Lect 3 - Hilbert Schemes

X - alg. vty, $\quad S^nX := \overbrace{X\times\cdots\times X}^{n\text{-times}}/S_n \quad$ space of unordered n-tuples of points in X

Ex 1) $S^n\mathbb{C} = \mathbb{C}^n$

$$\{z_1,\dots,z_n\} \longmapsto f(z) = (z-z_1)\cdots(z-z_n) = z^n + a_1z^{n-1} + \dots + a_n$$

2) $C = \underset{\uparrow\ \text{smooth}}{\text{alg.}}$ curve/Riemann surface $\Rightarrow$ S^nC is smooth

But this is not true for higher dimensions!

Example $S^n\mathbb{C}^2$ is not smooth! Even $S^2\mathbb{C}^2$ is not smooth (exercise).

The Hilbert scheme of n points in $\mathbb{C}^2$ is the moduli space of codim n ideals $I \subseteq \mathbb{C}[x,y]$ Notation $\mathrm{Hilb}^n\mathbb{C}^2$

$\mathrm{Hilb}^n\mathbb{C}^2 \ni I \rightsquigarrow$ subscheme of $\mathbb{C}^2$ of length n $\xrightarrow[\substack{\text{support of}\\ \text{this scheme}\\ \text{w/ multiplicities}}]{}$ $S^n\mathbb{C}^2$

Example $n=1$, $\mathrm{Hilb}^1\mathbb{C}^2 = \mathbb{C}^2 = S^1\mathbb{C}^2$

$n=2$. The support can be two distinct points $\cdot p_1$ $\cdot p_2$

or $2\,\text{pts}$, $I = (\underset{\uparrow\ \text{linear}}{l(x,y)}, x^2, y^2, xy)$ $l(x,y)=0$

So we get a point in $\mathbb{C}^2$ and a direction in $\mathbb{CP}^1$.

Fact $\mathrm{Hilb}^2\mathbb{C}^2 =$ blow-up of $S^2\mathbb{C}^2$ along the diagonal

Thm (Fogarty) $Hilb^n \mathbb{C}^2$ is smooth of dimension $2n$, and it is a resoln of singularities for $S^n\mathbb{C}^2$.

Before we prove this, we'll need:

ADHM construction

Prop
$$Hilb^n\mathbb{C}^2 = \left\{ (X,Y,v) \;\middle|\; \begin{array}{ll} X,Y \in Mat_{n\times n}(\mathbb{C}) & [X,Y]=0 \\ v \in \mathbb{C}^n & X^aY^bv \text{ span } \mathbb{C}^n \ \left(\begin{smallmatrix}\text{stability} \\ \text{condition}\end{smallmatrix}\right) \end{array} \right\} \Big/ G$$

where $G = GL_n$ acts by $g(X,Y,v) = (gXg^{-1}, gYg^{-1}, gv)$

Pf $Hilb^n\mathbb{C}^2 \ni I \implies \mathbb{C}[x,y]/I \cong \mathbb{C}^n$ We set $v = I \in \mathbb{C}[x,y]/I$
$X =$ mult. by x
$Y =$ mult. by y

Now, let (X,Y,v) satisfy the required condition. Then set

$$I := \{ f(x,y) \mid f(X,Y)v = 0 \}$$

It's easy to see that this only depends on the GL_n-orbit of (X,Y,v) □

Pf of Fogarty's thm The G-action is free, so it is enough to prove smoothness of the pre-quotient

$$\Phi : Mat(n) \times Mat(n) \longrightarrow Mat(n)$$
$$(X,Y) \longmapsto [X,Y]$$

$d\Phi(A,B) = [A,Y] + [X,B]$ Thus,

$$(\mathrm{Im}\, d\Phi)^{\perp} = \{C \mid \mathrm{Tr}(C([A,Y]+[X,B])) = 0 \ \forall A,B\} \nearrow \begin{cases} [C,Y]=0 \\ [C,X]=0 \end{cases}$$

If X,Y have common cyclic vector $v \Rightarrow C$ is determined by Cv

$\Rightarrow \dim((\mathrm{Im}(d\Phi))^{\perp}) = n \Rightarrow$ use implicit function thm to finish the proof. $\square$

Friends & Relatives of Hilb^n

1) $\mathrm{Hilb}^n(\mathbb{C}^2,0) := \pi^{-1}(n\{0\}) \qquad \pi: \mathrm{Hilb}^n\mathbb{C}^2 \to S^n\mathbb{C}^2$

Facts $\mathrm{Hilb}^n(\mathbb{C}^2,0)$ is
- reduced
- irreducible
- of dim $n-1$
- Cohen-Macaulay
- Singular

[Briançon-Haiman]

Examples $n=1$, $\mathrm{Hilb}^1(\mathbb{C}^2,0) = \{pt\}$

$n=2$, $\mathrm{Hilb}^2(\mathbb{C}^2,0) = \mathbb{CP}^1$

$n=3$, $\mathrm{Hilb}^3(\mathbb{C}^2,0) =$ projective cone over the twisted cubic in $\mathbb{CP}^3$ ($\Rightarrow$ singular)

ADHM description: same, but X,Y are nilpotent.

