Let's take for granted for now that $H \cong K^{G\times\mathbb{C}^*}(Z)$.

It is known that the center of H is $R(G\times\mathbb{C}^*) = \{\text{regular class functions on } G\times\mathbb{C}^*\}$

So one-dimensional reps of $Z(H) = R(G\times\mathbb{C}^*)$ may be viewed as evaluation maps at semisimple elements of $G\times\mathbb{C}^*$ - the conjugacy class of such a semisimple element is then determined. If a is s.s. elt. of $G\times\mathbb{C}^*$, write $\mathbb{C}_a$ for the corresponding rep of $Z(H)$. We will consider $H_a := \mathbb{C}_a \otimes_{Z(H)} H$. It is the quotient of $\mathbb{C}\otimes_{\mathbb{Z}} H$ by the ideal generated by the ann. of $\mathbb{C}_a$. Since $\mathbb{C}\otimes_{\mathbb{Z}} H$ is countable dimensional, by Schur's lemma any irrep. factors through some H_a. Notice H_a is actually finite-dimensional. Thus irreps of $H\otimes\mathbb{C}$ are all f.d, and factor through some H_a - this is why we consider H_a.

There is a nice geometric interpretation of H_a. Let A be the closed sbgp of $G\times\mathbb{C}^*$ gen. by a.

$$H_a \underset{①}{\cong} \mathbb{C}_a \otimes_{R(G\times\mathbb{C}^*)} K^{G\times\mathbb{C}^*}(Z) \underset{②}{\cong} \mathbb{C}_a \otimes_{R(A)} K^A(Z)$$

$$\underset{③}{\cong} \mathbb{C}_a \otimes_{R(A)} K^A(Z^A) \underset{④}{\cong} K_{\mathbb{C}}(Z^A) \underset{⑤}{\cong} H_\bullet(Z^A,\mathbb{C}) = H_\bullet(Z^a,\mathbb{C}).$$

I have to explain these isomorphisms. The first one is taken for granted. The second is because

$$R(A)\otimes_{R(G\times\mathbb{C}^*)} K^{G\times\mathbb{C}^*}(Z) = R(A)\otimes_{R(T\times\mathbb{C}^*)} R(T\times\mathbb{C}^*)\otimes_{R(G\times\mathbb{C}^*)} K^{G\times\mathbb{C}^*}(X)$$

where ~~RE~~ T is a max. torus s.t. $T\times\mathbb{C}^* \supset A$

Notice that

$$K^{T\times\mathbb{C}^*}(Z) \cong K^{B\times\mathbb{C}^*}(Z) = K^{G\times\mathbb{C}^*}(G\times\mathbb{C}^* \times_{B\times\mathbb{C}^*} Z)$$

$$\cong K^{G\times\mathbb{C}^*}(\mathcal{B}\times Z) \cong K^{G\times\mathbb{C}^*}(\mathcal{B}) \otimes K^{G\times\mathbb{C}^*}(Z)$$

6.1.19, 5.6.1(a) – Kunneth for $\mathcal{B}$

$$\cong R(T\times\mathbb{C}^*) \otimes K^{G\times\mathbb{C}^*}(Z)$$

Thus $R(A) \otimes_{R(G\times\mathbb{C}^*)} K^{G\times\mathbb{C}^*}(Z) \cong R(A) \otimes_{R(T\times\mathbb{C}^*)} K^{T\times\mathbb{C}^*}(Z)$

$$\cong K^{A}(Z) \quad \text{by cellular fibration lemma.}$$

(because Z is a cellular fibration / $\mathcal{B}$, & $\mathcal{B}$ is a cellular fibration over pt, and result is true replacing Z w/ pt).

To understand (3) come back to the setting of the Thom isom. thm:

$\pi: E \to X$ a vector bundle, $i: X \to E$ zero section. There is a flat resolution (Koszul complex) of $i_*\mathcal{O}_X$, given by

$$\cdots \to \pi^*\Lambda^2 E^\vee \to \pi^* E^\vee \to \mathcal{O}_E \to i_*\mathcal{O}_X \to 0.$$

Thus i^* (on K-theory) is given by multiplication by $\lambda(\pi^*E^\vee)$.

One may check also that $i^* i_*$ is given by multiplication by $\lambda(E^\vee)$.

One can also show (algebraic analogue of tubular nbhd thm) that if $X \hookrightarrow Y$ then $i^* i_*$ is given by multiplication by $\lambda(T^*_X Y)$.

Apply this to the case of reductive gp H acting on a smooth variety X, so that X^H is smooth.

Now suppose abelian reductive group A acts on E & trivially on X (E A-equivariant). For generic $a \in A$, we will have $X = E^a$. In that case, it is easy to check that $\lambda(E)$ is invertible in $K^A(X)_a$ (localisation in sense of $R(A)$-modules).

