak{S}_n$ acts on the direct sum $\bigoplus_\sigma A\sigma$, $\gamma(a\sigma) = \gamma(a)\gamma\sigma\gamma^{-1}$. Then, we get a $\mathfrak{S}_n$-action on $\bigoplus_\sigma \mathrm{HH}^i(A, A\sigma)$. We state without proof the following result.

**Proposition 1.12** *We have an isomorphism of graded vector spaces* $\mathrm{HH}^i(A\#\mathfrak{S}_n) \cong \left(\bigoplus_{\sigma\in\mathfrak{S}_n} \mathrm{HH}^i(A, A\sigma)\right)^{\mathfrak{S}_n}$.

So we need to compute $\mathrm{HH}^i(A, A\sigma)$ for $\sigma \in \mathfrak{S}_n$. For $\sigma \in \mathfrak{S}_n$, pick a basis $v_1, \dots, v_{2(n-1)}$ of $\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*$ such that $\sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_{2(n-1)})$. Note that every number $\sigma_i$ is a root of 1, so we can think of $\sigma_i$ as a member of a cyclic group acting on $\mathbb{C}v_i$. Then, we get:

$$A\sigma := \bigotimes_{i=1}^{2(n-1)} \mathbb{C}[v_i]\sigma_i.$$

And, by the Künneth formula,

$$\mathrm{HH}^\bullet(A, A\sigma) = \bigotimes \mathrm{HH}^\bullet(\mathbb{C}[v_i], \mathbb{C}[v_i]\sigma_i),$$

where the homological degree on the left hand side equals the sum of the homological degrees on the right hand side. So we have reduced the problem to computing the Hochschild cohomology $\mathrm{HH}^\bullet(\mathbb{C}[v_i], \mathbb{C}[v_i]\sigma_i)$. Now we can use the standard Koszul resolution of $\mathbb{C}[v_i]$ to compute these cohomology. Since this resolution has length 1, one immediately gets that $\mathrm{HH}^j(\mathbb{C}[v_i], \mathbb{C}[v_i]\sigma_i) = 0$ for $j \geq 2$, regardless of the value of $\sigma_i$. If $\sigma_i = 1$, then $\mathrm{HH}^0(\mathbb{C}[v_i], \mathbb{C}[v_i]) = \mathbb{C}[v_i]$ with its usual grading, and $\mathrm{HH}^1(\mathbb{C}[v_i], \mathbb{C}[v_i]) = \mathbb{C}[v_i]$, with its grading shifted by 1. If $\sigma_i \neq 1$, then $\mathrm{HH}^\bullet(\mathbb{C}[v_i], \mathbb{C}[v_i]\sigma_i)$ is 1-dimensional, concentrated in degree 1, and $\mathrm{HH}^1(\mathbb{C}[v_i], \mathbb{C}[v_i]\sigma_i)^{-1} = \mathbb{C}$.

Since the action of $\mathfrak{S}_n$ on $\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*$ is lifted from an action on $\mathfrak{h}^+$, any $\sigma \in \mathfrak{S}_n$ has an even number of eigenvalues different from 1. From here it follows easily that $\mathrm{HH}^2(A, A\sigma)^j = 0$ for $j < -2$, and $\mathrm{HH}^3(A, A\sigma)^j = 0$ for $j < -3$. It also follows that, unless $\sigma = s_{ij}$ for some $i, j$, every element in $\mathrm{HH}^\bullet(A, A\sigma)$ has homological degree at least 4. Then, only reflections are important when computing $\mathrm{HH}^2(\overline{A})^{-2}$. In fact, $\dim \left(\bigoplus_{i,j} \mathrm{HH}^2(A, As_{ij})\right)^{\mathfrak{S}_n} = 2$ see, for example, [7, Exercise 7.10]. In the computation, a very important fact one uses is that $\mathfrak{h}^+$ is an irreducible $\mathfrak{S}_n$-module.

So $\mathbb{C}[\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*]\#\mathfrak{S}_n$ does have a universal deformation over a vector space of dimension 2. It follows from our computations in the previous subsection that this deformation must coincide with $H^+_{\hbar,\overline{c}}$. Finally, note that, since $V$ and $\mathbb{C}\mathfrak{S}_n$ generate $H^+_{\hbar,\overline{c}}$ over $\mathbb{C}[\hbar, \overline{c}]$, any self-equivalence of $H^+_{\hbar,\overline{c}}$ as a deformation of $\mathbb{C}[\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*]\#\mathfrak{S}_n$ must be the identity. Hence, $H^+_{\hbar,\overline{c}}$ is a universal deformation.

## 1.6 Spherical RCA.

We return to the study of the rational Cherednik algebra $H_{t,c}$. Denote the spherical subalgebra $eH_{t,c}e$ by $B_{t,c}$. Consider the $H_{t,c} - B_{t,c}$-bimodule $H_{t,c}e$. An exercise is to show that $\mathrm{End}_{H_{t,c}}(H_{t,c}e) = B_{t,c}^{opp}$. It is clear that we have a map $H_{t,c} \to \mathrm{End}_{B_{t,c}}(H_{t,c}e)$. It turns out that this map is an isomorphism.

**Theorem 1.13 (Double centralizer property)** $H_{t,c} \cong \mathrm{End}_{B_{t,c}}(H_{t,c}e)$.

Let us outline an strategy to prove Theorem 1.13. First, we prove it in the associated graded case $t, c = 0$. We prove injectivity first, which is easier, and then surjectivity of the natural map $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]\#\mathfrak{S}_n \to \mathrm{End}_{\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{\mathfrak{S}_n}}(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*])$. After this, we can give a filtration to $\mathrm{End}_{B_{t,c,}}(H_{t,c}e)$ that will reduce the general case to the associated graded one.

Injectivity of the double centralizer map in the associated graded case is straightforward. Surjectivity is harder. To prove it, one observes that surjectivity would hold if the action of $\mathfrak{S}_n$ on $\mathfrak{h} \oplus \mathfrak{h}^*$ were free. Of course, this is not the case here. But we can restrict to a subset of codimension 2 where this holds, and the codimension claim implies that the map is surjective. We provide details in the Appendix, see Subsection 4.1.

Recall that for $t = 0$, we have the Dunkl embedding $\Theta_{0,c} : H_{0,c} \to \mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*]\#\mathfrak{S}_n$. Passing to spherical subalgebras, we get an inclusion of $B_{0,c}$ in $\mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*]^{\mathfrak{S}_n}$. In particular, we see that the algebra $B_{0,c}$ is commutative. So it follows that the Poisson bracket $\{\bullet, \bullet\}_{0,c}$ induced in $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$ is identically 0. Also note that the bracket $\{\bullet, \bullet\}_{t,c}$ induced by $B_{t,c}$ depends linearly on $(t, c)$. It follows that $\{\bullet, \bullet\}_{t,c} = \{\bullet, \bullet\}_{t,0}$. Finally, it is easy to see from the relations (2) that $\{\bullet, \bullet\}_{t,0} = t\{\bullet, \bullet\}$, where the latter bracket is the usual bracket on $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$.

Denote by $Z_{t,c}$ the center of the rational Cherednik algebra $H_{t,c}$. For any values of the parameters $(t, c)$, we have a natural map $Z_{t,c} \to B_{t,c}$ given by $m \mapsto em$. Using the double centralizer theorem we can prove that, for $t = 0$, this is actually an isomorphism.

**Theorem 1.14 (Satake isomorphism)** *The natural homomorphism $Z_{0,c} \to B_{0,c}$ is an isomorphism.*

*Proof.* We provide an inverse homomorphism. Since $B_{0,c}$ is commutative, for any $b \in B_{0,c}$ multiplication by $b$ provides an endomorphism of the right $B_{0,c}$-module $H_{0,c}e$, and this endomorphism commutes with any other endomorphism. By the double centralizer property, this endomorphism corresponds to a unique element $\varphi(b) \in Z_{0,c}$. It is easy to check that this is inverse to $Z_{0,c} \to B_{0,c}$, $m \mapsto em$. □

We remark that there is a natural Poisson bracket on $B_{0,c}$. Namely, it is easy to see that this algebra is the quasiclassical limit of $B_{\hbar,c}$, the spherical subalgebra of the algebra $H_{\hbar,c}$ introduced in Definition 1.5. Then, we can define the Poisson bracket as $\{a,b\} := \frac{1}{\hbar}[\overline{a},\overline{b}] \mod \hbar$, where $\overline{a},\overline{b}$ are lifts of $a$, $b$ to $H_{\hbar,c}$.

### 1.7 $\mathfrak{sl}_2$-actions on the rational Cherednik algebra.

To finish this section, let us mention some $\mathfrak{sl}_2$ actions on the rational Cherednik algebra $H_{t,c}$ and its spherical subalgebra that will be of importance later. First, assume $t \neq 0$, so we may as well asume $t = 1$. Consider the following elements in $H_{1,c}$:

$$\mathbf{E} := -\frac{1}{2}\sum_i x_i^2, \quad \mathbf{F} := \frac{1}{2}\sum_i y_i^2, \quad \mathbf{h} := \sum_i \frac{x_iy_i + y_ix_i}{2}.$$

The following can be seen via a direct calculation:

**Proposition 1.15** *The elements $(\mathbf{E},\mathbf{F},\mathbf{h})$ form an $\mathfrak{sl}_2$-triple in $H_{1,c}$, where the Lie bracket is the usual commutator.*

Moreover, the induced $\mathfrak{sl}_2$ action on $H_{1,c}$ is locally finite, this follows from our calculations below, see Section 3, where the element $\mathbf{h}$ is of special importance. Note that $[e,\mathbf{E}] = 0$, $[e,\mathbf{F}] = 0$, $[e,\mathbf{h}] = 0$, this is an exercise. It follows that the spherical subalgebra $eH_{1,c}e$ contains the $\mathfrak{sl}_2$ triple $(\mathbf{E}e, \mathbf{F}e, \mathbf{h}e)$, and the induced $\mathfrak{sl}_2$ action on $eH_{1,c}e$ is locally finite so, in particular, it integrates to an action of $\mathrm{SL}_2$.

We would like to get an $\mathfrak{sl}_2$-triple in the spherical subalgebra $B_{0,c}$. We've seen in the previous subsection that this is a Poisson algebra, with the Poisson bracket induced by the commutator in $B_{\hbar,c}$. We have the following result.