2) $\mathrm{Hilb}^n(\mathbb{C}^2,\mathbb{C}) = \pi^{-1}(S^n\{y=0\})$ [ideals set-theoretically supported at $\{y=0\}$]

Facts $\dim = n$, singular. **Q**: is it a complete intersection? (the char. poly. of Y should give equations...)

ADHM description: Take Y nilpotent

Flag Hilbert Scheme

$\{\mathbb{C}[x,y] \supseteq I_1 \supseteq \cdots \supseteq I_n\} =: FHilb^n(\mathbb{C}^2)$

I_k = ideal in $\mathbb{C}[x,y]$ of codim k.

Different versions: We can also consider $FHilb^n(\mathbb{C}^2, 0)$
$FHilb^n(\mathbb{C}^2, \{y=0\})$

ADHM description

$(X,Y,v)/B$, X,Y = $\overset{\text{commuting}}{\downarrow}$ lower triangular matrices (preserve the flag)
$v \in \mathbb{C}^n$ w/ stability condition
B = group of invertible lower tr. mat. acting as before

Flag: $\mathbb{C}[x,y]/I_1 \twoheadleftarrow \mathbb{C}[x,y]/I_2 \twoheadleftarrow \cdots \twoheadleftarrow \mathbb{C}[x,y]/I_n$

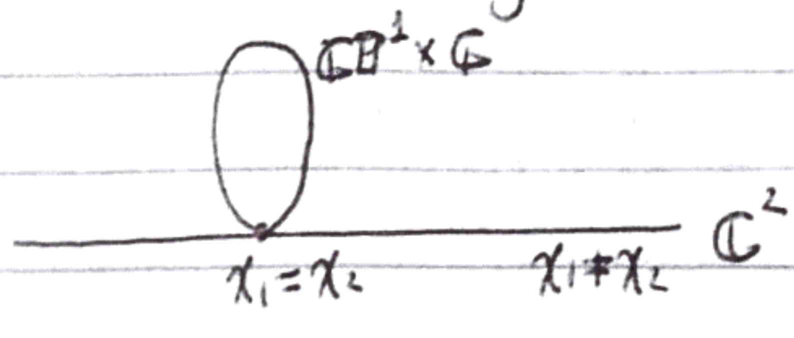
Example $FHilb^2(\mathbb{C}^2, \{y=0\})$

$$X = \begin{pmatrix} x_1 & 0 \\ z & x_2 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}, \quad v = (1,0)$$

$$[X,Y] = \begin{pmatrix} 0 & 0 \\ w(x_1 - x_2) & 0 \end{pmatrix}$$

Case 1 $x_1 \neq x_2 \Rightarrow w = 0$.
Stability cond $\Rightarrow z \neq 0$. May assume $z = 1$. So we get 

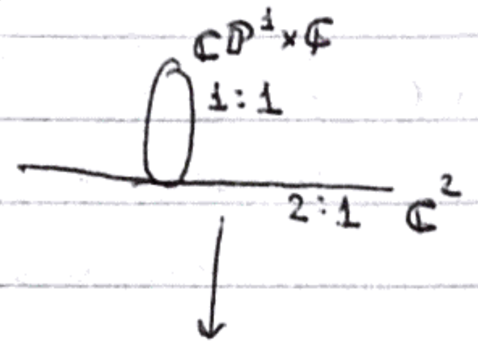
Case 2 $x_1 = x_2$ Then we get a $\mathbb{CP}^1$ w/ coordinate $[z:w]$



<u>Rmk</u> We always have a projection $FHilb^n(\mathbb{C}^2) \longrightarrow Hilb^n(\mathbb{C}^2)$

$$\{I_1 \supseteq \cdots \supseteq I_n\} \longmapsto I_n$$

So we get



$Hilb^2(\mathbb{C}^2, \{y=0\})$

<u>Note</u> $FHilb^n(\mathbb{C}^2)$

$$\downarrow$$

$$(\mathbb{C}^2)^n$$

<u>Bad news</u> $FHilb^n$ is singular for large n, reducible and dimension $\gg$ expected dimension

<u>Examples</u> $FHilb^2(\mathbb{C}^2, 0) = \mathbb{CP}^1$

$FHilb^3(\mathbb{C}^2, 0) =$ cubic Hirzebruch surface

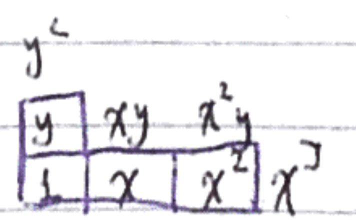
$=$ smooth resoln. of $Hilb^3(\mathbb{C}^2, 0)$

