Coherent sheaves on elliptic curves.

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Abstract

We describe the abelian category of coherent sheaves on an elliptic curve, and construct an action of a central extension of $\text{SL}_2(\mathbb{Z})$ on the derived category.

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1 Coherent sheaves on elliptic curve

Definition 1.1. An elliptic curve over a field $k$ is a nonsingular projective algebraic curve of genus 1 over $k$ with a fixed $k$-rational point.

Remark 1.2. If the characteristic of $k$ is neither 2 nor 3, an elliptic curve can be alternately defined as the subvariety of $\mathbb{P}^2_k$ defined by an equation $y^2z = x^3 - pxz^2 - qz^3$, where $p, q \in k$, and the polynomial $x^3 - pxz^2 - qz^3$ is square-free. In this case, the fixed point is $(0 : 1 : 0)$.

Remark 1.3. Over the field of complex numbers, there is even a simpler description. An elliptic curve is precisely a quotient $\mathbb{C}/\Lambda$ of $\mathbb{C}$ by a nondegenerate lattice $\Lambda \subset \mathbb{C}$ of rank 2.

Remark 1.4. Any elliptic curve carries a structure of a group, with the fixed point being the identity.
Fix an elliptic curve $X$ over a field $k$. We do not assume that $k$ is algebraically closed, since the main example is the finite field $\mathbb{F}_q$.

Recall that a coherent sheaf $F$ on $X$ is a sheaf of modules over $O$ such that for every open affine $U \subset X$ the restriction $F|_U$ is isomorphic to $\hat{N}$ for some finitely generated $O(U)$-module $N$.

Example 1.5. The structure sheaf $O$ is indeed a coherent sheaf. Also, one can consider the ideal sheaf $m_x = O(-x)$ corresponding to a closed point $x \in X$. Then the cokernel of the inclusion $O(-x) \to O$ is the so called skyscraper sheaf $O_x$, which is coherent as well.

Theorem 1.6. Coherent sheaves on $X$ form an abelian category $\text{Coh}(X)$.

Theorem 1.7 (Global version of Serre theorem). Any coherent sheaf $F$ on a smooth projective variety of dimension $n$ over a field $k$ admits a resolution $F_n \to F_{n-1} \to \ldots \to F_1 \to F_0$ where each $F_i$ is finitely generated and locally free ($\simeq$ vector bundle).

Theorem 1.8 (Grothendieck’s finiteness theorem). Any coherent sheaf $F$ on a smooth projective variety of dimension $n$ over a field $k$ has finite dimensional cohomologies over $k$.

Corollary 1.9. For any coherent sheaves $F$ and $G$ the space $\text{Hom}(F,G)$ has finite dimension over $k$, since $\text{Hom}(F,G) = \Gamma(\text{Hom}(F,G), X) = H^0(\text{Hom}(F,G), X)$.

Theorem 1.10 (Grothendieck’s vanishing theorem). Any coherent sheaf $F$ on a smooth projective variety of dimension $n$ over a field $k$ has no $i$-th cohomologies for $i > n$.

Definition 1.11. An abelian category $\mathcal{C}$ is called hereditary if $\text{Ext}^2(\cdot, \cdot) = 0$.

Corollary 1.12. The category $\text{Coh}(X)$ is hereditary.

2 (Semi)stable sheaves

To a coherent sheaf we can associate two numbers, the Euler characteristic $\chi(F)$ and the rank $\text{rk}(F)$.

Definition 2.1. The Euler characteristic $\chi(F)$ is the alternating sum $\sum_{i} (-1)^i \dim_k H^i(F, X)$. In our case, it is equal to $\dim_k H^0(F, X) - \dim_k H^1(F, X)$.

Definition 2.2. The rank $\text{rk}(F)$ is the dimension of the stalk $F_\xi$ of $F$ at a generic point $\xi$ of $X$ over the residue field. It is independent of $\xi$.

Example 2.3. We have $\chi(O) = 0$, $\text{rk}(O) = 1$, $\chi(O_x) = 1$, $\text{rk}(O_x) = 0$.

Proposition 2.4. Given a short exact sequence $0 \to F' \to F \to F'' \to 0$, we have $\chi(F) = \chi(F') + \chi(F'')$ and $\text{rk}(F) = \text{rk}(F') + \text{rk}(F'')$.

Definition 2.5. The slope $\mu(F)$ of a nontrivial coherent sheaf $F$ is the quotient $\chi(F)/\text{rk}(F)$. In the case $\text{rk}(F) = 0$ we set $\mu(F) = \infty$.