**Proposition 1.16** *The elements $(\mathbf{E},\mathbf{F},\mathbf{h})$ form an $\mathfrak{sl}_2$-triple in $B_{0,c}$, where the Lie bracket is the natural Poisson bracket on $B_{0,c}$. The induced $\mathfrak{sl}_2$-action on $B_{0,c}$ is locally finite, and it integrates to an $\mathrm{SL}_2$ action.*

## 2 Representation theory at $t = 0$.

### 2.1 Irreducible representations of $H_{0,c}$.

Note that the Satake isomorphism is filtered, where both $Z_{0,c}$ and $B_{0,c}$ have the inherited filtration from the one on $H_{0,c}$. By Corollary 1.9, it follows that $\mathrm{gr}\, Z_{0,c} = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$, so $\mathrm{gr}\, H_{0,c}$ is a finitely generated module over $\mathrm{gr}\, Z_{0,c}$. It follows that $H_{0,c}$ is finitely generated over $Z_{0,c}$, so every irreducible representation of $H_{0,c}$ is finite dimensional.

By Schur's lemma, the center $Z_{0,c}$ acts on every irreducible $H_{0,c}$-module by a character. For a central character $\chi : Z_{0,c} \to \mathbb{C}$, let $(\chi)$ be the ideal in $H_{0,c}$ generated by the kernel of $\chi$.

**Theorem 2.1** *Any irreducible representation of $H_{0,c}$ has dimension $n!$, and is isomorphic to the regular representation as an $\mathfrak{S}_n$-module.*

We divide the proof of the preceding theorem in two parts. First, we show that any irreducible representation of $H_{0,c}$ has dimension $\leq n!$. This is a consequence of the Amitsur-Levitski identity for $H_{0,c}$, which we show first. After that, we show that any irreducible representation of $H_{0,c}$ must be a multiple of the regular representation of $\mathfrak{S}_n$.

Recall that we have the Dunkl embedding $H_{0,c} \hookrightarrow \mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*]\#\mathfrak{S}_n$. Note that the latter algebra may be embedded in $\mathbb{C}(x_1,\ldots,x_n,y_1,\ldots,y_n)\#\mathfrak{S}_n$. But this algebra is isomorphic to the algebra of $n! \times n!$ matrices over $\mathbb{C}(x_1,\ldots,x_n,y_1,\ldots,y_n)^{\mathfrak{S}_n}$. It follows that any polynomial identity satisfied by $\mathrm{Mat}_{n!}(\mathbb{C}(x_1,\ldots,x_n,y_1,\ldots,y_n)^{\mathfrak{S}_n})$ is also satisfied by $H_{0,c}$. In particular, we have the following.

**Theorem 2.2 (Amitsur-Levitzki Identity)** *For any $N \times N$ matrices $A_1, \ldots, A_{2N}$ with entries in a commutative ring $R$, we have*

$$\sum_{\sigma \in \mathfrak{S}_{2N}} sgn(\sigma) A_{\sigma(1)} \cdots A_{\sigma(2N)} = 0.$$

*Moreover, this identity is not valid for matrices of size $> N$.*

**Lemma 2.3** *Any simple representation of $H_{0,c}$ has dimension $\leq n!$.*

*Proof.* Let $M$ be a simple representation of $H_{0,c}$ of dimension $m$. By the Density theorem, the matrix algebra $\mathrm{Mat}_m(\mathbb{C})$ must be a quotient of $H_{0,c}$. Then, the Amitsur-Levitzki identity (with $N = n!$) must be satisfied by $\mathrm{Mat}_m(\mathbb{C})$. But this identity is not valid in any matrix algebra of size larger than $n!$. Then, $m \leq n!$. □

We would like to give another proof of Lemma 2.3 that avoids making use of the Amitsur-Levitzki identity. This proof is due to Pavel Etingof. First, we see that Lemma 2.3 holds when $Z_{0,c}$ acts by a generic central character.

**Proposition 2.4** *If $\chi$ is a generic central character then $H_{0,c}/(\chi)$ is a matrix algebra of size $n!$. In particular, $H_{0,c}$ has a unique irreducible representation with central character $\chi$, and this representation is isomorphic to $\mathbb{C}\mathfrak{S}_n$ as an $\mathfrak{S}_n$ module.*

*Proof.* Recall the Dunkl embedding $\Theta_{0,c} : H_{0,c} \to \mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*]\#\mathfrak{S}_n$. Let $\delta = \prod_{i<j}(x_j - x_i)$. Note that $\delta^2$ is central in $H_{0,c}$, so we can localize $H_{0,c}[\delta^{-1}]$. Also note that $\Theta_{0,c}$ can be extended to an isomorphism $H_{0,c}[\delta^{-1}] \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*]\#\mathfrak{S}_n$. Since the action of $\mathfrak{S}_n$ on $\mathfrak{h}^{reg} \times \mathfrak{h}^*$ is free, for a generic central character $\chi$, the quotient $H_{0,c}/(\chi) = H_{0,c}[\delta^{-1}]/(\chi)$ is $\mathbb{C}[\mathbb{O}_\chi]\#\mathfrak{S}_n$, where $\mathbb{O}_\chi$ is the free orbit of $\chi$. It is easy to see that this last algebra is a matrix algebra of size $n! \times n!$, and its unique irreducible representation is $\mathbb{C}[\mathbb{O}_\chi]$, which is isomorphic to $\mathbb{C}\mathfrak{S}_n$ as $\mathfrak{S}_n$-modules. □

Now, for a finite dimensional algebra $A$ over a field $\mathbb{K}$, let $d(A)$ be the maximal $\mathbb{K}$-dimension of an irreducible $A$-module. The proof of Lemma 2.3 is based on the following result.

**Proposition 2.5** *Let $B$ be a $\mathbb{C}[t]$-algebra which is a free $\mathbb{C}[t]$-module of finte rank. For $a \in \mathbb{C}$, let $B_a := B/(t-a)B$. Then, the set of complex numbers $a$ such that $d(B_a) \leq d(B_0)$ is finite.*

*Proof.* Fix a $\mathbb{C}[t]$-isomorphism $B \xrightarrow{\sim} \mathbb{C}[t]^N$. Let $K := \overline{\mathbb{C}(t)}$ be the field of algebraic functions, and $L := \bigcup_n \mathbb{C}((t^{1/n}))$ be the algebraic closure of $\mathbb{C}((t))$.

Now let $B' := B \otimes_{\mathbb{C}[t]} K$, so that $B'$ is $N$-dimensional. We claim that $d(B_0) \leq d(B')$. To see this, let $B'' := B' \otimes_K L$. Note that $d_L(B'') = d_K(B')$. Now let $0 = V_0' \subseteq V_1' \subseteq \cdots \subseteq V_m' = B'$ be a Jordan-Hölder filtration of $B'$ (as a left $B'$-module), and let $V_i'' = V_i' \otimes_K L$, so that each $V_i''$ can be seen as a point in the Grassmanian $\mathrm{Gr}_L(\dim_K V_i', N)$. Note that each of these points is actually defined over some subfield $\mathbb{C}((t^{1/k}))$ of $L$. So we can take the limit when $t \to 0$, to get a filtration of $B_0$ by the limiting submodules $V_i^0$. Then, $\dim(V_i^0/V_{i-1}^0) = \dim(V_i'/V_{i-1}') \leq d(B')$. The claim follows since any simple $B_0$-module appears in any Jordan-Hölder filtration of $B_0$.

The filtration $V_i'$ is defined over some finite extension, say $R$, of $\mathbb{C}[t, (t-a_1)^{-1}, \ldots, (t-a_s)^{-1}]$ for some $a_1, \ldots, a_s \in \mathbb{C}$, that is, $V_i' = V_i \otimes_R K$, where $V_i$ is a filtration of $B_R := B \otimes_{\mathbb{C}[t]} R$. This is an exercise. By the Density theorem, for each $i$ the map $B' \to \mathrm{End}_K(V_i'/V_{i-1}')$ is surjective. It follows (after extending the set $\{a_1, \ldots, a_s\}$ if needed), that the action map $B_R \to \mathrm{End}_R(V_i/V_{i-1})$ is surjective.

Now pick $a \in \mathbb{C}$, $a \neq a_1, \ldots, a_s$. Fix a maximal ideal $a'$ of $R$ over (the maximal ideal of $\mathbb{C}[t]$ corresponding to) $a$. Reduce the filtration $V_i \mod a'$. So we get a filtration $V_i^a$ of $B_a$ such that the action map $B_a \to \mathrm{End}_{\mathbb{C}}(V_i^a/V_{i-1}^a)$ is surjective. Then, $V_i^a$ is a Jordan-Hölder filtration. Moreover, $\dim(V_i^a/V_{i-1}^a) = \dim(V_i'/V_i'')$, so $d(B_a) = d(B')$. Then, $d(B_0) \leq d(A)$. □

To derive Lemma 2.3 from Proposition 2.5, observe that the invariant subalgebras $\mathbb{C}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$, $\mathbb{C}[y_1, \ldots, y_n]^{\mathfrak{S}_n}$ are central in $H_{0,c}$. It follows by the PBW theorem that $H_{0,c}$ is a $S = \mathbb{C}[x_1, \ldots, x_n]^{\mathfrak{S}_n} \otimes \mathbb{C}[y_1, \ldots, y_n]^{\mathfrak{S}_n}$-algebra which is a free $S$-module of finite rank $(n!)^3$. For $P \in \mathrm{Spec}(S)$, we can restrict to a generic line $\ell$ passing through $P$. We can introduce coordinate a coordinate $t$ on $\ell$ so that $t(P) = 0$. The result then follows by Propositions 2.4 and 2.5.

Now we show the second part of Theorem 2.1.

**Lemma 2.6** *Any irreducible representation of $H_{0,c}$ is isomorphic to a multiple of the regular representation of $\mathfrak{S}_n$.*

*Proof.* We show that the trace of every element $1 \neq \sigma \in \mathfrak{S}_n$ at a finite dimensional representation of $H_{0,c}$ is 0. Take $j \in \{1, \ldots, n\}$, with $\sigma(j) = i \neq j$. Then, in $H_{0,c}$ we have:

$$[y_j, x_i s_{ij}\sigma] = x_i(y_j s_{ij}\sigma - s_{ij}\sigma y_i) + [y_j, x_i]s_{ij}\sigma = 0 + cs_{ij}s_{ij}\sigma = c\sigma.$$

So that $\sigma$ is a commutator and therefore has trace 0 on every finite dimensional $H_{0,c}$ representation. It follows that any such representation is a multiple of the regular representation of $\mathfrak{S}_n$. $\square$

## 2.2 Generalized Calogero-Moser space.

**Definition 2.7** *For $c \neq 0$, the generalized Calogero-Moser space is $V = \mathrm{Spec}(Z_{0,c}) = \mathrm{Spec}(B_{0,c})$.*

In the next subsection, we will see that the word 'generalized' is superfluous.

Note that, since $\mathrm{gr}\, B_{0,c} = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$, the generalized Calogero-Moser space is reduced. Also note that $B_{0,c}$ admits a graded quantization: the spherical subalgebra $B_{\hbar,c}$ of the rational Cherednik algebra $H_{\hbar,c}$ (that is, we replace '$t$' by a variable), so that $B_{0,c}$ has a natural structure of a Poisson algebra. Our goal for this section is to prove the following.