$\subseteq \mathrm{Fun}(A)$

Hence $i_*: K^A(X)_a \longrightarrow K^A(E)_a$ is an isom (follows from Thom isom. thm)
$i_* = \pi^*(\lambda(E^\vee))$.

<u>Localisation thm</u> A, X as above, $a \in A$, $i: X^a \hookrightarrow X$. When is i_{*a} an isom.?
$K^A(X^a)_a \longrightarrow K^A(X)_a$?

<u>Answer</u>: always, due to Thomason – though in Chriss-Ginzburg they only need that it holds for cellular fibrations if for the base, which has an easier proof.

In ptic. $\quad i_*: \overset{K_{\mathbb{C}}(X^A)}{\overset{\|}{\mathbb{C}_a \otimes_{R(A)} K^A(X^A)}} \longrightarrow \mathbb{C}_a \otimes_{R(A)} K^A(X)$ is isom, a X-regular

Let $\lambda_A = \lambda(T^*_{X^A} X) \in K^A(X^A)$ and λ_a be its evaluation at a
$\in K_{\mathbb{C}}(X^A)$.

Then $i^* i_*$ is mult. by λ_a, thus the inverse to i_* is ~~eqv~~ explicitly given as $\lambda_a^{-1} i^* = \mathrm{res}_a$. (res$_a$ commutes w/ proper pushforward)

$$\begin{tikzcd}
 & K^A(X^A) \arrow[dr, "\lambda_a^{-1}\circ ev_a"] & \\
K^A(X) \arrow[ur, "i^*"] \arrow[dr] & & K_{\mathbb{C}}(X^A) \\
 & \mathbb{C}_a \otimes_{R(A)} K^A(X) \arrow[ur, "\lambda_a^{-1}\mathbb{C}_a \otimes_{R(A)} i^*"'] &
\end{tikzcd}$$

~~This explains~~. <u>Convolution</u>: $i: Z_{12} \hookrightarrow X_1 \times X_2$ etc.
Now if we define $r_a: \mathbb{C}_a \otimes_{R(A)} K^A(Z_{12}) \longrightarrow K_{\mathbb{C}}(Z_{12}^A)$ by $1 \boxtimes \lambda_2^{-1} \cdot i^*$.

<u>Check</u>. r_a is isom.

r_a commutes with convolution !!!

This explains points (3) & (4).

(5) is the bivariant Riemann-Roch Theorem

Bivariant RR is $(1 \boxtimes Td_{M_2}) \cup$ chern map. It commutes w/ convolution.

Here, $H_\bullet$ is Borel-Moore Homology.

It is defined for reasonable spaces (locally compact, homotopy CW complex with a closed embedding into a contable at ∞ manifold such that it becomes a homotopy retract of an open nbhd) $X \hookrightarrow M$

then $H_\bullet(X) = H^{m-\bullet}(M, M \setminus X)$. Ring structure on RHS gives one on LHS, mult. is written as $\cap$.

Convolution in Borel-Moore homology is defined as for K-theory:

$$c_{12} * c_{23} = p_{13*}\left(c_{12} \boxtimes [M_3] \cap [M_1] \boxtimes c_{23} \right).$$

In our setting, we have reduced to understanding the irreducible representations of the convolution algebra $H_\bullet(Z^a, \mathbb{C})$

(recall — $Z^a \hookrightarrow T^*\mathcal{B}^a \times T^*\mathcal{B}^a$, $Z^a \circ Z^a = Z^a$).

$$\begin{array}{ccc} \| & & \| \\ \tilde{\mathcal{N}}^a \times_{\mathcal{N}^a} \tilde{\mathcal{N}}^a & \hookrightarrow & \tilde{\mathcal{N}}^a \times \tilde{\mathcal{N}}^a \end{array}$$

We study the more general situation where $\mu: X \to Y$ is a proper map, X smooth (e.g. $\tilde{\mathcal{N}}^a \to \mathcal{N}^a$) and $Z = X \times_Y X$ (so $Z \circ Z = Z$).

We work in bounded derived category w/ constructible cohomology.

$f: X \to Y$. It is known there are morphisms $f_*, f_!, f^*, f^!$

dualizing complex ω_X, Verdier duality $\mathbb{D}_X$

I can give an overview of these functors if required.

- f_* is right-derived functor of usual pushforward
- f^* is ~~left~~ " " " usual pullback (already exact)
- $f_!$ is right-derived " of proper pushforward
- $f^!$ is right-adjoint to $f_!$. It is constructed as follows:

if f is smooth, $f^! = f^*[2(\dim X - \dim Y)]$
if f is closed embedding, $f^!$ is the right-derived functor of the "pullback with support" map.

The reason why this makes sense is that in both those cases the adjointness criteria holds. (Note that in the first case this is essentially Poincaré duality: exercise).