Lemma 2.6. Given a short exact sequence $0 \to F' \to F \to F'' \to 0$, we have three options:
\[ \mu(F') < \mu(F) < \mu(F''); \]
\[ \mu(F') = \mu(F) = \mu(F''); \]
\[ \mu(F') > \mu(F) > \mu(F''). \]

**Proof.** We have
\[
\begin{align*}
\mu(F') &= \frac{\chi(F')}{\text{rk}(F')}, \\
\mu(F'') &= \frac{\chi(F'')}{\text{rk}(F'')}, \\
\mu(F) &= \frac{\chi(F)}{\text{rk}(F)} = \frac{\chi(F') + \chi(F'')}{\text{rk}(F') + \text{rk}(F'')}.
\end{align*}
\]
Since both \( \text{rk}(F') \) and \( \text{rk}(F'') \) are nonnegative, we indeed get the lemma.

**Definition 2.7.** A coherent sheaf \( F \) is called **stable** (resp. **semistable**) if for any nontrivial short exact sequence \( 0 \to F' \to F \to F'' \to 0 \) we have \( \mu(F') < \mu(F) \) (resp. \( \mu(F') \leq \mu(F) \)).

General theory gives us the following

**Theorem 2.8** ([1] Harder-Narasimhan filtration). For a coherent sheaf \( F \), there is a unique filtration
\[
0 = F_0 \subset F_1 \subset \ldots \subset F_n \subset F_{n+1} = F
\]
such that all \( A_i = F_{i+1}/F_i \) are semistable and \( \mu(A_i) > \mu(A_{i+1}) \) for each \( i \).

In our case, we can derive much stronger proposition. Before stating it, note two useful statements.

**Proposition 2.9.** If \( F \) and \( G \) are semistable sheaves, and \( \mu(F) > \mu(G) \), then \( \text{Hom}(F, G) = 0 \).

**Proof.** Suppose we have a nontrivial map \( f : F \to G \). Then \( \mu(F) \leq \mu(F/\ker f) = \mu(\text{im } f) \leq \mu(G) \). Contradiction.

Another property of \( \text{Coh}(X) \) we will need is

**Proposition 2.10** (Calabi-Yau property). For any two coherent sheaves \( F \) and \( G \), there is an isomorphism \( \text{Hom}(F, G) \simeq \text{Ext}^1(G, F)^* \).

**Proof.** From Remark 1.4 we know that the canonical bundle \( K \) is trivial, \( K \simeq \mathcal{O} \). Also by Serre duality we get
\[
\text{Hom}(F, G) = \text{Ext}^0(F, G) \simeq \text{Ext}^1(G, F \otimes K)^* = \text{Ext}^1(G, F)^*.
\]

We are ready to prove
Theorem 2.11. Any nontrivial coherent sheaf is a direct sum of indecomposable semistable sheaves.

Proof. We only need to prove that any indecomposable sheaf is semistable. Suppose some indecomposable sheaf $F$ is not semistable. Then the Harder-Narasimhan filtration of $F$ is nontrivial. Consider only the case of length 1 filtration, it captures the main idea. So, we have a short exact sequence $0 \to F' \to F \to F'' \to 0$, where both $F'$ and $F''$ are semistable, and $\mu(F') > \mu(F'')$. By Proposition 2.9 we get $\text{Hom}(F', F'') = 0$. By Proposition 2.10 we obtain $\text{Ext}^1(F'', F') = \text{Hom}(F', F'')^* = 0$. Therefore the exact sequence splits, contradiction with the assumption that $F$ is indecomposable.

Definition 2.12. Denote the full subcategory of semistable coherent sheaves on $X$ of slope $\mu$ by $C_\mu$.

Proposition 2.13. The category $C_\mu$ is abelian, artinian, and closed under extensions. The simple objects in $C_\mu$ are stable sheaves of slope $\mu$.

Corollary 2.14. $\text{Coh}(X)$ is the direct sum of all $C_\mu$ (on the level of objects).

3 Euler form

Since $\text{rk}$ and $\chi$ are well defined on $K_0(\text{Coh}(X))$, we can consider

Definition 3.1. The Euler form $\langle \mathcal{F}, \mathcal{G} \rangle$ of two elements $\mathcal{F}, \mathcal{G} \in K_0(\text{Coh}(X))$ is equal to $\dim \text{Hom}(\mathcal{F}, \mathcal{G}) - \dim \text{Ext}^1(\mathcal{F}, \mathcal{G})$.