**Theorem 2.8** *The generalized Calogero-Moser space is smooth.*

Since $V = \mathrm{Spec}(B_{0,c})$, Theorem 2.8 is equivalent to the statement that the global dimension of $B_{0,c}$ is finite. On the other hand, we know that the global dimension of $H_{0,c}$ is finite: its associated graded is $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]\#\mathfrak{S}_n$, which has finite global dimension, and for any filtered algebra $\mathcal{A}$, its global dimension cannot exceed the global dimension of $\mathrm{gr}\, \mathcal{A}$. Hence, Theorem 2.8 is deduced from the following result.

**Proposition 2.9** *The algebras $H_{0,c}$ and $B_{0,c}$ are Morita equivalent.*

*Proof.* By Morita theory, the statement of the proposition is equivalent to $e \in H_{0,c}$ being a full idempotent, that is, $H_{0,c}eH_{0,c} = H_{0,c}$. Assume this is not true, so that $H_{0,c}/H_{0,c}eH_{0,c} \neq 0$. Let $M$ be a simple module over this latter algebra. Then, $M$ is a simple $H_{0,c}$-module such that $eM = 0$. This is a contradiction with Theorem 2.1.$\square$

A consequence of Proposition 2.9 is that simple $H_{0,c}$ modules are in correspondence with simple $B_{0,c} = Z_{0,c}$-modules, which, in turn, are in correspondence with central characters. It follows that for any central character $\chi$, the algebra $H_{0,c}/(\chi)$ is a matrix algebra of size $n!$, and therefore there is a unique $H_{0,c}$-module where $Z_{0,c}$ acts by $\chi$.

Now we see that $V$ is actually symplectic. Note that the set of points where $\bigwedge^{top} \Pi$ vanishes has codimension at least 2, where $\Pi$ is the Poisson bivector. This is so because it also holds in the degeneration $\mathrm{Sym}_n(\mathbb{C}^2)$ of $V$. Since $V$ is smooth and $\bigwedge^{top} \Pi$ is a regular function, this implies the following.

**Corollary 2.10** *The generalized Calogero-Moser space $V$ is symplectic.*

## 2.3 Calogero-Moser space.

In this subsection, we introduce the so-called Calogero-Moser space, and prove that it is isomorphic to the generalized Calogero-Moser space of the previous subsection.

Let $G = \mathrm{PGL}_n(\mathbb{C})$, and $\mathcal{M} = T^* \mathrm{Mat}_n(\mathbb{C})$. Using the trace form, we may identify $\mathcal{M} = \mathrm{Mat}_n(\mathbb{C}) \oplus \mathrm{Mat}_n(\mathbb{C})$. Also, note that $\mathrm{Lie}(G) = \mathfrak{sl}_n(\mathbb{C})$. The group $G$ acts on $\mathcal{M}$ by simultaneous conjugation. This action is Hamiltonian, with moment map $\mu : (X, Y) \mapsto [X, Y]$, cf. [3, Section 4.4]. Let $\mathbb{O} := \{A \in \mathfrak{sl}_n : \mathrm{rank}(A + I) = 1\}$, where $I$ denotes the identity matrix. Note that this is a single conjugacy class, namely $\mathbb{O} = GL_n.\mathrm{diag}(n-1, -1, -1, \ldots, -1)$.

**Definition 2.11** *The Calogero-Moser space $\mathcal{C}_n$ is the scheme $\mu^{-1}(\mathbb{O})//G$.*

In other words, $\mathcal{C}_n$ is the Hamiltonian reduction of $\mathcal{M}$ at the orbit $\mathbb{O}$, so $\mathcal{C}_n = \mathrm{Spec}((\mathbb{C}[\mathcal{M}]/\mathbb{C}[\mathcal{M}]\mu^* I_{\mathbb{O}})^{\mathrm{ad}\, \mathfrak{g}})$, where $I_{\mathbb{O}}$ is the ideal in $S\mathfrak{g}$ corresponding to the closed coadjoint orbit $\mathbb{O}$.

**Proposition 2.12** *The action of $G$ on $\mu^{-1}(\mathbb{O})$ is free.*

*Proof.* Let $(X,Y) \in \mu^{-1}(\mathbb{O})$. So $(X,Y)$ determines a representation of the quiver with one vertex and two loops. This representation is simple because no proper subcollection of $(-1,-1,\dots,n-1)$ adds up to 0. Details of this are left as an exercise. From here, the result follows by Schur's lemma.□

It follows, see [3, Theorem 4.4], that $\mathcal{C}_n$ is a smooth symplectic variety, of dimension $\dim(\mathcal{M}) - 2\dim(\mathrm{PGL}_n) + \dim(\mathbb{O}) = 2n$. The space $\mathcal{C}_n$ is closely related to a Hamiltonian reduction considered by Barbara and Yi. Let $\mathcal{M}' = T^*(\mathrm{Mat}_n(\mathbb{C}) \oplus \mathbb{C}^n)$. Recall that $G' = \mathrm{GL}_n$ acts on $\mathcal{M}'$ in a Hamiltonian way, with moment map $\mu' : T^*(\mathrm{Mat}_n(\mathbb{C}) \oplus \mathbb{C}^n) \to \mathfrak{gl}_n$, $\mu'(X,Y,i,j) = [X,Y] + ij$. Barbara and Yi considered the Hamiltonian reduction of $\mathcal{M}'$ at 0, $\mu'^{-1}(0)//G'$. In the next result, we see that $\mathcal{C}_n$ is the Hamiltonian reduction of $\mathcal{M}'$ at $-I$ (which is a single orbit on $\mathfrak{gl}_n$).

**Proposition 2.13** *The reductions $\mu'^{-1}(-I)//G'$ and $\mu^{-1}(\mathbb{O})//G'$ are naturally identified.*

*Proof.* We have a natural projection $\rho : \mathcal{M}' \to \mathcal{M}$, $(X,Y,i,j) \mapsto (X,Y)$. It is clear that $\rho(\mu'^{-1}(-I)) = \mu^{-1}(\mathbb{O})$. Now, if $(X,Y,i,j), (X,Y,i',j') \in \rho^{-1}(X,Y)$ where $(X,Y) \in \mu^{-1}(\mathbb{O})$, then there exists a unique $t \in \mathbb{C}^\times$ such that $i' = ti$, $j' = t^{-1}j$. It follows that $\rho$ identifies $\mu^{-1}(\mathbb{O}) = \mu'^{-1}(-I)//\mathbb{C}^\times$, where $\mathbb{C}^\times$ acts on $\mu^{-1}(\mathbb{O})$ as the center of $\mathrm{GL}_n$. From here, the result follows. □

The description of $\mathcal{C}_n$ as a Hamiltonian reduction of the variety $\mathcal{M}'$ is useful because the moment map $\mu'$ is flat. On the other hand, we will need to see $\mathcal{C}_n$ as a Hamiltonian reduction of $\mathcal{M}$ when looking at the connection of $\mathcal{C}_n$ with $V$, the generalized Calogero-Moser space from the previous subsection.

**Corollary 2.14** *The Calogero-Moser space $\mathcal{C}_n$ is connected.*

*Proof.* Recall that the moment map $\mu'$ is flat. This was proved by Barbara in her lecture. It then follows that we have a filtration on $\mathbb{C}[\mu'^{-1}(-I)//G'] = [\mathbb{C}[\mathcal{M}']/\mathbb{C}[\mathcal{M}']\mu'^*(\mathcal{I})]^{G'}$, where $\mathcal{I}$ is the ideal in $S \gg$ corresponding to the closed orbit $-I$, whose associated graded is $\mathbb{C}[\mu^{-1}(0)//G']$. Barbara proved that we have an isomorphism $\mu^{-1}(0)//G \cong \mathrm{Sym}^n(\mathbb{C}^2)$, which is connected. It follows that $\mathrm{gr}(\mathbb{C}[\mathcal{C}_n])$ doesn't have zero divisors, hence $\mathcal{C}_n$ is irreducible. Since we know it's smooth, it must be connected. □

Note that $\mathcal{C}_n$ carries an $\mathrm{SL}_2$ action, given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(X,Y) = (aX+bY, cX+dY)$. This action will be important later.

We describe, in coordinates, a dense open subset of $\mathcal{C}_n$. Namely, consider the set $U$ of conjugacy classes of pairs of matrices $(X,Y)$ such that $X$ is diagonalizable with pairwise distinct eigenvalues, say $X = \mathrm{diag}(x_1,\dots,x_n)$, $x_i \neq x_j$. We have a map $\varphi : T^*(\mathfrak{h}^{reg}/\mathfrak{S}_n) \to U$, given by the formula $(x_1,\dots,x_n,y_1,\dots,y_n) \mapsto (X,Y)$, where $X = \mathrm{diag}(x_1,\dots,x_n)$, $y_{ij} = 1/(x_i - x_j)$, $i \neq j$, and $y_{ii} = y_i$. We show in the Appendix that this map is surjective. Actually, it is an isomorphism of Poisson varieties.

**Proposition 2.15** *The map $\varphi$ is an isomorphism of symplectic varieties.*

To prove Proposition 2.15, we take a closer look at the symplectic structure on $\mathcal{C}_n$. Recall that this is given as follows. Pick an element $\alpha \in \mathbb{O}$. Then, $\mu^{-1}(\alpha)//G_\alpha = \mu^{-1}(\mathbb{O})/G$, where $G_\alpha$ is the stabilizer of $\alpha$ in $\mathrm{PGL}_n$. Similarly to a result shown in Barbara's lecture, we see that there exists a unique 2-form $\underline{\omega}$ (that turns out to be symplectic) on $\mu^{-1}(\alpha)//G_\alpha$ such that $\iota^*\omega = \pi^*\underline{\omega}$, where $\iota : \mu^{-1}(\alpha) \to \mathcal{M}$ is the inclusion and $\pi : \mu^{-1}(\alpha) \to \mu^{-1}(\alpha)//G_\alpha$ is the projection.

Take $\alpha$ to be the anti-identity matrix, that is, the matrix that has 0's on the diagonal and 1's everywhere else. Clearly, $\alpha \in \mathbb{O}$ and we have a map $\overline{\varphi} : T^*\mathfrak{h}^{reg} \to \mu^{-1}(\alpha)$, given by the same formula as above. This map is clearly a morphism, and hence so it is $\pi \circ \overline{\varphi} : T^*(\mathfrak{h}^{reg}) \to \mu^{-1}(\alpha)//G_\alpha$. Note that $\pi \circ \overline{\varphi}$ is $\mathfrak{S}_n$-invariant, so it descends to $\varphi : T^*(\mathfrak{h}^{reg}/\mathfrak{S}_n) \to \mu^{-1}(\alpha)//G_\alpha$. It is an exercise to check that this map is injective. We show in the appendix that its image is precisely $U$.

