QH: Quiver Hecke category \rightarrow \text{type } \mathfrak{sl}_n, \; I = \{1, \ldots, N-1\}

Objects:
- objects $I$ (so general object is of the form $i = i_1 \cdots i_l$)
- morphisms $i \rightarrow j$

1) Relations $X = \sum_{i,j} \delta_{i,j} i_{j-1} = 1$

2) $X - X = X - X = \delta_{i,j}$

3) $X - X = \sum_{i,j,k} \delta_{i,j} i_{j-1}$ if $i = k = j \neq 1$

Def (Chuang-Rouquier) Let $C$ be a "nice" abelian category ("nice" = arbitrary, $P$ = weight lattice of $\mathfrak{sl}_n$)

A categorical self-action on $C$ is data:
- $C = \bigoplus_{P \in \mathfrak{h}} C_P$ "weight decmp" ($P$ = weight lattice of $\mathfrak{sl}_n$)
- $\forall P, \exists$ functors $\varphi: C_P \rightarrow C_{\varphi P}$ bidual
- 3 strict monoidal functors $\Phi: QH \rightarrow \text{End}(C)$
  with $i \mapsto E_i, \; i \mapsto x, \; x \mapsto t$

... with axioms!
The axioms: $e_i = E_i$, $f_i = F_i$

- on $K_0(C_q)$, need $[e_i f_j] = e_i(h_j) f_j$

- $X$ should be locally nilpotent, i.e. $V$ object $M$, $X_M \otimes M$ is nilpotent.

Next part of the talk will focus on:

**Consequences of $T$**

- Focus on a single $i \in I$. $i^n$ is object in $\mathcal{A}H$ ($n$-fold tensor product), and we define

  $N\mathcal{H}_n := \text{End}_{\mathcal{A}H}(i^n)$ (nil-Hecke algebra)

Think about the relation in $\mathcal{A}H$ in the one-color case:

Let $\chi_r$ denote \[ r \quad \text{if} \quad (\text{dot on } r^{th} \text{ strand}) \quad (1 \leq r \leq n) \]

and $T_r$ similarly the crossing of $r^{th} + 1$ strands, $1 \leq r \leq n$. The $T_r$ satisfy the braid relations, and $T_r^2 = 1$.

Because the braid relations hold, can deduce $T_r \in W(S_n)$.

To understand $N\mathcal{H}_n$, we give it a polynomial representation:

$N\mathcal{H}_n \otimes \mathbb{C}[x_1, \ldots, x_n]$

$x_i f = x_i f_i$

and $T_r$ acts by permuting \[ f_r \quad \text{for} \quad 1 \leq r \leq n \]

Let $\text{Sym} \mathbb{C}[x_1, \ldots, x_n]$ be the $S_n$-invariant polys.

Claim: $\mathbb{C}[x_1, \ldots, x_n]$ is a free $\text{Sym} \mathcal{H}_n$-module of rank $n!$
For fact, we can give a basis. Consider
\[ \{ b_w := T_{\omega} x_1^{n-1} x_2 \ldots x_{n-1} \}_{\omega \in S_n} \]

this gives a basis for Poln over \( S_n \).

Idea of proof: use induction, show \( b_n = 1 \), show
the basis are linearly independent, then compute graded
dimension via Poincaré polynomial for \( S_n \).

So we get an algebra homomorphism
\[ (b) \quad NH_n \rightarrow \text{End}_{S_n}(\text{Poln}) \cong \text{Mat}_n(\text{Sym}_n) \]

You then prove that this map is injective. This is done
using the basis \( \{ b_3 \} \) and a triangularity argument.
Even better: it's an isomorphism! This is done by
comparing (graded) dimensions (they coincide).

In particular, \( \mathbb{Z}(NH_n) \cong \text{Sym}_n \) (so \( NH_n \) is free of finite rank
over its field)

Define \( T_n = x_1^{n-1} \ldots x_{n-1} T_{\omega} \).

Note \( T_{\omega} \cdot 1 = 1 \), and \( T_{\omega} \cdot b_w = 0 \) for \( \omega \in S_n, \omega \neq 1 \), by
degree considerations.
This is analogous (wrt (b)) to matrix unit \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \),
a primitive idempotent.

We see \( NH_n \cong (NH_n T_n) \oplus \mathbb{N} \) as left \( NH_n \)-modules.

So now, let \( \mathbb{C} \) have a categorical action as above.

(next page)
Let $C$ have a categorical action as we've defined it.

1. Define $E_i(n) = \bigoplus (\pi_n)E_i^n$ (giving a projection of $E_i^n$ to a direct summand).

So we see (from the $\text{Nth} \times (\text{Nth} \times n) \oplus$ that

$E_i^n \cong (E_i(n)) \oplus n!$ (get divided power functors)

2. $x$ locally nilpotent $\Rightarrow$ on any $E_i(x) \neq 0$, $E_i^n = 0$ for $n \gg 0$.

(similarly for the $E_i$ by adjunction)

Why is this true? Well, it's easy to show $E_i(0) = 0$

or $E_i^n$ for sufficiently large $n$ (depending on $E_i$).

But $E_i(n) = \bigoplus (\pi_n)^n E_i^n$, get $0 \Rightarrow M$ by

the 1--1 part.