Proposition 3.2. We have $\langle \mathcal{F}, \mathcal{G} \rangle = \text{rk}(\mathcal{F})\chi(\mathcal{G}) - \chi(\mathcal{F})\text{rk}(\mathcal{G})$.

Proof. First notice that the RHS only depends on the classes of $\mathcal{F}$ and $\mathcal{G}$ in the Grothendieck group $K_0(\text{Coh}(X))$. Therefore it is sufficient to check the equality for some generators of the Grothendieck group, for example, for locally free sheaves. If $\mathcal{F}$ is locally free, the LHS reduces to $\chi(\mathcal{F}^\vee \otimes \mathcal{G})$. Note that in the case of elliptic curve, the Hirzebruch-Riemann-Roch theorem gives us that $\chi(\mathcal{E}) = \deg(\mathcal{E})$ for any coherent sheaf $\mathcal{E}$. Applying it here, we get

$$LHS = \chi(\mathcal{F}^\vee \otimes \mathcal{G}) = \deg(\mathcal{F}^\vee \otimes \mathcal{G}) = \text{rk}(\mathcal{F})\deg(\mathcal{G}) - \deg(\mathcal{F})\text{rk}(\mathcal{G}) = \text{rk}(\mathcal{F})\chi(\mathcal{G}) - \chi(\mathcal{F})\text{rk}(\mathcal{G}) = RHS.$$ 

Definition 3.3. The charge map is $Z = (\text{rk}, \chi): K_0(\text{Coh}(X)) \to \mathbb{Z}^2$.

It is surjective, since we have both $(1, 0)$ and $(0, 1)$ in the image. We have a canonical nondegenerate volume form on $\mathbb{Z}^2$, $\langle (a, b), (c, d) \rangle = ad - bc$, and it is equal to the push-forward of the Euler form.

Proposition 3.4. The kernel of the Euler form coincides with the kernel of $Z$, equivalently, $K_0(\text{Coh}(X))/\ker \langle , \rangle \cong \mathbb{Z}^2$. 

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Also we can now write some relations between different $C_\mu$ and $C_\mu'$.

**Proposition 3.5.** Suppose $\mathcal{F}$ and $\mathcal{F}'$ are indecomposable, and $Z(\mathcal{F}) = (r, \chi)$, $Z(\mathcal{F}') = (r', \chi')$.

- If $\chi/r > \chi'/r'$, then $\text{Hom}(\mathcal{F}, \mathcal{F}') = 0$, $\dim \text{Ext}^1(\mathcal{F}, \mathcal{F}') = \chi r' - \chi' r$;
- If $\chi/r < \chi'/r'$, then $\dim \text{Hom}(\mathcal{F}, \mathcal{F}') = \chi' r - \chi r'$, $\text{Ext}^1(\mathcal{F}, \mathcal{F}') = 0$.

**Proof.** By Proposition 2.10 and Proposition 2.9 we know that

- if $\chi/r > \chi'/r'$, then $\text{Hom}(\mathcal{F}, \mathcal{F}') = 0$;
- if $\chi/r < \chi'/r'$, then $\text{Ext}^1(\mathcal{F}, \mathcal{F}') = 0$.

Proposition 3.2 concludes the proof. □

### 4 Derived category of coherent sheaves

Let us show that Corollary 1.12 implies a neat description of the derived category $D^b(\text{Coh}(X))$ of bounded complexes of coherent sheaves on $X$.

**Theorem 4.1.** Suppose $\mathcal{C}$ is a hereditary abelian category. Then any object $L \in D^b(\mathcal{C})$ is isomorphic to the sum of its cohomologies, i.e. $L = \bigoplus_i H^i L[-i]$.