Let us check that $\varphi$ is compatible with symplectic forms. This is somewhat technical but completely straightforward. Let $\omega_{\mathcal{M}}$ be the canonical symplectic form on $\mathcal{M} = T^*\,\mathrm{Mat}_n$, so that $\omega_{\mathcal{M}} = \sum_{i,j} dx_{ij} \wedge dy_{ji}$, and let $\omega_\alpha$ be the form on the reduction. Also, let $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ be the symplectic form on $T^*\mathfrak{h}^{reg}$, and $\underline{\omega}$ be the induced form on $T^*(\mathfrak{h}^{reg}/\mathfrak{S}_n) = T^*(\mathfrak{h}^{reg})/\mathfrak{S}_n$. In other words, $\omega = \pi^*_{\mathfrak{S}_n}\underline{\omega}$. Now,

$$\overline{\varphi}^*\omega_{\mathcal{M}} = \sum_{i,j} d\overline{\varphi}^*(x_{ij}) \wedge d\overline{\varphi}^*(y_{ij}) = \sum_{i=1}^n dx_i \wedge dy_i + \sum_{i\neq j} 0 \wedge d\frac{1}{x_i - x_j} = \omega.$$

So it follows that:

$$\pi^*_{\mathfrak{S}_n}\underline{\omega} = \omega = \overline{\varphi}^*\omega_{\mathcal{M}} = \overline{\varphi}^*\pi^*\omega_\alpha = \pi^*_{\mathfrak{S}_n}\varphi^*\omega_\alpha.$$

Then, $\underline{\omega} = \varphi^*\omega_\alpha$ and $\varphi$ is indeed a morphism of symplectic varieties.

We have constructed two deformations of $\mathrm{Sym}_n(\mathbb{C}^2)$. One is the generalized Calogero-Moser space from the previous subsection, the other one is the Calogero-Moser space. It turns out these are the same.

**Theorem 2.16** *The spaces $V$ and $\mathcal{C}_n$ are isomorphic as Poisson varieties.*

*Proof.* Without loss of generality, we may assume $V = \mathrm{Spec}(Z_{0,1})$. By the results of Subsection 2.1, we can regard $V$ as the moduli space of irreducible representations of $H_{0,1}$. Let $E$ be an irreducible representation of $H_{0,1}$. Recall that $\dim(E) = n!$, and that, as a $\mathfrak{S}_n$-module, $E$ is isomorphic to the regular representation of $\mathfrak{S}_n$. Now let $\mathfrak{S}_{n-1} \subseteq \mathfrak{S}_n$ be the subgroup of elements that do not permute the element 1. Note that $E^{\mathfrak{S}_{n-1}}$ has dimension $n$, as these are just functions on the cosets $\mathfrak{S}_n/\mathfrak{S}_{n-1}$. We will need a description of a basis on $E^{\mathfrak{S}_{n-1}}$: if we identify $E$ with $\mathbb{C}\mathfrak{S}_n$, then $E^{\mathfrak{S}_{n-1}}$ is generated by $v_1 = \underline{e} = \frac{1}{(n-1)!}\sum\{\sigma \in \mathfrak{S}_n : \sigma(1) = 1\}$, $v_2 = \underline{e}s_{12}, \dots, v_n = \underline{e}s_{1n}$. Also note that, since in $H_{0,1}$ the elements $x_1, y_1$ commute with any element of $\mathfrak{S}_{n-1}$, the action of $x_1, y_1$ induces operators $X_1, Y_1 : E^{\mathfrak{S}_{n-1}} \to E^{\mathfrak{S}_{n-1}}$. What we will see now is that the operators $(X_1, Y_1)$ define an element of $\mathcal{C}_n$. To do so, we need to check that the rank of $[X_1, Y_1] + I$ is 1.
By the relations defining $H_{0,1}$ we have that:

$$[X_1, Y_1] = \sum_{i\neq 1} s_{1i}|_{E^{\mathfrak{S}_{n-1}}}.$$

Note that $\sum_{i\neq 1} s_{1i}$ commutes with $\underline{e}$. It follows that $[X_1, Y_1]v_i = \sum_{j\neq i} v_j$. Then, $[X_1, Y_1] + I$ has rank 1. Also note that, since $E$ is only well-defined up to isomorphism, the pair $(X_1, Y_1)$ is only well-defined up to the action of $\mathfrak{S}_n$. Then, this defines a point on $\mathcal{C}_n$. Let us look more closely at this map.

Consider the open set $\mathfrak{U} \subseteq V$ consisting on representations on which $x_i - x_j$ acts invertibly for every $i, j$. Recall the Dunkl embedding $\Theta_{0,1} : H_{0,1} \to \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n, \frac{1}{x_i - x_j}]\#\mathfrak{S}_n$. So $\mathfrak{U}$ consists of those representations that can be obtained by restricting a representation of the latter algebra via $\Theta_{0,1}$. It follows that any $E \in \mathfrak{U}$ has the form $E_{(\lambda,\mu)}$, $(\lambda, \mu) \in \mathfrak{h}^{reg} \times \mathfrak{h}^*$, where $E_{(\lambda,\mu)}$ is the space of functions on the $\mathfrak{S}_n$-orbit $\mathbb{O}_{(\lambda,\mu)}$, and the action of $H_{0,c}$ is given by Dunkl operators: if $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n)$ are such that $(a, b) \in \mathbb{O}_{(\lambda,\mu)}$ and $F$ is a function on $\mathbb{O}_{(\lambda,\mu)}$:

$$(x_iF)(a, b) = a_iF(a, b), \quad (y_iF)(a, b) = b_iF(a, b) + \sum_{j\neq 1} \frac{F(s_{ij}a, s_{ij}b)}{a_i - a_j}, \quad (\sigma F)(a, b) = F(\sigma^{-1}a, \sigma^{-1}b).$$

Now, the space $E^{\mathfrak{S}_{n-1}}_{(\lambda,\mu)}$ has as a basis the characteristic functions of $\mathfrak{S}_{n-1}$ orbits on $\mathbb{O}_{(\lambda,\mu)}$. It is clear from the presentation above that, in this basis, the matrices of the operators $X_1, Y_1$ are given by:

$$X_1 = \mathrm{diag}(\lambda_1, \dots, \lambda_n), \quad (Y_1)_{ij} = \frac{1}{\lambda_i - \lambda_j}, j \neq i, \quad (Y_1)_{ii} = \mu_i. \tag{9}$$

Then, we have an isomorphism $f : \mathfrak{U} \to U$. Let us check that this is actually an isomorphism of symplectic varieties. Let $\delta = \prod_{i<j}(x_i - x_j)$, so that $\delta^2$ is $\mathfrak{S}_n$-invariant. Note that, by definition, $\mathfrak{U} = \mathrm{Spec}(B_{0,1}[\delta^{-2}e])$. Via the Dunkl homomorphism $\Theta_{0,1}$, $B_{0,1}[\delta^{-2}e] = \mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*]^{\mathfrak{S}_n}$. We claim that the Poisson bracket on $B_{0,1}[\delta^{-2}e]$ is the natural Poisson bracket on $\mathbb{C}[\mathfrak{h}^{reg}\times\mathfrak{h}^*]^{\mathfrak{S}_n}$ induced from the symplectic structure on $T^*(\mathfrak{h}^{reg}/\mathfrak{S}_n)$. Recall that the Poisson structure on $B_{0,c}$ is induced from the commutator on $B_{\hbar,c}$. So we need to take a small detour and consider the algebra $H_{\hbar,c}$.

Note that the adjoint action of the element $\delta^2 \in H_{\hbar,c}$ is locally nilpotent. Then, the localization $H_{\hbar,c}[\delta^{-1}]$ makes sense. Now consider the algebra $\mathcal{D}_\hbar(\mathfrak{h})$ of homogenized differential operators on $\mathfrak{h}$, see Yi's notes, [9, Subsection 1.3] . The localization $\mathcal{D}_\hbar(\mathfrak{h})[\delta^{-1}]$ also makes sense, and actually $\mathcal{D}_\hbar(\mathfrak{h})[\delta^{-1}] = \mathcal{D}_\hbar(\mathfrak{h}^{reg})$. Moreover, we have a Dunkl embedding $\Theta_{\hbar,c} : H_{\hbar,c} \hookrightarrow \mathcal{D}_\hbar(\mathfrak{h}^{reg})\#\mathfrak{S}_n$, defined by a similar formula to the usual Dunkl embedding $\Theta_{t,c}$. This embedding identifies $H_{\hbar,c}[\delta^{-1}] = \mathcal{D}_\hbar(\mathfrak{h}^{reg})$. It follows that $B_{\hbar,c}[\delta^{-2}e]$ gets identified with $\mathcal{D}_\hbar(\mathfrak{h}^{reg})^{\mathfrak{S}_n}$. Since $\mathfrak{S}_n$ acts freely on $\mathfrak{h}^{reg}$, the latter algebra is $\mathcal{D}_\hbar(\mathfrak{h}^{reg}/\mathfrak{S}_n)$. It follows that the bracket on $B_{0,c}[\delta^{-2}e]$ is the usual one on $\mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*]^{\mathfrak{S}_n}$. For more details, see [8].

Now the claim that $f : \mathfrak{U} \to U$ is an isomorphism of symplectic varieties is basically Proposition 2.15.

Since $\mathfrak{U}$ is dense in $V$ and $U$ is dense in $\mathcal{C}_n$, $f$ is actually a birational symplectomorphism, hence an open embedding. We claim now that $f^* : \mathbb{C}[\mathcal{C}_n] \to B_{0,1}$ is surjective. Indeed, we can see from (9) that, $f^*(\mathrm{Tr}(X^p)) = \sum_{i=1}^n x_i^p$. Also note that $f$ commutes with the $\mathrm{SL}_2$-actions on $V$ and $\mathcal{C}_n$. Thus, $f^*(\mathrm{Tr}(Y^p)) = \sum_{i=1}^n y_i^p$. Since $f^*$ is a Poisson isomorphism and $\sum_i x_i^p$, $\sum_i y_i^p$, $p > 0$, are Poisson generators of $B_{0,1}$. Now $f^*$ is surjective, and $\mathbb{C}[\mathcal{C}_n]$, $B_{0,1}$ are integral domains of the same dimension. It follows that $f^*$, hence $f$, is an isomorphism. □

We'd like to indicate some future directions. We have seen that the spherical subalgebra $B_{0,c}$ can be obtained via classical Hamiltonian reduction along an orbit on $T^*(\mathrm{Mat}_n(\mathbb{C}))$. It turns out that the algebra $B_{1,c}$ can be obtained via quantum Hamiltonian reduction. Recall the sheaf of algebras $\mathfrak{A}_\lambda$ quantizing $\mathrm{Hilb}_n(\mathbb{C}^2)$ constructed by Ivan in his lecture. It was an exercise in his lecture to see that $\Gamma(\mathfrak{A}_\lambda)$ quantizes the algebra of regular functions on $\mathrm{Hilb}_n(X)$ (this is a consequence of the Grauert-Riemenschneider vanishing theorem). Since the Hilbert-Chow morphism is a resolution of singularities, this is precisely $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$. In this lecture, we have constructed an algebra quantizing this, namely, the spherical rational Cherednik algebra $B_{1,\lambda}$.