Dualising sheaf. is $\omega_X = c^! \mathbb{C}_{pt}$ $\quad c : X \to \{pt\}$
X smooth $\Rightarrow \omega_X = \mathbb{C}_X[2\dim X]$. $\quad \mathbb{C}_{pt}$ const. sheaf

Why care? Because you can define a duality

$$\mathbb{D}_X = R\mathcal{H}om(-, \omega_X) \text{ and it satisfies}$$

$\mathbb{D}_Y f_* = f_! \mathbb{D}_X$, $\mathbb{D}_X f^* = f^! \mathbb{D}_Y$.

You can check (key point) that $H_i^{BM}(X) = \underbrace{H^{-i}(X, \omega_X)}$.

$:= \mathcal{H}^{-i} c_* \omega_X$

- this follows from the description of $f^!$ for closed embedding.

So.

$\mu_i : X_i \to Y$ proper

Consider:

$$\begin{tikzcd} Z = X_1 \times_Y X_2 \arrow[r, "\tilde{i}"] \arrow[d, "\tilde{p}"'] \arrow[dr, phantom, "\square"] & X_1 \times X_2 \arrow[d, "p"] \\ Y \arrow[r, "i"'] & Y \times Y \end{tikzcd}$$

$$H^\bullet(Z, \tilde{i}^!(\omega_X \boxtimes \mathcal{C}_X)) = H^\bullet(Y, \tilde{p}_* \tilde{i}^!(\omega_X \boxtimes \mathcal{C}_X))$$
$$= H^\bullet(Y, i^! p_*(\omega_X \boxtimes \mathcal{C}_X))$$
$$= H^\bullet(Y, i^!(\mu_* \omega_X \boxtimes \mu_* \mathcal{C}_X))$$
$$=$$

$$H^\bullet(Z, \tilde{i}^!(\omega_{X_1} \boxtimes \mathcal{C}_{X_2})) = H^\bullet(Y, \tilde{p}_* \tilde{i}^!(\omega_{X_1} \boxtimes \mathcal{C}_{X_2})$$
$$= H^\bullet(Y, i^! p_*(\omega_{X_1} \boxtimes \mathcal{C}_{X_2}))$$
$$= H^\bullet(Y, i^!(\mu_{1*} \omega_{X_1} \boxtimes \mu_{2*} \mathcal{C}_{X_2}))$$
$$= H^\bullet(Y, i^!(\mathbb{D}_Y \mu_{1*} \mathcal{C}_{X_1} \boxtimes \mu_{2*} \mathcal{C}_{X_2}))$$
$$= H^\bullet(Y, \mathcal{H}om(\mu_{1*} \mathcal{C}_{X_1}, \mu_{2*} \mathcal{C}_{X_2}))$$
$$= \mathrm{Ext}^\bullet_{D^b(Y)}(\mu_{1*} \mathcal{C}_{X_1}, \mu_{2*} \mathcal{C}_{X_2})$$

$$\text{So } H_{-j}(Z) = H^j(\omega_Z) = H^{j+d_1+d_2}(Z, \tilde{i}^!(\mathcal{C}_{X_1} \boxtimes \mathcal{C}_{X_2}))$$
$$= H^{j+d_1+d_2}(Y, \tilde{p}_* \tilde{i}^!(\mathcal{C}_{X_1} \boxtimes \mathcal{C}_{X_2}))$$
$$= H^{j+d_1+d_2}(Y, i^! p_*(\mathcal{C}_{X_1} \boxtimes \mathcal{C}_{X_2}))$$
$$= H^{-}(Y, i^!(\mu_{1*} \mathcal{C}_{X_1} \boxtimes \mu_{2*} \mathcal{C}_{X_2}))$$
$$= H^{-}(Y, i^!(\mathbb{D}_Y \mu_{1*} \omega_{X_1} \boxtimes \mu_{2*} \mathcal{C}_{X_2}))$$
$$= H$$

Any ~~of~~ this translated category has an abelian subcategory, category of <u>perverse sheaves</u>: complexes $\mathcal{F}$ satisfying

$$\dim \operatorname{supp} \mathcal{H}^i \mathcal{F} \le -i$$
$$\dim \operatorname{supp} \mathcal{H}^i (\mathbb{D}\mathcal{F}) \le -i \quad \text{for all } i.$$

If $Y \subset X$ is smooth locally closed subvariety of complex dim. d_Y ~~the~~ and $\mathcal{L}$ a local system on Y, you can construct an intersection cohomology sheaf $IC(Y, \mathcal{L}) \in D^b(X)$
It satisfies

$$\mathcal{H}^i (IC(Y,\mathcal{L})) = 0 \quad \text{for } i < -d$$
$$\mathcal{H}^{-d} (IC(Y,\mathcal{L}))|_Y = \mathcal{L}$$
$$\dim \operatorname{supp} \mathcal{H}^i (IC(Y,\mathcal{L})) < -i \quad \text{if } i > -d$$
$$\dim \operatorname{supp} \mathcal{H}^i (\mathbb{D}\, IC(Y,\mathcal{L})) < -i \quad \text{if } i > -d$$

and $IC(Y,\mathcal{L}))$ is supported on $\overline{Y}$.