<u>Torus actions</u> $\mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathbb{C}^2$ by dilating coordinates

$$(z_1, z_2) \longmapsto (q z_1, t z_2)$$

This action lifts to $Hilb^n$, $FHilb^n$

<u>Fixed points</u> An ideal $I \in Hilb^n$ is fixed by $\mathbb{C}^* \times \mathbb{C}^*$ iff it is monomial.
These correspond to Young diagrams



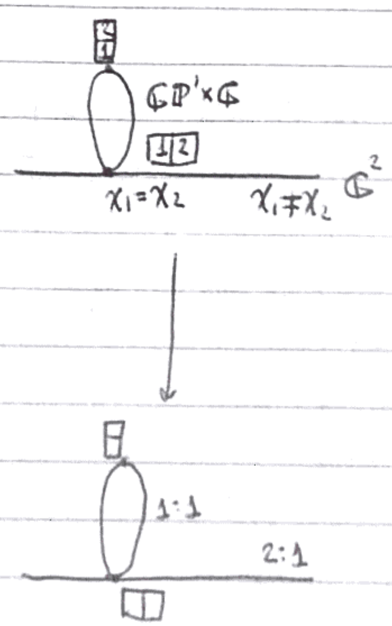
$\longmapsto I = \langle y^2, xy, x^2y, x^3 \rangle$

So, fixed pts. of $\mathbb{C}^* \times \mathbb{C}^*$ on $Hilb^n(\mathbb{C}^2, *) \longleftrightarrow$ Young diagrams

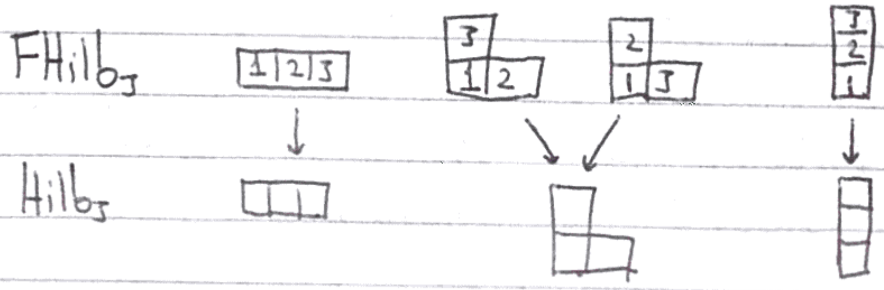
Fixed points of $\mathbb{C}^* \times \mathbb{C}^*$ on $FHilb^n(\mathbb{C}^2, A) \longleftrightarrow$ Standard Young tableaux
[need to keep track on how we add boxes]

(Recall that Standard Young tableaux = filling of a Young diagram with numbers $1, \dots, n$ that is increasing in rows and columns)

So, in the previous example ($FHilb(\mathbb{C}^2, \text{line})$)



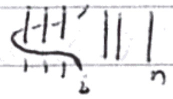
For $n = 3$,



Lecture 4

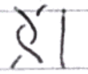
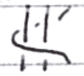
Main conjecture

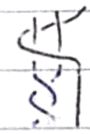
We need some notation first.

$L_i =$ 

Lemma The L_i commute with each other in Br_n.

For example, in 3 strands

$L_2 = =$  $L_3 =$ 

$L_2 L_3 =$  $L_3 L_2 =$  slide the L_2 up using braid relations

So we get

$C_n := \langle L_1, \dots, L_n \rangle =$ comm. subalgebra in H_n generated by L_i

$\mathcal{C}_n := \langle L_1, \dots, L_n \rangle =$ subcategory of $\underbrace{\text{homotopy caty of SBim}}_{K^{b}?(SBim)}$ generated by L_i

[Recall from yesterday that $L_i \to$ complex of Soergel bimodules]

Conjecture There exists a pair of adjoint functors

$\boxed{? \neq ?'}$ $\quad K^{?}_{q\text{-}g + \text{hom.gr.}}(SBim) \underset{i^*}{\overset{i_*}{\rightleftarrows}} D^{b?'}_{\mathbb{C}^* \times \mathbb{C}^*} Coh(FH_n)$, $\quad FH_n := FHilb_n(\mathbb{C}^2, \text{line})$

with the following properties

1) i^* is monoidal and i_* is not.
2) $i_* \mathbb{1} = \mathcal{O}_{FH_n}$; $\mathbb{1} =$ identity bimodule

and $i^*\mathcal{L}_i = L_i$

3) $i_* L_i = \mathcal{L}_i$, where $\mathcal{L}_i = I_i / I_{i+1}$ - tautological line bundle on FH_n.