**Theorem 2.17** *The algebras $B_{1,\lambda}$ and $\Gamma(\mathfrak{A}_\lambda)$ are isomorphic.*

A version of this algebra was also considered by Yi. Namely, Yi considered the quantum Hamiltonian reduction $R(\mathfrak{g}, \mathcal{D}(\mathfrak{g} \oplus \mathbb{C}^n), 0)$, where $\mathfrak{g} = \mathfrak{gl}_n$. Ivan proved that if we consider a diferent fiber of the moment map, say $R_\lambda = R(\mathfrak{g}, \mathcal{D}(\mathfrak{g} \oplus \mathbb{C}^n), \lambda)$, then $\Gamma(\mathfrak{A}_\lambda)$ and $R_\lambda$ are isomorphic. It follows that $B_{1,\lambda} = R(\mathfrak{g}, \mathcal{D}(\mathfrak{g} \oplus \mathbb{C}^n), \lambda)$. Proof of Theorem 2.17 will be given in a subsequent lecture.

# 3 Representation theory at $t \neq 0$.

In this section, we assume $t \neq 0$. Then, by Remark 1.7, we may assume $t = 1$. Denote $H_c := H_{1,c}$.

## 3.1 Category $\mathcal{O}$.

By the PBW theorem, we have a decomposition $H_c \cong S(\mathfrak{h}^*) \otimes \mathbb{C}\mathfrak{S}_n \otimes S(\mathfrak{h})$ as vector spaces. Compare this decomposition with the decomposition $\mathcal{U}(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{n}^-) \otimes \mathcal{U}(\mathfrak{t}) \otimes \mathcal{U}(\mathfrak{n})$ of the universal enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$ with triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}$. Then, we will call $H_c \cong S(\mathfrak{h}^*) \otimes \mathbb{C}\mathfrak{S}_n \otimes S(\mathfrak{h})$ the triangular decomposition of $H_c$. Motivated by this, we can define a category $\mathcal{O}$ for $H_c$, which is analogous to the Berstein-Gelfand-Gelfand category $\mathcal{O}$ for $\mathfrak{g}$ that we saw in Kostya's talk last semester, [10, Section 1].

**Definition 3.1** *The category $\mathcal{O}$ for the algebra $H_c$ is the full subcategory of the category of all $H_c$ modules consisting of all modules $M$ satistying the conditions*

*($\mathcal{O}$1) $M$ is finitely generated.*

*($\mathcal{O}$2) $\mathfrak{h}$ acts on $M$ by locally nilpotent operators.*

Since, by Theorem 1.8 $\mathrm{gr}\, H_c$ is a Noetherian algebra, it follows that $H_c$ is also Noetherian. Then, $\mathcal{O}$ is a Serre (closed under submodules, quotient modules and extensions) subcategory of the category of all $H_c$-modules.

Again by analogy with the Lie algebra case, there is a notion of Verma modules for $H_c$. Namely, let $\tau$ be an irreducible representation of $\mathfrak{S}_n$. We let $\mathfrak{h}$ act on $\tau$ by 0, so that $\tau$ becomes a $S(\mathfrak{h})\#\mathfrak{S}_n$-module. Then, define the Verma module

$$\Delta(\tau) = H_c \otimes_{S(\mathfrak{h})\#\mathfrak{S}_n} \tau.$$

Note that, by the triangular decomposition of $H_c$, we have that as vector spaces $\Delta(\tau) = \mathbb{C}[\mathfrak{h}] \otimes \tau$. By definition, we have the following result.

**Proposition 3.2 (Frobenius reciprocity)** *For any module $V \in \mathcal{O}$, $\mathrm{Hom}_{H_c}(\Delta(\tau), V) = \mathrm{Hom}_{\mathfrak{S}_n}(\tau, V^{\mathfrak{h}})$, where $V^{\mathfrak{h}} := \{v \in V : \mathfrak{h}v = 0\}$.*

We would like to see that Verma modules actually live in $\mathcal{O}$. To see this, we introduce the following element, called the *grading* or *Euler* element of $H_c$:

$$\mathbf{h} := \sum_{i=1}^n x_i y_i + \frac{n}{2} - \sum_{i<j} c s_{ij}.$$

**Proposition 3.3** *We have*

$$\mathbf{h} = \sum_{i=1}^{n} \frac{x_i y_i + y_i x_i}{2}.$$

*So that the notation used here is consistent with the notation used in Subsection 1.7.*

*Proof.* We have $x_i y_i + y_i x_i = [y_i, x_i] + 2x_i y_i = 1 - \sum_{j \neq i} c s_{ij} + 2x_i y_i$. From here, the result follows. □

The reason why this element is interesting is because the adjoint action of $\mathbf{h}$ gives a grading on $H_c$, as follows:

**Proposition 3.4** *We have*

$$(i)\ [\mathbf{h}, x] = x, x \in \mathfrak{h}^*, \quad (ii)\ [\mathbf{h}, y] = -y, y \in \mathfrak{h} \quad (iii)\ [\mathbf{h}, \sigma] = 0, \sigma \in \mathfrak{S}_n.$$

*Proof.* A direct computation. □

Note that the previous proposition implies that any finite dimensional module is in $\mathcal{O}$. In fact, we can decompose such a module in the direct sum of its generalized $\mathbf{h}$-eigenspaces. Parts (i) and (ii) tell us that $\mathfrak{h}$ and $\mathfrak{h}^*$ act by locally nilpotent endomorphisms.

Now, let $N$ be an irreducible $\mathfrak{S}_n$-submodule of an $H_c$-module that is annihilated by every $y \in \mathfrak{h}$ so that, in particular $\mathbf{h}$ acts on $N$. Then, by Proposition 3.4 (iii), the action of $\mathbf{h}$ on $N$ is by a scalar. For an irreducible $\mathfrak{S}_n$-module $\tau$, let $c_\tau$ be the scalar determined by the action of $\mathbf{h}$ on $1 \otimes \tau \subseteq \Delta(\tau)$. In other words,

$$c_\tau = \frac{n}{2} - \sum_{i<j} c s_{ij}|_\tau.$$

Note that $\mathbf{h}$ acts semisimply on the Verma module $\Delta(\tau)$. In fact, by Proposition 3.4 (i), the eigenvalues of $\mathbf{h}$ on $\Delta(\tau)$ are of the form $c_\tau + m, m \in \mathbb{Z}_{\geq 0}$, and the $(c_\tau + m)$-eigenspace of $\Delta(\tau)$ is $\Delta(\tau)[c_\tau + m] = \mathbb{C}[\mathfrak{h}]^m \otimes \tau$. Also, by Proposition 3.4 (ii), the action of $y \in \mathfrak{h}^*$ maps $\Delta(\tau)[c_\tau + m]$ to $\Delta(\tau)[c_\tau + m - 1]$. Then, $y$ acts locally nilpotently on $\Delta(\tau)$. Since $\Delta(\tau)$ is clearly finitely generated, we get that $\Delta(\tau) \in \mathcal{O}$. Analyzing the $\mathbf{h}$-action on $\Delta(\tau)$, we get the following proposition, which is analogous to the Lie algebra case.

**Proposition 3.5** *A Verma module has a unique simple quotient.*

*Proof.* Exercise □

For a Verma module $\Delta(\tau)$, we denote by $L(\tau)$ its unique simple quotient. Note that $1 \otimes \tau \subseteq \Delta(\tau)$ projects injectively to $L(\tau)$. Moreover, if $N$ is an irreducible module in $\mathcal{O}$ then, by ($\mathcal{O}$2), $N^{\mathfrak{h}} \neq 0$. By Frobenius reciprocity, there exists a Verma module $\Delta(\tau)$ with $\mathrm{Hom}_{H_c}(\Delta(\tau), N) \neq 0$. So $N = L(\tau)$. In other words, the $L(\tau)$ form a complete list of irreducible objects in $\mathcal{O}$. Also note that $L(\tau) \not\cong L(\mu)$ if $\tau \not\cong \mu$.

**Proposition 3.6** *Category $\mathcal{O}$ is artinian, that is, every object has finite length.*

*Proof.* First, we claim that Verma modules have finite length. In fact, the multiplicity of an irreducible $L(\mu)$ in a Verma module $\Delta(\tau)$ is bounded by the dimension of a certain graded component of $\Delta(\tau)$ (which one?). From here,the claim follows. It also follows that any module in $\mathcal{O}$ must have a simple submodule. Now use the fact that $H_c$ is Noetherian to construct a Jordan-Hölder series for any object of $\mathcal{O}$. We leave details to the reader. □

Finally, we show that $\mathbf{h}$ acts locally finitely, with finite dimensional generalized eigenspaces.

**Proposition 3.7** *The Euler element* $\mathbf{h}$ *acts locally finitely on every object of $\mathcal{O}$ and each generalized* $\mathbf{h}$*-eigenspace is finite dimensional. In particular, every module in $\mathcal{O}$ is graded with finite dimensional homogeneous components.*

*Proof.* This follows from the fact that category $\mathcal{O}$ is artinian, as the claim clearly holds for simples $L(\tau)$. □

This last result tells us that the definition of category $\mathcal{O}$ for rational Cherednik algebra is analogous to that of category $\mathcal{O}^-$ for Lie algebras $\mathfrak{g}$ with nondegenerate $\mathbb{Z}$-grading, cf. [4]. It also allows us to define a notion of character for a module in $\mathcal{O}$. Namely, for $M \in \mathcal{O}$, its character is the following formal series on $t$:

$$\mathrm{ch}_M(\sigma, t) = \sum_{\beta \in \mathbb{C}} t^\beta \, \mathrm{Tr}_{M[\beta]}(\sigma) = \mathrm{Tr}(\sigma t^{\mathbf{h}}), \sigma \in \mathfrak{S}_n.$$

so that, if $\sigma = 1$, we recover a more familiar notion of a character, $\sum_{\beta \in \mathbb{C}} \dim(M[\beta]) t^\beta$. By their construction, characters of Verma modules are easy to compute.