The point is, that these describe the simple perverse sheaves.

~~There is also~~ If X is smooth with irreducible components X_i, define the <u>constant perverse sheaf</u> $\mathcal{C}_X$ by $\mathcal{C}_X|_{X_i} = \mathbb{C}_{X_i}[\dim_{\mathbb{C}} X_i]$ – it is self-dual.

Now consider

$$\begin{array}{ccc} Z = X_1 \times_Y X_2 & \xrightarrow{\tilde{i}} & X_1 \times X_2 \\ \tilde{p} \downarrow & \square & \downarrow p \\ Y & \xrightarrow{i} & Y \times Y_2 \end{array} \qquad \mu_i : X_i \to Y \text{ proper}$$

~~Xi~~ X_i smooth.

So $\mathcal{C}_{X_1 \times X_2} = \mathbb{C}_{X_1 \times X_2}[d_1 + d_2]$, $\omega_{X_1 \times X_2} = \mathcal{C}_{X_1 \times X_2}[d_1 + d_2]$

hence. $$H_{-j}(Z) = H^j(\omega_Z) = H^{j+d_1+d_2}(Z, \tilde{i}^!(\mathcal{C}_{X_1} \boxtimes \mathcal{C}_{X_2}))$$
$$= H^{j+d_1+d_2}(Y, \tilde{p}_* \tilde{i}^!(\mathcal{C}_{X_1} \boxtimes \mathcal{C}_{X_2}))$$
$$= H^{j+d_1+d_2}(Y, i^! p_*(\mathcal{C}_{X_1} \boxtimes \mathcal{C}_{X_2}))$$
$$= H^{j+d_1+d_2}(Y, i^!(\mu_{1*}\mathcal{C}_{X_1} \boxtimes \mu_{2*}\mathcal{C}_{X_2}))$$
$$= H^{j+d_1+d_2}(Y, \mathcal{H}om(\mu_{1*}\mathcal{C}_{X_1}, \mu_{2*}\mathcal{C}_{X_2}))$$
$$= \mathrm{Ext}^{j+d_1+d_2}_{D^b(Y)}(\mu_{1*}\mathcal{C}_{X_1}, \mu_{2*}\mathcal{C}_{X_2})$$

Neat! One proves (this is quite technical) that in case $X_1 = X_2$, so $Z \circ Z = Z$ and $H_\bullet(Z)$ is a convolution algebra, that this isom.

$$H_\bullet(Z) \cong \mathrm{Ext}^\bullet_{D^b(Y)}(\mu_*\mathcal{C}_X, \mu_*\mathcal{C}_X)$$

is actually an algebra isom. (mult. on LHS is convolution; mult. on RHS is "composition").

There is a theorem (very deep - called the Decomposition theorem) saying that proper pushforward of a simple perverse sheaf is a direct sum of degree-shifted simple perverse sheaves. In the present case, it tells us that

$$\mu_*\mathcal{C}_X = \bigoplus_{\substack{i\in\mathbb{Z}\\ \phi=(Y',\mathcal{L})}} L_\phi(i) \otimes IC_\phi[i]$$ for some (f.d.) vector spaces $L_\phi(i)$

thus $$\mathrm{Ext}^\bullet_{D^b(Y)}(\mu_*\mathcal{C}_X, \mu_*\mathcal{C}_X) = \bigoplus_{\substack{i,j,k\in\mathbb{Z}\\ \phi,\psi}} \mathrm{Hom}_{\mathbb{C}}(L_\phi(i), L_\psi(j)) \otimes \underbrace{\mathrm{Ext}^k(IC_\phi[i], IC_\psi[j])}_{\mathrm{Ext}^{k+j-i}(IC_\phi, IC_\psi)}$$

$$= \bigoplus_{\substack{k\geq 0\\ \phi,\psi}} \mathrm{Hom}(L_\phi, L_\psi) \otimes \mathrm{Ext}^k(IC_\phi, IC_\psi)$$

$$= \left(\bigoplus_{\phi} \mathrm{End}\, L_\phi\right) \oplus \underbrace{\left(\bigoplus_{\substack{\phi,\psi \\ k>0}} \mathrm{Hom}(L_\phi, L_\psi) \otimes \mathrm{Ext}^k(IC_\phi, IC_\psi)\right)}_{\text{radical!}}$$

(clearly nilpotent & quotient is semisimple).