**Proof.** Let $L$ be a complex $\ldots \to L_i \to L_{i+1} \to \ldots$. Fix any $i$. We have a short exact sequence $0 \to \ker d^{-1} \to L^{-1} \to \text{im } d^{-1} \to 0$. Apply $R\text{Hom}(H^i L, -)$. This gives rise to an exact sequence $\text{Ext}^1(H^i L, L^{-1}) \to \text{Ext}^1(H^i L, \text{im } d^{-1}) \to \text{Ext}^2(H^i L, \ker d^{-1})$. Since $\text{Coh}(X)$ is hereditary, we obtain a surjection from $\text{Ext}^1(H^i L, L^{-1})$ to $\text{Ext}^1(H^i L, \text{im } d^{-1})$. In particular, there exists $M^i$ such that the following diagram commutes

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & L^{-1} & \longrightarrow & M^i & \longrightarrow & H^i L & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{im } d^{-1} & \longrightarrow & \ker d^i & \longrightarrow & H^i L & \longrightarrow & 0
\end{array}
\]

Then the following morphism

\[
\begin{array}{ccccccccc}
& \ldots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^i L & \longrightarrow & 0 & \longrightarrow & \ldots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& \ldots & \longrightarrow & 0 & \longrightarrow & L^{-1} & \longrightarrow & M^i & \longrightarrow & 0 & \longrightarrow & \ldots
\end{array}
\]

of complexes is a quasi-isomorphism. If we compose its inverse with the morphism

\[
\begin{array}{ccccccccc}
& \ldots & \longrightarrow & 0 & \longrightarrow & L^{-1} & \longrightarrow & M^i & \longrightarrow & 0 & \longrightarrow & \ldots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& \ldots & \longrightarrow & L^{-2} & \longrightarrow & L^{-1} & \longrightarrow & L^i & \longrightarrow & L^{i+1} & \longrightarrow & \ldots
\end{array}
\]

we get a morphism $H^i L[-i] \to L$ in $D^b(\text{Coh}(X))$ which is isomorphism in the $i$-th cohomology, and zero elsewhere. Therefore, if we sum up all this morphisms, we obtain an isomorphism $\bigoplus_i H^i L[-i] \to L$. □
Corollary 4.2. The derived category $D^b(\text{Coh}(X))$ is the direct sum of $\mathbb{Z}$ copies of $\text{Coh}(X)$, a sheaf $\mathcal{F}$ in the $i$-th copy goes to $\mathcal{F}$.

Since $K_0(D^b(\text{Coh}(X))) = K_0(\text{Coh}(X))$, $Z$ is defined on $K_0(D^b(\text{Coh}(X)))$ as well. Note that $Z(\mathcal{F}[i]) = (-1)^i Z(\mathcal{F})$.

Remark 4.3. The corollary works for any smooth projective curve $X$. Another example of a hereditary category is the category of representations of a quiver.

5 SL$_2(\mathbb{Z})$ action

Proposition 3.2 suggests to define $\langle L, M \rangle = \sum (-1)^i \dim \text{Hom}(L, M[i])$ for any two objects $L, M \in D^b(\text{Coh}(X))$. Therefore the Euler form is preserved by any autoequivalence of $D^b(\text{Coh}(X))$. In other words, any autoequivalence $f \in \text{Aut}(D^b(\text{Coh}(X)))$ gives a corresponding automorphism of $\mathbb{Z}^2$ preserving the volume form, i.e. gives an element $\pi(f) \in \text{SL}_2(\mathbb{Z})$.

Definition 5.1. Say that an object $\mathcal{E} \in D^b(\text{Coh}(X))$ is spherical if $\text{Hom}(\mathcal{E}, \mathcal{E}) = k$ (and consequently $\text{Hom}(\mathcal{E}, \mathcal{E}[1]) = k$).

Example 5.2. The structure sheaf $\mathcal{O}$ and the skyscraper sheaf at a rational $k$-point are spherical.

Definition 5.3. A Fourier-Mukai transform with a kernel $\mathcal{L} \in D^b(\text{Coh}(X \times Y))$ is a functor $\Phi_\mathcal{L}: D^b(\text{Coh}(X)) \to D^b(\text{Coh}(Y))$ which sends an object $\mathcal{F} \in D^b(\text{Coh}(X))$ to $R\pi_{2*}(\pi_1^*\mathcal{F} \otimes^L \mathcal{L})$, where $\pi_1: X \times Y \to X$ and $\pi_2: X \times Y \to Y$ are the natural projections.

Definition 5.4. For a spherical object $\mathcal{E} \in D^b(\text{Coh}(X))$, which is a complex of locally free sheaves, we can define a twist functor $T_\mathcal{E}: D^b(\text{Coh}(X)) \to D^b(\text{Coh}(X))$ to be equal to a Fourier-Mukai transform with the kernel $\text{cone}(\mathcal{E}^\vee \boxtimes \mathcal{E} \to \mathcal{O}_\Delta) \in D^b(\text{Coh}(X \times Y))$.