**Proposition 3.8** *The character of the Verma module* $\Delta(\tau)$ *is*

$$\mathrm{ch}_{\Delta(\tau)}(\sigma,t)=\frac{\chi_\tau(\sigma)t^{c_\tau}}{\det_{\mathfrak{h}^*}(1-t\sigma)},$$

*where* $\chi_\tau$ *is the usual character associated to the* $\mathfrak{S}_n$*-representation* $\tau$*.*

Proposition 3.8 follows easily from the fact that, since $\sigma:\mathfrak{h}^*\to\mathfrak{h}^*$ is diagonalizable, $\sum_{n\geq 0}t^n\,\mathrm{Tr}(S^n\sigma)=1/\det(1-t\sigma)$.

## 3.2 Highest weight structure on $\mathcal{O}$.

Consider the Verma module $\Delta(\tau)$. Assume that a simple module $L(\mu)$ appears as a constituent in a Jordan-Hölder filtration of $\Delta(\tau)$. Since all the $\mathbf{h}$-eigenvalues of $L(\mu)$ must also be $\mathbf{h}$-eigenvalues of $\Delta(\tau)$, it follows that $c_\mu=c_\tau+m$ for some $m\in\mathbb{Z}_{\geq 0}$. Recall that the $\mathbf{h}$-eigenvalues on the radical (= unique maximal submodule) of $\Delta(\tau)$ are of the form $c_\tau+m$ with $m>0$. So we have that, if $\tau\neq\mu$, then $c_\mu=c_\tau+m$ for some $m>0$. This motivates the following definition.

**Definition 3.9** *We define a partial order on the set of irreducible* $\mathfrak{S}_n$*-representations as follows:* $\mu<_c\tau$ *if* $c_\mu-c_\tau$ *is a positive integer.*

Note that, by the first paragraph of this subsection, if $\mathrm{Hom}_{H_c}(\Delta(\mu),\Delta(\tau))\neq 0$, then $\mu\leq_c\tau$. Also note that every endomorphism of $\Delta(\tau)$ is determined by its values on $1\otimes\tau$, and it has to send $1\otimes\tau$ to $1\otimes\tau$. It follows, then, that $\mathrm{End}_{H_c}(\Delta(\tau))=\mathbb{C}$.

Let us examine the partial order $\leq_c$ more carefully. Recall that irreducible representations of $\mathfrak{S}_n$ are labelled by Young diagrams of size $n$ (i.e. partitions of $n$). For a Young diagram $\tau$, a basis of the corresponding irreducible $\mathfrak{S}_n$-module is given by the set of standard Young tableaux of shape $\tau$, that is, an enumeration of the boxes of $\tau$ that is increasing along each row and column.

For a box $(i,j)\in\tau$, define its content:

$$\mathrm{ct}(i,j)=j-i.$$

By looking at the action of $\mathbf{h}$ on a basis element corresponding to a standard Young tableaux of shape $\tau$, we have that

$$c_\tau=\frac{n}{2}-c\sum_{(i,j)\in\tau}\mathrm{ct}(i,j).$$

Note that, if $c=r/m$, with $r,m\in\mathbb{Z}$, $(r;m)=1$ and $0<m\leq n$, then the partial order induced by $c$ is not trivial. We also remark that if $\leq_c$ is trivial, then category $\mathcal{O}$ is semisimple. However, category $\mathcal{O}$ may be semisimple even if $\leq_c$ is not trivial.

**Proposition 3.10** *Assume that for any two irreducible representations* $\tau,\mu$ *of* $\mathfrak{S}_n$*,* $c_\tau-c_\mu\notin\mathbb{Z}_{>0}$*. Then, category* $\mathcal{O}$ *is semisimple.*

*Proof.* This follows from the fact that, if $c_\tau-c_\mu\notin\mathbb{Z}_{>0}$, $\mathrm{Ext}^1_{H_c}(L(\tau),L(\mu))=0$. This is an exercise. (A hint: take an extension $0\to L(\mu)\to M\to L(\tau)\to 0$. Look at the highest weight vectors of $L(\mu)$, $L(\tau)$ in $M$). □

The partial order $\leq_c$ also endows category $\mathcal{O}$ with many upper-triangularity properties. This is made precise in the following notion.

**Definition 3.11** *Let* $\mathcal{C}$ *be an abelian,* $\mathbb{C}$*-linear, artinian category, with finitely many simples and enough projectives. Let* $\Lambda$ *be a labeling of the set of simples. For* $\lambda\in\Lambda$*, denote by* $L(\lambda)$ *the corresponding simple object and by* $P(\lambda)$ *its projective cover. A highest weight (or quasi-hereditary) structure on* $\mathcal{C}$ *is given by a partial order on* $\Lambda$ *and a set of objects (the standard objects)* $\{\Delta(\lambda)\}_{\lambda\in\Lambda}$ *satisfying the following conditions:*

*(HW1) If* $\mathrm{Hom}_{\mathcal{C}}(\Delta(\mu),\Delta(\lambda))\neq 0$*, then* $\mu\leq\lambda$*.*

*(HW2)* $\mathrm{End}_{\mathcal{C}}(\Delta(\lambda))=\mathbb{C}$ *for every* $\lambda\in\Lambda$*.*

*(HW3) There exists an epimorphism* $P(\lambda)\twoheadrightarrow\Delta(\lambda)$ *whose kernel admits a filtration by standard objects* $\Delta(\mu)$ *with* $\mu>\lambda$*.*

So far, we have seen that $\mathcal{O}$ is abelian, $\mathbb{C}$-linear and artinian, and that, together with the partial order introduced above, it satisfies conditions $(HW1), (HW2)$, this is a consequence of Frobenius reciprocity. It remains to see that category $\mathcal{O}$ has enough projectives, and that projectives admit a required filtration.

**Theorem 3.12** *Category $\mathcal{O}$ is a highest weight category.*

*Proof.* Let us construct a projective cover for $L(\tau)$. First of all, consider the induced modules

$$\Delta_m(\tau) = H_c \otimes_{S(\mathfrak{h})\#\mathfrak{S}_n} (\tau \otimes S(\mathfrak{h})/(\mathfrak{h}^m)).$$

Of course, $\Delta(\tau) = \Delta_1(\tau)$. Note that the $\mathbf{h}$-action on $\Delta_m(\tau)$ is locally finite with eigenvalues $c_\tau - m + k, k \in \mathbb{Z}_{\geq 0}$. It follows that $\Delta_m(\tau) \in \mathcal{O}$. Also, by Frobenius reciprocity, for any $H_c$-module $M$, $\mathrm{Hom}_{H_c}(\Delta_m(\tau), M) = \mathrm{Hom}_{\mathfrak{S}_n}(\tau, M^{\mathfrak{h}^m})$. Consider the module $\overline{\Delta}(\tau) := \varprojlim_m \Delta_m(\tau)$. For any module $M \in \mathcal{O}$, we have that $\varinjlim_m M^{\mathfrak{h}^m} = M$, so,

$$\mathrm{Hom}_{H_c}(\overline{\Delta}(\tau), M) = \varinjlim_m \mathrm{Hom}_{H_c}(\Delta_m(\tau), M) = \mathrm{Hom}_{\mathfrak{S}_n}(\tau, \varinjlim_n M^{\mathfrak{h}^n}) = \mathrm{Hom}_{\mathfrak{S}_n}(\tau, M).$$

So wee see that $\overline{\Delta}(\tau)$ is projective relative to objects in $\mathcal{O}$. However, $\overline{\Delta}(\tau)$ doesn't live in $\mathcal{O}$. To get an object from $\mathcal{O}$, we need to make a more careful construction. In fact, we will obtain a direct summand of $\overline{\Delta}(\tau)$ that actually lives in $\mathcal{O}$. This is somewhat technical, and we'll try to do it as painless as possible.

Note that $H_c$ is $\mathbb{Z}$-graded by eigenvalues of $\mathrm{ad}(\mathbf{h})$, so that the zeroth graded component is $\mathbb{C}\mathfrak{S}_n$, the positive component is $S(\mathfrak{h}^*)$ and the negative component is $S(\mathfrak{h})$. The modules $\Delta_m(\tau)$ are all $\mathbb{Z}$-graded by eigenvalues of $\mathbf{h}$, too. In general, consider a $\mathbb{Z}$-graded $H_c$-module $M = \bigoplus M^i$. Since $\mathbf{h} \in H_c$ sits in degree 0, it acts on the graded components of $M$. For each $i \in \mathbb{Z}$, $a \in \mathbb{C}$, let $M^i[a]$ be the generalized $\mathbf{h}$-eigenspace of $M^i$ with eigenvalue $a$. Note that the action of $\mathfrak{h}$ sends $M^i[a]$ to $M^{i-1}[a-1]$, and the action of $\mathfrak{h}^*$ sends $M^i[a]$ to $M^{i+1}[a+1]$. It follows that $W_a(M) = \bigoplus M^i[a+i]$ is a graded submodule of $M$.

Now let $\widetilde{\Delta}_m(\tau) := W_{c_\tau}(\Delta_m(\tau))$. Note that $\widetilde{\Delta}_1(\tau) = \Delta(\tau)$. Since there is a graded epimorphism $\Delta_{m+1}(\tau) \twoheadrightarrow \Delta_m(\tau)$ for every $m$ and the functor $W_{c_\tau}(\bullet)$ is exact, we get a natural epimorphism $\widetilde{\Delta}_{m+1}(\tau) \twoheadrightarrow \widetilde{\Delta}_m(\tau)$. We claim that this is an isomorphism for $m \gg 0$, and leave the proof of this claim as an exercise for the reader.

Let $\widetilde{\Delta}(\tau)$ be the module $\widetilde{\Delta}_m(\tau)$ for $m \gg 0$. Clearly, $\widetilde{\Delta}(\tau) \in \mathcal{O}$. We claim that it is projective in $\mathcal{O}$. Let $M \in \mathcal{O}$. Since any morphism must respect $\mathbf{h}$-eigenvalues, $\mathrm{Hom}_{\mathcal{O}}(\widetilde{\Delta}(\tau), M) = \mathrm{Hom}_{\mathcal{O}}(\widetilde{\Delta}(\tau), W_{c_\tau+\mathbb{Z}}(M))$, so we may assume $M = W_{c_\tau+\mathbb{Z}}(M)$. We can give a grading to $M$ by letting the $i$-th graded component be the generalized $\mathbf{h}$-eigenspace with eigenvalue $c_\tau + i$. An exercise now is to show that $\mathrm{Hom}(\widetilde{\Delta}(\tau), M) = \mathrm{Hom}_{\mathfrak{S}_n}(\tau, M_{c_\tau})$. Since the functor $M \mapsto M_{c_\tau}$ is exact, this shows that $\widetilde{\Delta}(\tau)$ is projective in $\mathcal{O}$.