4) Projection formula

$$i_*(A \otimes i^* B) = i_*(A) \otimes B$$

[Cor $i_*(A \otimes L_1^{a_1} \cdots L_n^{a_n}) = i_*(A) \otimes \mathcal{L}_1^{a_1} \cdots \mathcal{L}_n^{a_n}$ $\forall A \in K^?(SBim)$, $\forall a_i$]

5) i_*, i^* are equivalences between $\mathcal{C}_n$ and $D^{?'}_{\mathbb{C}^\times \times \mathbb{C}^\times}(FH_n)$

$D^{?'}_{\mathbb{C}^\times \times \mathbb{C}^\times} Coh(FH_n)$

Corollary For every braid β, there is an object $\beta \to$ complex of Soergel bimodules $\to i_*(\beta)$

$$KhR(\beta) = \mathrm{Hom}_{K(SBim)}(\mathbb{1}, \beta) = \mathrm{Hom}_{DCoh(FH_n)}(\mathcal{O}_{FH_n}, i_*(\beta))$$
$$= H^*(FH_n, i_*\beta)$$

$\|$

HOMFLY homology
q-grading - from SBim grading
t-grading - homological grading
+ grading b/c we take RHom [we don't take total complex!]

graded module over $\mathbb{C}[x_1, \dots, x_n]$

also a module over $\mathbb{C}[x_1, \dots, x_n]$

Proof for $n=2$ Can assume $x_1 + x_2 = 0$.

$$R = \mathbb{C}[x,y]/x=y, \quad B = \mathbb{C}[x,y]/x^2=y^2$$

$$B \otimes B = B \oplus qB$$

$$\times = [B \to R] \quad L_2 = \times\!\!\!\times = [B \to R]^2 = [B^2 \to \begin{smallmatrix} B \\ \oplus \\ B \end{smallmatrix} \to R]$$

$(B^2 = B \oplus qB)$

$$= [qB \to B \to R]$$

Rmk $(Y_1)^n = [\overbrace{B \to B \to \cdots \to B}^{n} \to R]$ w/ grading shifts!

$$\mathrm{Hom}(\mathbb{1}, L_2) = \mathrm{Hom}(R, L_2)$$

$$\begin{tikzcd} qB \arrow[r] & B \arrow[r] & R \\ & R \arrow[ul, "z"] \arrow[ur, "w"'] & \end{tikzcd}$$

So we get $z, w : \mathbb{1} \to L_2$

FACT $wx = 0$ (or $w(x_1 - x_2) = 0$ in general)

Now, $KhR(L_2) = \mathrm{Span}_R(z, w) / w(x_1 - x_2) = 0 = \langle w, z, zx, zx^2, \ldots \rangle$

Now let

$$A := \bigoplus_{k=0}^{\infty} \mathrm{Hom}(\mathbb{1}, L_2^k) \qquad \text{graded algebra (by } k\text{)}$$

$$= \mathbb{C}[\underbrace{x_1, x_2}_{\deg 0}, \overbrace{z, w}^{\deg 1}] / w(x_1 - x_2) = 0$$

$M \in K^b(SBim_2)$

$\bigoplus_{k=0}^{\infty} \mathrm{Hom}(\mathbb{1}, M \otimes L_2^k)$ - graded A-module

$= \text{sheaf on } \mathrm{Proj} A = FH_2$

Recall that $FH_2 = \left\{ \begin{pmatrix} x_1 & 0 \\ z & x_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix} \right\}$

And define $i_* M := \bigoplus_{k=0}^{\infty} \mathrm{Hom}(\mathbb{1}, M \otimes L_2^k)$

Note that by definition $i_* L_2 = \mathcal{O}(1)$

Remark Intuitively, if $K^?(SBim)$ were $D^b Coh(Y)$, then i^*, i_* would follow from the existence of a birational morphism $Y \xrightarrow{i} FH_n$.

(10)

$i\mathbb{1} = \mathbb{1}$ $\quad i_* \mathbb{1} = \mathcal{O}_{FH_2}$ follows trivially by construction

$i_*(Y_1) = \mathcal{O}_{FHilb^2(\mathbb{C}^2,0)}$ $[FHilb^2(\mathbb{C}^2,0) = \mathbb{CP}^1 \subseteq FH_2$

$i_*((Y_1)^{2k}) = \mathcal{O}(k)$ on FH_2

$i_*((Y_1)^{2k+1}) = \mathcal{O}(k) \otimes \mathcal{O}_{FHilb^2(\mathbb{C}^2,0)}$

infinite complex!

$$\mathcal{O} \xleftarrow{x_1 - x_2} \mathcal{O} \xleftarrow{w} \mathcal{O}(-1) \xleftarrow{x_1 - x_2} \cdots$$