Thus the nonzero vector spaces L_ϕ are precisely the irreducible reps of $H_\bullet(Z)$.

OK great but can we get a more representation-theoretic interpretation?

<u>Standard modules</u> Recall the setup. $a=(s,t) \in G \times \mathbb{C}^*$ s.s.

$\mathcal{N}^a = \{x \in \mathcal{N} \mid s x s^{-1} = t x\}$ ($\mathbb{C}^*$ acting through t^{-1})

$\tilde{\mathcal{N}}^a = \{(x, b) \in \mathcal{N}^a \times \mathcal{B}^a \mid x \in b\}$.

Fiber of $x \in \mathcal{N}^a$ under $\tilde{\mathcal{N}}^a \to \mathcal{N}^a$ is denoted $\mathcal{B}^s_x$.

It is {Borel subalgebras containing x & fixed by s}.

$\mathcal{B}^s_x = \tilde{\mathcal{N}}^a \times_{\mathcal{N}^a} \{x\}$, $Z^a \circ \mathcal{B}^s_x = \mathcal{B}^s_x$

$\rightsquigarrow H_\bullet(\mathcal{B}^s_x)$ is a convolution module for $H_\bullet(Z^a)$

Write $G(s,x)$ for the centralizer in G of s and x

(i.e. intersection of ~~stab~~ centralizer of s with adjoint stabilizer of x)

It acts on $\mathcal{B}^s_x$ (it is somehow the biggest subgp of G with this)

$\rightsquigarrow$ action on $H_\bullet(\mathcal{B}^s_x)$ ~~(which factors through the identity component)~~ (and the identity component actually acts trivially, so action factors through $C(s,x)$). Action of $C(s,x)$ commutes with action of $H_\bullet(Z^a)$ (easy exercise) and if x_1, x_2 are $G(s)$-conjugate then $H_\bullet(\mathcal{B}^s_{x_1}) \cong H_\bullet(\mathcal{B}^s_{x_2})$ as $H_\bullet(Z^a)$-mods. (easy also).

So for each $G(s)$-conjugacy class$_x$ in $\mathcal{N}^a$ break up $H_\bullet(\mathcal{B}^s_x)$ into:

$$\bigoplus_{\substack{\chi \text{ irrep} \\ \text{of } C(s,x)}} \underbrace{\mathrm{Hom}_{C(s,x)}(\chi, H_\bullet(\mathcal{B}^s_x))}_{H_\bullet(Z^a)\text{-rep}} \otimes \chi$$

Call this $K_{a,x,\chi}$ a <u>standard rep</u>

<u>Costandard modules</u> Give $G(s)$-orbit in $\mathcal{N}^a$ a name $\mathbb{O}$. Then you can take a transverse slice S in $\mathfrak{g}^a$ to $\mathbb{O}$ at $x \in \mathbb{O}$, and let $\tilde{S}$ be the preimage in $\tilde{\mathcal{N}}^a$. You can do it in such a way that $\tilde{S}$ retracts to $\mathcal{B}^s_x$, $\tilde{S}$ is $K(s,x)$-invariant
$\Rightarrow$ get action of $C(s,x)$ on $H_\bullet(\tilde{S})$ and $Z \circ \tilde{S} = \tilde{S}$
$\Rightarrow$ get action of $H_\bullet(Z)$ w/ commutes w/ action of $C(s,x)$

Proper $\mathcal{B}^s_x \hookrightarrow \tilde{S} \Rightarrow H_\bullet(\mathcal{B}^s_x) \to H_\bullet(\tilde{S})$ as $H_\bullet(Z)$-mods & $C(s,x)$-mods

Get <u>costandard reps</u> $K^\vee_{a,x,\chi}$ and define

$L_{a,x,\chi}$ as image of $K_{a,x,\chi}$ in $K^\vee_{a,x,\chi}$. ok!

We want to compare the $L_{a,x,\chi}$ to the L_ϕ.

<u>Aside.</u> Forgot to talk about G-equivariant version of Decomp. thm.
~~Suppose~~ $\mu: X \to Y$ proper, Y has a stratification $Y = \coprod Y_i$ s.t. $\mu_i : \mu^{-1} Y_i \to Y_i$ is topologically a fibration, then you can take the Y_i for the locally closed subspaces of Y appearing in decomp. thm.
So e.g. if μ is G-equivariant, and Y has $< \infty$ G-orbits, can take the G-orbits, ~~[illegible]~~ and can assume that the local systems are G-equivariant. Let $x \in \mathbb{O}$
Of course a local system is just a rep. of $\pi_1(\mathbb{O}, x) = \pi_1(G/G_x, x)$
have the map $\pi_1(G/G_x) \to \pi_0(G_x) = G_x / G_x^\circ$

then G-equivariance of the local system is equivalent to the rep being pulled back from a G_x/G_x°-rep.