Theorem 5.5 ([2]). For a spherical object $\mathcal{E} \in D^b(\text{Coh}(X))$ the twist functor $T_\mathcal{E}$ is an exact equivalence which sends an object $\mathcal{F}$ to $\text{cone}(\text{RHom}(\mathcal{E}, \mathcal{F}) \otimes^L \mathcal{E} \boxtimes \mathcal{F})$.

Remark 5.6. The evaluation works by applying $\text{ev}: \text{Ext}^i(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E}[-i] \to \mathcal{F}$ on each grading.

Let us see how $T_\mathcal{E}$ acts on Grothendieck group.

Proposition 5.7. The action of $T_\mathcal{E}$ on $K_0(D^b(\text{Coh}(X)))$ is given by $[\mathcal{F}] \mapsto [\mathcal{F}] - \langle \mathcal{E}, \mathcal{F} \rangle [\mathcal{F}]$.

Proof. Indeed, $[T_\mathcal{E}(\mathcal{F})] = [\mathcal{F}] - [\text{RHom}(\mathcal{E}, \mathcal{F}) \otimes^L \mathcal{E}] = [\mathcal{F}] - \langle \mathcal{E}, \mathcal{F} \rangle [\mathcal{F}]$. □

Corollary 5.8. $\pi(T_\mathcal{O}) = \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right)$, $\pi(T_{\mathcal{O}_x}) = \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right)$.

Proof. Since $\mathbb{Z}^2$ are generated by the charges of $\mathcal{O}$ and $\mathcal{O}_x$, we can check this on $\mathcal{O}$ and $\mathcal{O}_x$ only.

$T_\mathcal{O}([\mathcal{O}]) = [\mathcal{O}] - \langle \mathcal{O}, \mathcal{O} \rangle [\mathcal{O}] = [\mathcal{O}]$,

$T_\mathcal{O}([\mathcal{O}_x]) = [\mathcal{O}_x] - \langle \mathcal{O}_x, \mathcal{O}_x \rangle [\mathcal{O}_x] = [\mathcal{O}_x] - [\mathcal{O}]$,

$T_{\mathcal{O}_x}([\mathcal{O}]) = [\mathcal{O}] - \langle \mathcal{O}_x, \mathcal{O} \rangle [\mathcal{O}_x] = [\mathcal{O}] + [\mathcal{O}_x]$,

$T_{\mathcal{O}_x}([\mathcal{O}_x]) = [\mathcal{O}_x] - \langle \mathcal{O}_x, \mathcal{O}_x \rangle [\mathcal{O}_x] = [\mathcal{O}_x]$. □
Proposition 5.9. $T_{O_x}$ is in fact just the tensor product with $O(x)$.

Proof. The formula for the adjoint of a Fourier-Mukai transform gives that the inverse of $T_{O_x}$ is the Fourier-Mukai transform with the kernel $\text{cocone}(O_\Delta \to O_{(x,x)})$. The map inside a cocone is nonzero. But any nonzero map $O_\Delta \to O_{(x,x)}$ is a nonzero multiple of the natural surjection $O_\Delta \to O_{(x,x)}$. Therefore the cocone is equal to the kernel of this map, or just $O_\Delta \otimes \pi_1^*(O(-x))$. Now note that the sheaf $O_\Delta$ in the kernel trivializes all pullbacks and pushforwards we do to the identity maps between sheaves on $X$ and on $\Delta \simeq X$. The proposition follows.

The matrices $\pi(T_O)$ and $\pi(T_{O_x})$ generate $\text{SL}_2(\mathbb{Z})$, therefore, $\pi: \text{Aut}(D^b(\text{Coh}(X))) \to \text{SL}_2(\mathbb{Z})$ is surjective.

6 Classification of indecomposable sheaves

Note that indecomposable torsion sheaves lie in $C_\infty$, and generate $C_\infty$. Moreover, we have

Theorem 6.1. Indecomposable torsion sheaves are parametrized by a positive integer $s > 0$ and a closed point $x \in X$. The corresponding torsion sheaf is $O/O(-sx)$.

Proof. Indeed, we reduce to the case of one point, then the local ring is PID, and the claim follows.

In addition to that, $\text{SL}_2(\mathbb{Z})$ action allows us to prove

Theorem 6.2. For each $\mu \in \mathbb{Q}$ we have a canonical isomorphism $C_\mu \simeq C_\infty$.