Note that $\widetilde{\Delta}(\tau)$ has a required filtration by Verma modules. Indeed, it is filtered with succesive quotients being $W_{c_\tau}(\Delta_m(\tau)/\Delta_{m-1}(\tau)) = W_{c_\tau}(\Delta(\tau \otimes S^m\mathfrak{h}))$, where $S^m\mathfrak{h}$ denotes the $m$-th symmetric product of $\mathfrak{h}$. This module splits into a certain sum of Verma modules, say $\Delta(\tau')$. Note that we must have $c_{\tau'} = c_\tau - m$, so $\tau < \tau'$. Note, however, that the module $\widetilde{\Delta}(\tau)$ need not be a projective cover for $L(\tau)$. The existence of a required filtration for the projective cover of $L(\tau)$ is guaranteed by Proposition 3.13 below. □

**Proposition 3.13** *Let $\tau, \mu$ be irreducible representations of $\mathfrak{S}_n$. Assume that $\tau \not<_c \mu$. Then,* $\mathrm{Ext}^1_{H_c}(\Delta(\tau), \Delta(\mu)) = 0$.

We leave the proof of Proposition 3.13 as an exercise, but we do check that this implies that a direct summand of a standardly filtered object is again standardly filtered. In fact, assume $M = M_1 \oplus M_2$ is standardly filtered, with succesive subquotients being $\Delta(\tau_1), \ldots, \Delta(\tau_m)$. Let $i$ be such that $\tau_i$ is maximal (with respect to $\leq_c$) among the $\{\tau_1, \ldots, \tau_m\}$. Then, by Proposition 3.13, we have an exact sequence $0 \to \Delta(\tau_i)^{\oplus m} \to M \to M' \to 0$, where $M'$ is standardly filtered and contains no $\Delta(\tau_i)$. It follows that $\mathrm{Hom}(\Delta(\tau_i), M) = \mathrm{Hom}(\Delta(\tau_i), \Delta(\tau_i)^m)$. Note that any nonzero map here is injective. Since $\mathrm{Hom}(\Delta(\tau_i), M) = \mathrm{Hom}(\Delta(\tau_i), M_1) \oplus \mathrm{Hom}(\Delta(\tau_i), M_2)$, we must have an embedding $\Delta(\tau_i) \hookrightarrow M_1$ or $\Delta(\tau_i) \hookrightarrow M_2$. Now we can mod out by the image of $\Delta(\tau_i)$. By induction on the composition length of $M$, we get the result.

## 3.3 Finite dimensional representations and spherical values.

In this subsection, we construct finite dimensional representations of the algebra $H_c^+ := H_{1,c}^+$, see Subsection 1.4. The reason why we have to restrict to this algebra is the following. Consider the elements $\underline{x} = \sum_{i=1}^n x_i, \underline{y} = \sum_{i=1}^n y_i \in H_c$.

Note that these elements commute with $H_c^+$. Also note that the subalgebra of $H_c$ generated by $\underline{x}, \underline{y}$ is isomorphic to $\mathcal{D}(\mathbb{C})$. It then follows by the PBW theorem that we have a decomposition $H_c = H_c^+ \otimes \mathcal{D}(\mathbb{C})$, so the algebra $H_c$ does not admit finite dimensional representations. The decomposition $H_c = H_c^+ \otimes \mathcal{D}(\mathbb{C})$ shouldn't be too surprising, it is a consequence of the decomposition $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathbb{C}$ as a $\mathfrak{S}_n$-module, where $\mathfrak{h}^+$ is the reflection representation and $\mathbb{C}$ the trivial representation of $\mathfrak{S}_n$.

Now, recall that $\overline{\mathfrak{h}}^+ = \mathfrak{h}/\mathbb{C}$, where $\mathbb{C}$ gets identified with the diagonal in $\mathfrak{h}$. Then, the representation $\Delta(\text{triv})$ may be identified with the space of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ that are invariant under simultaneous translation $x_i \mapsto x_i + a$. The following proposition can be proved by a computation.

**Proposition 3.14** *Assume $c = r/n$, where $r$ is a positive integer not divisible by $n$. Then, $\Delta(\text{triv})^{\mathfrak{h}^+}$ contains a copy of the reflection representation of $\mathfrak{S}_n$. This copy sits in degree $r$ and it is spanned by the functions:*

$$f_i(x_1, \ldots, x_n) = \text{Res}_\infty[(z - x_1) \ldots (z - x_n)]^{r/n} \frac{dz}{z - x_i}.$$

We remark that the space spanned by the $f_i's$ is indeed $n-1$ dimensional, as $\sum_i f_i = 0$. Then, we have a morphism $\Delta(\text{refl}) \to \Delta(\text{triv})$. Let $I$ denote the image of this morphism, and let $V$ be the quotient $\Delta(\text{triv})/I$. Note that $V$ can be regarded as a $\mathbb{C}[\mathfrak{h}^+]$ module.

**Proposition 3.15** *Let $d$ be the greatest common divisor of $r$ and $n$, $d = \gcd(r, n)$. Then, the support of $V$ is the union of the $\mathfrak{S}_n$-translates of the subspaces of $\mathfrak{h}/\mathbb{C}$, defined by the equations:*

$$x_1 = x_2 = \cdots = x_{\frac{n}{d}}, \ \ x_{\frac{n}{d}+1} = \cdots = x_{2\frac{n}{d}}, \ldots, x_{(d-1)\frac{n}{d}+1} = \cdots = x_n. \tag{10}$$

*Proof.* Since, as a $\mathbb{C}[\mathfrak{h}^+]$-module, $I$ is generated by $f_1, \ldots, f_n$, a point $(x_1, \ldots, x_n) \in \mathbb{C}^n$ is in the support of $V$ if and only if $f_i(x_1, \ldots, x_n) = 0$ for every $i = 1, \ldots, n$. This hapens if and only if $\sum_1^n \lambda_i f_i(x_1, \ldots, x_n) = 0$ for every $\lambda_i$, that is:

$$\text{Res}_\infty \left( \prod_{j=1}^n (z - x_j)^{r/n} \sum_{i=1}^n \frac{\lambda_i}{z - x_i} \right) dz = 0.$$

Assume the $x_1, \ldots, x_n$ take distinct values $a_1, \ldots, a_p$ with multiplicities $m_1, \ldots, m_p$. Then, the point $(x_1, \ldots, x_n)$ is in the common zero set of $f_1, \ldots, f_n$ if and only if:

$$\text{Res}_\infty \left( \prod_{j=1} p(z - a_j)^{m_j \frac{r}{n} - 1} \left( \sum_{i=1}^p \nu_i (z - a_1) \cdots \widehat{(z - a_i)} \cdots (z - a_p) \right) \right) dz = 0$$

for every collection of $\nu_i \in \mathbb{C}$, where a hat means that we omit the respective factor. Since the $\nu_i$ can be arbitrary, this happens if and only if

$$\text{Res}_\infty \left( \prod_{j=1}^p (z - a_j)^{m_j \frac{r}{n} - 1} z^i dz \right) = 0, \quad \text{for } i = 0, \ldots, p - 1.$$

We claim that this implies that $\prod_{j=1}^p (z - y_j)^{m_j \frac{r}{n} - 1}$ is a polynomial. Note that this holds if and only if every $m_j$ is divisible by $n/d$, and therefore the claim implies the proposition. We prove this claim in the following lemma. □

**Lemma 3.16** *Let $a(z) = \prod_{i=1}^p (z - a_i)^{\mu_i}$, where $\mu_i \in \mathbb{C}$, $\sum_{i=1}^p \mu_i \in \mathbb{Z}$ and $\sum_{i=1}^p \mu_i > -p$. Moreover, suupose that*

$$\text{Res}_\infty \, a(z) z^i dz = 0, \quad \textit{for } i = 0, \ldots, p - 2.$$

*Then, $a(z)$ is a polynomial.*

*Proof.* Let $g(z)$ be a polynomial. Then,

$$0 = \text{Res}_\infty \, d(g(z)a(z)) = \text{Res}_\infty (g'(z)a(z) + g(z)a'(z))dz.$$

Recalling the form of $a(z)$, this says that:

$$\text{Res}_\infty\left(g'(z)+\sum_i \frac{\mu_i}{z-a_i}g(z)\right)a(z)dz=0.$$

Pick the polynomial $g(z)=z^l\prod_{i=1}^p(z-y_i)$. Then, $g'(z)+\sum_i\frac{\mu_i}{z-a_i}g(z)$ is a polynomial of degree at most $l+p-1$. Note that the coefficient of $z^{l+p-1}$ is $l+p+\sum_i\mu_i\neq 0$ since $\sum\mu_i>-p$. Then, $\text{Res}_\infty z^{l+p-1}a(z)dz$ is a linear combination of residues of the form $\text{Res}_\infty z^q a(z)$, with $q<l+p-1$. Using the hypothesis of the lemma, by induction this implies that $\text{Res}_\infty z^m a(z)=0$ for every $m\geq 0$. Then, $a(z)$ is a polynomial. □

Note that, in our case, we have that $\sum_i m_i\frac{r}{n}-1=r-p$, because $\sum_i m_i=m$. Then, Lemma 3.16 does imply Proposition 3.15.

**Corollary 3.17** *Assume that* $\gcd(r,n)=1$*. Then, the representation* $V=\Delta(\text{triv})/I$ *is finite dimensional.*

*Proof.* In this case, the support of $V$ is a single point. Since $V$ is finitely generated over $\mathbb{C}[\mathfrak{h}^+]$, it must be finite dimensional. □

Now, since $V$ is finite dimensional, we can construct a resolution of $V$ by Verma modules and therefore compute its character. Namely, we claim that we have a resolution

$$0\to\Delta(\wedge^{n-1}\text{refl})\to\cdots\to\Delta(\wedge^2\text{refl})\to\Delta(\text{refl})\to\Delta(\text{triv})\to V\to 0. \tag{11}$$

In fact, consider $S(\text{refl})\subseteq S(\mathfrak{h}^*)$, where the former algebra is the one generated by the functions $f_i$ constructed in Proposition 3.14. Since $V$ is finite dimensional, $S(\mathfrak{h}^*)$ is a finitely generated $S(U)$-module. By a theorem of Serre, this module is free, of rank $r^{n-1}$ (recall here that $c=r/n$). Consider the Koszul resolution associated to the $S(U)$-module $S(\mathfrak{h}^*)$. It has precisely the form (11). It only remains to check that the differential maps are actually maps of $H_c$-modules. But this follows easily by Frobenius reciprocity. From the resolution (11) and the formula for characters of Vermas given in Proposition 3.8, it follows that the character of $V$ is:

$$\text{ch}_V(\sigma,t)=t^{\frac{(1-r)(n-1)}{2}}\frac{\det_{(\mathfrak{h}^+)^*}(1-\sigma t^r)}{\det_{(\mathfrak{h}^+)^*}(1-\sigma t)}.$$

It turns out that we have constructed all irreducible finite dimensional representations of $H_c^+$, for positive $c$. Namely, the following result is proved in [1, Theorem 1.2].