Thus we may consider the data ϕ as an orbit $\mathbb{O}$ and a G_x/G_x°-rep, $x \in \mathbb{O}$.

So we should <u>really hope</u> that $L_{a,x,\chi} = L_\phi$. It is!

Here's some idea why.
First of all suppose M is smooth (alg. var) $\mu: M \to N$ proper, $x \in N$

$$\begin{tikzcd} M_x \arrow[r, hook, "i"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\square"] & M \arrow[d, "\mu"] \\ \{x\} \arrow[r, hook, "i_x"'] & N \end{tikzcd}$$

$\omega_M = \mathcal{C}_M[m]$.

So $H_\bullet(M_x) = H^{-\bullet}(M_x, \omega_{M_x}) = H^{-\bullet}(\mu_* \omega_{M_x})$

$= H^{-\bullet}(\mu_* i^! \mathcal{C}_M[m]) = H^{m-\bullet}(i_x^! \mu_* \mathcal{C}_M)$

similarly $H^\bullet(M_x) \cong H^{\bullet-m}(i_x^* \mu_* \mathcal{C}_M)$

(If $i: Z \subset X$ locally closed, there is a canonical map $i^! \longrightarrow i^*$.)

We get a big commutative diagram:

$$\begin{tikzcd}
H^\bullet(i_x^! \mu_* \mathcal{C}_M) \arrow[r, equal] \arrow[d] & H^\bullet(i^! \mathcal{C}_M) \arrow[r, "\simeq"] \arrow[d] & H^\bullet(\mathbb{D}_{M_x}[-m]) \arrow[r, equal] & H_{m-\bullet}(M_x) \arrow[d, "i^* \circ \text{poincaré} \circ i_*"'] \\
H^\bullet(i_x^* \mu_* \mathcal{C}_M) \arrow[r, equal] & H^\bullet(i^* \mathcal{C}_M) \arrow[r, "\simeq"] & H^\bullet(\mathbb{C}_{M_x}[m]) \arrow[r, equal] & H^{m+\bullet}(M_x)
\end{tikzcd}$$

$x \in U \subset N$, $i: M_x \hookrightarrow \tilde{U}$ homotopy equivalence so $H^\bullet(i_x^* \mu_* \mathcal{C}_M) = H^\bullet(U, \mu_* \mathcal{C}_M)$

$= H^\bullet(\tilde{U}, \mathbb{C}_M[m]) = H^{m+\bullet}(\tilde{U})$

$i^*: H^\bullet(\tilde{U}) \xrightarrow{\sim} H^\bullet(M_x)$, $\quad i_*: H_\bullet(M_x) \xrightarrow{\sim} H^{ord}_\bullet(\tilde{U})$

$\rightsquigarrow$

$$\begin{tikzcd}
H_{m-\bullet}(M_x) \arrow[r, equal] \arrow[d] & H^{m+\bullet}(\tilde{U}, \tilde{U} \setminus M_x) \arrow[r, equal] \arrow[d] & H_{m-\bullet}(M_x) \arrow[r, equal] \arrow[d] & H^{ord}_{m-\bullet}(\tilde{U}) \arrow[d] \\
H^{m+\bullet}(M_x) \arrow[r, equal] & H^{m+\bullet}(\tilde{U}) \arrow[r, dash] & H_{m-\bullet}(\tilde{U}) \arrow[r, equal] & H_{m-\bullet}(\tilde{U})
\end{tikzcd}$$

OK so $\mu: M \to N$ projective, $N = \coprod N_\alpha$ alg. stratification
$\mu: \mu^{-1}(N_\alpha) \to N_\alpha$ top. fibration

$$\rightsquigarrow \mu_* \mathcal{C}_M = \bigoplus_{k, \phi = (N_\alpha, \mathcal{L}_\alpha)} L_\phi(k) \otimes IC_\phi[k]$$

$$H^{\circ + m}(M_x) \cong \bigoplus_{k,\phi} L_\phi(k) \otimes H^{\circ + k}(i_x^* IC_\phi) \doteq \bigoplus_\phi L_\phi \otimes H^\circ(i_x^* IC_\phi)$$

If $\{x\}$ is a stratum there is only one irred. loc. system $\mathbb{C}_x$ & its H° is dim 1 $\therefore$ corresponding L_ϕ, if non-zero, appears once. — call it L_x.

<u>Proposition</u> Consider $i_x: H_\circ(M_x) \to H_\circ(\tilde{U})$ $(\doteq H^\circ(M_x))$
Then image of i_x is L_x
kernel is radical of intersection pairing on $H_\circ(M_x)$ in smooth ambient $\tilde{U}$.