Proof. Indeed, let $\mu$ be equal to $a/b$ for coprime $a$ and $b$. Choose some $\gamma \in \text{SL}_2(\mathbb{Z})$ which sends $(a,b)$ to $(0,1)$, and lift it to an autoequivalence $\bar{f} \in \text{Aut}(D^b(\text{Coh}(X)))$ of the derived category. Take any indecomposable sheaf $\mathcal{F} \in C_\mu$. Then $\bar{f}(\mathcal{F})$ is an indecomposable object in $D^b(\text{Coh}(X))$ with the slope $\infty$. Therefore, it is of form $\mathcal{G}[k]$, where $\mathcal{G}$ is a torsion sheaf, and $k$ is some integer. Denote by $\bar{f}: C_\mu \to C_\infty$ a map which sends an indecomposable sheaf $\mathcal{F}$ to a sheaf $\mathcal{G}$ defined in this way. It is easy to see that if we begin with the inverse matrix $f^{-1}$, then we get a map $\bar{f}^{-1}: C_\infty \to C_\mu$ which is inverse to $\bar{f}$. Also $\bar{f}$ does not depend on a lift $\bar{f}$. So $C_\mu$ and $C_\infty$ are canonically isomorphic.

Summarizing, we have

Theorem 6.3. Indecomposable sheaves are parametrized by a pair $(\text{rk}, \chi)$ in the right half of $\mathbb{Z}^2$ and a closed point $x \in X$.

Let us show how this describes indecomposable sheaves with charges $(1,1)$ and $(1,0)$.

Proposition 6.4.

$T_O(O) = O, \ T_{O}(O(x)) = O_x, \ T_{O_x}(O) = O(x), \ T_{O_x}(O_x) = O_x.$
Proof. The second line is a consequence of Proposition 5.9. The first line is an easy computation based on Theorem 5.5.

Proposition 6.5. The indecomposable sheaves of charge \((1, 1)\) are the sheaves \(\mathcal{O}(x)\). The indecomposable sheaves of charge \((1, 0)\) are the sheaves \(\mathcal{O}(x - y)\).

Proof. The autoequivalence \(T^{-1}_{\mathcal{O}}\) maps the charge \((0, 1)\) to \((1, 1)\), so we can use it to obtain the indecomposables of charge \((1, 1)\). Given an indecomposable \(\mathcal{O}_x\) of charge \((0, 1)\), its image is \(\mathcal{O}(x)\) by Proposition 6.4. The first part follows.

Then we can apply \(T^{-1}_{\mathcal{O}}\) to the latter indecomposables. We get that the indecomposables of charge \((1, 0)\) are \(\mathcal{O}(x - y)\).

7 Braid group relations

For matrices \(A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\) and \(B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\) we have the following relations

\[
ABA = BAB \\
(AB)^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\]

We expect similar relations to hold for \(T_{\mathcal{O}}\) and \(T_{\mathcal{O}_x}\).

Theorem 7.1 ([2]).

\[
T_{\mathcal{O}}T_{\mathcal{O}_x}T_{\mathcal{O}} \simeq T_{\mathcal{O}_x}T_{\mathcal{O}}T_{\mathcal{O}_x} \\
(T_{\mathcal{O}}T_{\mathcal{O}_x})^3 \simeq i^*[1],
\]

where \(i: X \to X\) is the inverse map of \(X\).

We can prove the braid relation using the following

Proposition 7.2 ([2]). Given two spherical objects \(E_1\) and \(E_2\), we have

\[
T_{E_1}T_{E_2} = T_{E_2(E_1)}T_{E_1}
\]

Proof. Using the computations in Proposition 6.4, we can write

\[
T_{\mathcal{O}}T_{\mathcal{O}_x}T_{\mathcal{O}} = T_{\mathcal{O}}T_{\mathcal{O}_x}T_{\mathcal{O}_x} = T_{\mathcal{O}_xT_{\mathcal{O}(x)}}T_{\mathcal{O}_x} = T_{\mathcal{O}_xT_{\mathcal{O}(x)}}T_{\mathcal{O}_xT_{\mathcal{O}_x}} = T_{\mathcal{O}_xT_{\mathcal{O}_xT_{\mathcal{O}_x}}}.
\]

This shows that \(T_{\mathcal{O}}\) and \(T_{\mathcal{O}_x}\) generate the group \(\widetilde{\text{SL}}_2(\mathbb{Z})\) in \(Aut(D^b(Coh(X)))\), the central extension of \(\text{SL}_2(\mathbb{Z})\) by \(\mathbb{Z}\).

References