**Theorem 3.18** *The only values of* $c\in\mathbb{C}$ *for which nontrivial finite dimensional representations of* $H_c^+$ *exists, are* $c=\pm r/n$*, where* $\gcd(r,n)=1$*. If* $c>0$*, the unique irreducible finite dimensional representation of* $H_c^+$ *is* $L(\text{triv})$*. If* $c<0$*, the unique irreducible finite dimensional representation of* $H_c^+$ *is* $L(\text{sign})$*.*

Note that the claims for $c>0$ and $c<0$ are equivalent, since we have an isomorphism $H_c^+\to H_{-c}^+$, $x_i\mapsto x_i$, $y_i\mapsto y_i$, $\sigma\mapsto\text{sign}(\sigma)\sigma$, so that, in the induced equivalence $\mathcal{O}_{-c}\to\mathcal{O}_c$, $L(\text{triv})$ gets sent to $L(\text{sign})$.

Finally, let us mention results regarding to spherical values, that is, those $c$ for which the rational Cherednik algebra $H_c$ is Morita equivalent to its spherical subalgebra, $eH_ce$. These are precisely the values of $c$ for which $eH_ce$ is smooth, that is, it has finite global dimension. This is a result of Bezrukavnikov, see [5]. It is possible to show that, if $c=-r/n$, $0<r<n, \gcd(r,n)=1$, then the idempotent $e\in H_c^+$ kills the finite dimensional representation $L(\text{sign})$, so that the algebras $H_c$ and $eH_ce$ cannot be Morita equivalent. This is clear, for example, when $c=-\frac{1}{n}$: the representation $L(\text{sign})$ is 1 dimensional, isomorphic to sign as a $\mathfrak{S}_n$-module and with zero action of both $\mathfrak{h}$ and $\mathfrak{h}^*$. Using an inductive argument based on parabolic restriction functors for rational Cherednik algebras, Bezrukavnikov and Etingof proved the following, [2, Corollary 4.2].

**Theorem 3.19** *The parameter* $c$ *is spherical if and only if* $c$ *is not a rational number in* $(-1,0)$ *with denominator* $\leq n$*.*

# 4 Appendix

In this Appendix, we give some of the most technical proofs that are in the text.

### 4.1 Proof of the Double Centralizer Property

*Injectivity of the natural map* $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]\#\mathfrak{S}_n \to \mathrm{End}_{\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{\mathfrak{S}_n}}(\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*])$. Note that $\mathfrak{h}^{reg}\times\mathfrak{h}^*$ is open and dense in $\mathfrak{h}\oplus\mathfrak{h}^*$, and the action of $\mathfrak{S}_n$ here is free. Let $\sum_{\sigma\in\mathfrak{S}_n} f_\sigma\sigma$ be in the kernel of $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]\#\mathfrak{S}_n \to \mathrm{End}_{\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{\mathfrak{S}_n}}(\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*])$. Then, for every $g\in\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]$, $\sum f_\sigma\sigma(g)=0$. Now pick $v\in\mathfrak{h}^{reg}\times\mathfrak{h}^*$. Since $\mathfrak{S}_n$ acts freely on $v$, for any collection of complex numbers $\{z_\sigma\}_{\sigma\in\mathfrak{S}_n}$ we can find a regular function $g$ such that $g(\sigma^{-1}v)=z_\sigma$, so $\sum f_\sigma(v)z_\sigma=0$. Then, $f_\sigma(v)=0$ for every $\sigma\in\mathfrak{S}_n$. But $\mathfrak{h}^{reg}\times\mathfrak{h}^*$ is dense in $\mathfrak{h}\oplus\mathfrak{h}^*$, so $f_\sigma=0$ for every $\sigma\in\mathfrak{S}_n$.

*Surjectivity of the natural map* $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]\#\mathfrak{S}_n \to \mathrm{End}_{\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{\mathfrak{S}_n}}(\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*])$. First, we make the following observation: *If a group* $\Gamma$ *acts freely on a smooth affine variety* $X$*, then the homomorphism* $\mathbb{C}[X]\#\Gamma\to\mathrm{End}_{\mathbb{C}[X]^\Gamma}(\mathbb{C}[X])$ *is an isomorphism.* This follows because both algebras are locally free over $\mathbb{C}[X]^\Gamma$ of rank $|\Gamma|^2$, so to show bijectivity, it is enough to show injectivity fiberwise, and this can be done as in the previous step of this proof.

Now, since $\mathfrak{S}_n$ acts on $\mathbb{C}^{2n}=T^*\mathfrak{h}$ by symplectomorphisms, the fixed point locus of any permutation $\sigma\in\mathfrak{S}_n$ has codimension at least 2. So the codimension of $\mathbb{C}^{2n}\setminus\mathbb{C}^{2n,reg}$ is at least 2. For a point $v\in\mathbb{C}^{2n,reg}$, we can find an invariant function $f_v\in\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{\mathfrak{S}_n}$ with $f_v(v)\neq 0$, $f|_{\mathbb{C}^{2n}\setminus\mathbb{C}^{2n,reg}}=0$. Now let $\mathbb{C}^{2n,reg}_v:=\{x: f(x)\neq 0\}$. This is an affine open set on $\mathbb{C}^{2n,reg}$, $\mathfrak{S}_n$-stable since $f_v$ is $\mathfrak{S}_n$-invariant. Cover $\mathbb{C}^{2n,reg}$ by a finite number of sets of the for $\mathbb{C}^{2n,reg}_v$, say $\mathbb{C}^{2n,reg}=\bigcup V_i$, with $f_i$ the function used to define $V_i$. An observation here is that $\mathrm{End}_{\mathbb{C}[V_i]^{\mathfrak{S}_n}}(\mathbb{C}[V_i])$ is just the localization of $\mathrm{End}_{\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{\mathfrak{S}_n}}(\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*])$ at $f_i$. In particular, we have a morphism $\iota_i:\mathrm{End}_{\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{\mathfrak{S}_n}}(\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*])\to\mathrm{End}_{\mathbb{C}[V_i]^{\mathfrak{S}_n}}(\mathbb{C}[V_i])$. Since $f_i$ is $\mathfrak{S}_n$-invariant, $\iota_i$ is injective.

Now let $f\in\mathrm{End}_{\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{\mathfrak{S}_n}}(\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*])$. Consider $\iota_i(f)$. Since the action of $\mathfrak{S}_n$ on $V_i$ is free, there exist $f_i^\sigma\in\mathbb{C}[V_i]$ such that $\iota_i(f)=\sum_{\sigma\in\mathfrak{S}_n} f_i^\sigma\sigma$. We claim that, for every $\sigma,i,j$, $f_i^\sigma|_{V_i\cap V_j}=f_j^\sigma|_{V_i\cap V_j}$. This follows, again, because the action of $\mathfrak{S}_n$ on $V_i\cap V_j$ is free. So the $f_i^\sigma$ glue to form a regular function $f^\sigma$ on $\mathbb{C}^{2n,reg}$. Since the codimension of $\mathbb{C}^{2n}\setminus\mathbb{C}^{2n,reg}$ is at least 2, $f^\sigma$ is actually regular in $\mathbb{C}^{2n}=\mathfrak{h}\oplus\mathfrak{h}^*$. We are done with surjectivity.

Since $H_{t,c}e$ is a finitely generated $eH_{t,c}e$-module, we can equip it with a filtration compatible with that of $eH_{t,c}e$ making $\mathrm{gr}(H_{t,c}e)$ a finitely generated $\mathrm{gr}(eH_{t,c}e)$-module, and we can use this filtration to equip $\mathrm{End}_{eH_{t,c}e}(H_{t,c}e)$ wih a filtration such that $\mathrm{gr}\,H_{t,c}\to\mathrm{gr}\,\mathrm{End}_{eH_{t,c}e}(H_{t,c}e)$ is precisely the isomorphism $H_{0,0}\to\mathrm{gr}\,\mathrm{End}_{eH_{0,0}e}(H_{0,0}e)$. This can be done as follows. Set $\mathrm{End}_{eH_{t,c}e}(H_{t,c}e)^{\leq p}:=\{\psi\in\mathrm{End}_{eH_{t,c}e}(H_{t,c}e):\varphi(H_{t,c}e^{\leq q}\subseteq H_{t,c}e^{\leq p+q}$ for every $q\}$. It is an exercise to show that this filtration satisfies the claims above. It follows that the original morphism $H_{t,c}\to\mathrm{End}_{eH_{t,c}e}(H_{t,c}e)$ is an isomorphism.

### 4.2 An isomorphism of symplectic varieties.

Here, we show that the map $\varphi: T^*(\mathfrak{h}^{reg}/\mathfrak{S}_n)\to U$ constructed in Subsection 2.3 is surjective. Recall that $U$ is the set of conjugacy classes of pairs of matrices $(X,Y)$ such that $X$ is diagonalizable with pairwise distinct eigenvalues, say $X=\mathrm{diag}(x_1,\ldots,x_n)$, $x_i\neq x_j$.Then, the entries of $XY-YX$ are $(x_i-x_j)y_{ij}$. It follows that the diagonal entries are 0. Recall that $T+I$ has rank one, so that $T+I=a\otimes b$ for some vectors $a=(a_1,\ldots,a_n)$, $b=(b_1,\ldots,b_n)^T$. Since the diagonal entries of $T+I$ are 1, we have that $a_i=b_i^{-1}$, so the $(i,j)$-entry of $T+I$ is $a_ia_j^{-1}$. If we conjugate the pair $(X,Y)$ by $\mathrm{diag}(a_1,\ldots,a_n)$, we get that $T+I$ is the matrix whose every entry is 1. Then, $(x_i-x_j)y_{ij}=1$ for $i\neq j$, so $y_{ij}=1/(x_i-x_j)$. The diagonal entries of $Y$ are unconstrained. This implies the claim on surjectivity of $\varphi$.

We describe, in coordinates, a dense open subset of $\mathcal{C}_n$. Namely, consider the set $U$ of conjugacy classes of pairs of matrices $(X,Y)$ such that $X$ is diagonalizable with pairwise distinct eigenvalues, say $X=\mathrm{diag}(x_1,\ldots,x_n)$, $x_i\neq x_j$. We have a map $\varphi:T^*(\mathfrak{h}^{reg}/\mathfrak{S}_n)\to U$, given by the formula $(x_1,\ldots,x_n,y_1,\ldots,y_n)\mapsto(X,Y)$, where $X=\mathrm{diag}(x_1,\ldots,x_n)$, $y_{ij}=1/(x_i-x_j)$, $i\neq j$, and $y_{ii}=y_i$. As we show in the text, this turns out to be an isomorphism of symplectic varieties.

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