Let's prove it (yay!). The diagrams we drew show that the map $H_\circ(M_x) \to H_\circ(\tilde{U})$ is just $H^\circ(i_x^! \mu_* \mathcal{C}_M) \to H^\circ(i_x^* \mu_* \mathcal{C}_M)$
By decomp. thm. such a map is a sum of maps.

$$\bigoplus_\phi L_\phi \otimes (H^\circ(i_x^! IC_\phi) \to H^\circ(i_x^* IC_\phi))$$

You can show (from def^n of IC) that $i_x^! IC_\phi \to i_x^* IC_\phi$ vanishes unless $Y = \{x\}$, where it's isom.

Thus image is L_ϕ.

We can also identify with $H_\circ^{ord}(\hat{U}) \to H_\circ(\tilde{U})$ and the intersection pairing on $H_\circ(M_x)$ gets identified with the standard $\cap$-product on $H_\circ^{ord}(\hat{U})$!

$$H^{ord}_{m+\circ}(\tilde{U}) \times H^{ord}_{m-\circ}(\hat{U}) \longrightarrow H^{ord}_{m+\circ}(\hat{U}) \times H_{m-\circ}(\hat{U}) \underset{\text{non-deg. by Poincaré duality}}{\longrightarrow} \mathbb{C}$$

$\rightsquigarrow$ radical of pairing is kernel of map. $\square$

Why is this useful? Because in the equivariant setting, we will ~~use~~ pick orbit $\mathbb{O} \subset N$, $x \in \mathbb{O}$, transversal slice S blah blah as before and observe that the orbit-stratification on N induces one on S in which $\{x\}$ is the unique point stratum.

The END!

Hint of pf of original claim $s \in S$

Y_s

$\overline{Y}_s = Y_s \amalg \mathcal{B}_\Delta$

$\overline{Y}_s \xrightarrow{p_1} \mathcal{B}$ $\quad \mathbb{P}^1$-bundle

$\Omega^1_{\overline{Y}_s/\mathcal{B}}$

$\pi_s : T^*_{\overline{Y}_s}(\mathcal{B} \times \mathcal{B}) \longrightarrow \overline{Y}_s$

smooth irred. comp. of Z

$Q_s = \pi_s^* \, \Omega^1_{\overline{Y}_s/\mathcal{B}}$

$T_s \longrightarrow -([q Q_s] + [\mathcal{O}_0])$

$Z \subset \tilde{N} \times \tilde{N} = T^*\mathcal{B} \times T^*\mathcal{B} \longrightarrow T^*\mathcal{B} \times \mathcal{B}$ is $\hookrightarrow$ so proper

$T^*\mathcal{B} \hookrightarrow T^*\mathcal{B} \times \{pt\}$ so $\longrightarrow T^*\mathcal{B}$ also proper

$Z \circ T^*\mathcal{B} \subset T^*\mathcal{B}$ is above map

$\rightsquigarrow$ convolution $K^G(Z) \otimes K^G(T^*\mathcal{B}) \to K^G(Z \circ T^*\mathcal{B}) \to K^G(T^*\mathcal{B})$

$\rightsquigarrow K^G(\mathcal{B})$ has $K^G(Z)$-alg. structure. ... ok!

There is another more concrete description of the L_ϕ (the point of the work earlier was to prove that these are *all* the irreps)

Call fibre of x under $\tilde{\mathcal{N}}^a \to \mathcal{N}^a$ $\quad \mathcal{B}_x^s = \{ b \mid x \in b, \ sb = b \}$

$$\tilde{\mathcal{N}}^a \times_{\mathcal{N}^a} \{x\}, \quad Z^a \circ \mathcal{B}_x^s = \mathcal{B}_x^s$$

$\Rightarrow H_\bullet(\mathcal{B}_x^s)$ convolution module for $H_\bullet(Z^a)$.

$G(s,x)$ acts on $\mathcal{B}_x^s$ $\quad \to$ acts on $H_\bullet(\mathcal{B}_x^s)$

$\to C(s,x)$ acts on $H_\bullet(\mathcal{B}_x^s)$

x, y $G(s)$-conj. $\Rightarrow \mathcal{B}_x^s \cong \mathcal{B}_y^s$.

$$H_\bullet(\mathcal{B}_x^s) = \bigoplus_{\substack{\chi \text{ irrep} \\ \text{of } C(s,x)}} \underbrace{\mathrm{Hom}_{C(s,x)}(\chi, H_\bullet(\mathcal{B}_x^s))}_{H_\bullet(Z^a)\text{-rep}} \otimes \chi$$

K_ϕ ::

So for each datum ϕ get standard rep $K_{a,x,\chi}$

It can be shown that L_ϕ is the ~~unique simple quo~~ head of K_ϕ. (Some more details about this in the notes)

Recall fixed $a = (s,t) \in G \times \mathbb{C}^*$ (semisimple)

and saw that $H_a \cong H_\bullet(Z^a, \mathbb{C})$ convolution algebra in BM-homology

Thus reps w/ central char a

$\longleftrightarrow$ reps of $H_\bullet(Z^a, \mathbb{C})$

$\mu : \tilde{N}^a \to \dot{N}^a$ springer map, it is proper.

$\mathcal{C}$ the constant perverse sheaf on $\tilde{N}^a$.

We saw that $H_{-j}(Z^a) = \mathrm{Ext}^{j+2d}_{D^b(N)}(\mu_* \mathcal{C}, \mu_* \mathcal{C})$

$d = \dim \tilde{N}$

Conv. Algebra structure on $H_\bullet(Z^a)$ agrees w/ Yoneda product structure on $\mathrm{Ext}^\bullet_{D^b(N)}(\mu_* \mathcal{C}, \mu_* \mathcal{C})$

But $\mu_* \mathcal{C} = \bigoplus_{\substack{i \in \mathbb{Z} \\ \phi}} L_\phi(i) \otimes IC_\phi[i]$

$\Rightarrow \mathrm{Ext}^\bullet_{D^b(N^a)}(\mu_* \mathcal{C}, \mu_* \mathcal{C}) = \left(\bigoplus_{\phi} \mathrm{End}\, L_\phi\right) \oplus \underbrace{\left(\bigoplus_{\substack{\phi, \psi \\ k > 0}} \mathrm{Hom}(L_\phi, L_\psi) \otimes \mathrm{Ext}^k(IC_\phi, IC_\psi)\right)}_{\text{radical}}$

Thus the L_ϕ are the irreps of $H_\bullet(Z^a)$.

ϕ is an $G(s)$-orbit on N^a, say $\phi = G(s) \cdot x$ and a rep. of

$G(s,x)/G(s,x)^0$.

$\overset{\shortparallel}{C(s,x)}$

__Next up__: How the isom $K^{G\times\mathbb{C}^*}(Z) \cong \mathbb{H}$ works.

Recall that $Z \subseteq T^*(\mathcal{B}\times\mathcal{B})$

is the union of the conormal bundles Z_w to the G-diagonal orbits Y_w of $\mathcal{B}\times\mathcal{B}$.

The conormal bundles to $\overline{Y_w}$ are the irred. components of Z.

We took a total order on W extending Bruhat and observed that $Z_{\le w} := \coprod_{v\le w} Z_v$ is closed - used this in conjunction with the cellular fibration lemma to show $K^{G\times\mathbb{C}^*}(Z)$ is free of rank $|W|$ over $R(T\times\mathbb{C}^*)$.

Notice $Z_1 \cong T^*\mathcal{B}$ and so $K^{G\times\mathbb{C}^*}(Z_1) = R(T\times\mathbb{C}^*)$

We have the embeddings $\overline{T^*_{Y_w}(\mathcal{B}\times\mathcal{B})} \hookrightarrow Z$

inducing $K^{G\times\mathbb{C}^*}(\underline{\qquad\qquad}) \to K^{G\times\mathbb{C}^*}(Z)$

which (by cellular fibration stuff) are injective.

Define $\Theta: S \to K^{G\times\mathbb{C}^*}(Z)$

$$\lambda \mapsto e^{-\lambda}$$
$$s \mapsto ?$$

Notice $\overline{Y_s}$ is $\mathbb{P}^1$-bundle over $\mathcal{B}$. Let Q_s be pullback of $\Omega^1_{\overline{Y_s}/\mathcal{B}}$ & stick it inside $K^{G\times\mathbb{C}^*}(Z)$; $\Theta(s) := -([qQ_s]+[\mathcal{O}_\Delta])$.

Now let $e = \sum_{w\in W} T_w$, then $\mathbb{H}\cdot e = \mathrm{Ind}_{H_W}^{\mathbb{H}} \varepsilon$.

$$\cong R(T\times\mathbb{C}^*)$$

Also note that

$$\begin{array}{c} \subseteq T^*\mathcal{B}\times T^*\mathcal{B} \\ Z \circ T^*\mathcal{B} = T^*\mathcal{B} \\ \| \\ T^*\mathcal{B}\times\{pt\} \end{array}$$

$$\Rightarrow K^{G\times\mathbb{C}^*}(T^*\mathcal{B}) \cong R(T\times\mathbb{C}^*) \text{ is module}$$

Claim the maps

$$\begin{tikzcd}
K^{G\times\mathbb{C}^*}(T^*\mathcal{B}) \arrow[rr] & & \mathrm{End}_{\mathbb{Z}[q,q^{-1}]}(R(T\times\mathbb{C}^*)) \\
& \mathbb{H} \arrow[ur] & \\
S \arrow[uu] \arrow[ur] & &
\end{tikzcd}$$

are injective. & the diagram commutes

From this the isom. follows.