3. Cube Theorem $(K, L, P)$

$E_i$ satisfy some relation of set

(and too for $f; i$)

Idea of proof: Now we need to think about the case

with $>1$ color. It's basically enough to deal with

$i; j = 1$. There's a (split) short exact sequence

$0 \rightarrow E_i E_i^{(2)} \rightarrow E_i E_i E_i \rightarrow E_i E_i^{(2)} E_i \rightarrow 0$

(thats already enough).

Let $\{L(b) : b \in B\}$ be a full set of irreducibles in $C$.

$C = \bigoplus_{E_i \in P} C_i \Rightarrow B = \bigoplus_{E_i \in P} B_i$.

For $b \in B$, set $E_i b = 0$ if $E_i L(b) = 0$,

some form of symmetry

and otherwise it turns out (theorem) $E_i L(b)$ has a

simple socle and head, say $L(E_i b)$, so we and

and isomorphic.

Similarly for $f_i$ with $F_i$. 
Theorem (Cheng-Bongvan)

\((B = \mathbb{R} \otimes \mathbb{R}_k, \bar{e}_i, \bar{f}_i)\) is a normal crystal.

Theorem For \(\xi \in \mathbb{R} \otimes \mathbb{R}_k\), there is a derived equivalence

\(\Theta_i : D^b(\mathcal{C}_k) \to D^b(\mathcal{C}_{k, s \eta(\xi)}) \) where \(s \eta W = SN\)

(\(s^\mu\)th simple reflection)

\(\Rightarrow \) \(W\)-orbit of \(\mathcal{C}_k\) is all derived equivalent.

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Let \(n = \xi(h_i) > 0\). In mid 90's, it was shown \(W\) complex

\[ 0 \to F_i^{(n)} \to F_i^{(n+m)} \to \cdots \]

(this \(i\) built on my given \(M\))

(Regard complex)

This complex of functors defines a functor between the derived cats, and this was a candidate for \(\Theta_i\),

To show this defines an equivalence, the full power of Cheng-Bongvan's work was needed.

Example \(\text{gln}(\mathbb{C}) = \mathfrak{g}\). This is a Lie superalgebra (\(\mathfrak{gl}\)-graded)

\((m+n) \times (m+n)\), \(m(\text{even}), n(\text{all})\), supercommute (on homogeneous elements)

\(\text{is} \ [x,y] = xy - (-1)^{\text{deg}y}\text{deg}x\).

To get Cartan subalgebra \(\mathfrak{t} = \begin{pmatrix} 0 \\ \mathfrak{g} \end{pmatrix}\) = dings, weights \(\mathfrak{t}^*\)

Borel subalgebra \(\mathfrak{b} = \begin{pmatrix} \mathfrak{g} \\ 0 \end{pmatrix}\)

special weight \(\delta^i \in \mathfrak{t}^*\) coordinate of \(i\)th dink's row,

get a pairing \(\langle \delta_i, \delta_j \rangle = (-1)^{\text{deg}i}\text{deg}j\), with \(\text{deg} j = \sum_{i=1}^{\infty} \delta_{ij} \)
(this is just the supertrace form).

Have \[ \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \oplus \cdots + (1-m)\mathfrak{g}_m \oplus (m-1)\mathfrak{g}_{m+1} \oplus \cdots + (m-n)\mathfrak{g}_{m+n}. \]

Define \( \mathcal{O} \) to be (full subcategory of supermodules) set of

\( \) finitely generated \( \mathfrak{g} \)-supermodules, which are locally finite

\( \) over \( \mathfrak{g} \) and semisimple over \( \mathfrak{g} \) with \( \mathcal{M} = \bigoplus_{\lambda \in \mathbb{C}^*} \mathcal{M}_\lambda \)

Assume all weight are integral

\[ \lambda \text{-weight space is in party } (\lambda, \delta_{m+1} \mathbb{C} \oplus \cdots + \delta_{m+n}). \]

Construct categorical \( \mathfrak{g} \)-action on \( \mathcal{O} \) (\( \mathcal{O} \) is a

\( \) highest weight cat, in its "generic" \( \Rightarrow \) "nice" \( \Rightarrow \)

Let \( \mathcal{B} = \{ b = \begin{bmatrix} b_1 & \cdots & b_r \\ b_{r+1} & \cdots & b_m \end{bmatrix} \} \) (2-row tableaux,\n
\( \) entries \( \in \mathbb{C} \))

Given \( \mathfrak{b} \in \mathcal{B} \), define a Verma module in the usual way:

\[ \mathcal{M}(\mathfrak{b}) = \mathfrak{g} \otimes \mathcal{O}_\infty \text{ with } \lambda \text{ acting by} \]

\[ [\mathfrak{b}, \lambda \delta_{m+n}] = \mathfrak{b} \]

\( \) (pair height at space in the right party).

As usual, \( \mathcal{M}(\mathfrak{b}) \) has a unique irreducible quotient \( \mathcal{L}(\mathfrak{b}), \)

\( \) \( \) \( \) is a complete set of irreps in \( \mathcal{O} \).

Also \( \mathfrak{L}(\mathfrak{b}) \rightarrow \mathcal{M}(\mathfrak{b}) \) projective nice, then have

Verma filtrations, etc all highest weight cat stay

\( \) as usual.

Set \( \mathcal{O} = \bigoplus_{\mathfrak{p} \in \mathcal{P}} \mathcal{O}_\mathfrak{p} \) (dont confuse the Lie alg \( \mathfrak{g} \) = gl(m) \)

\( \) whose modules weve considered \( \Rightarrow \) the Lie alg \( \mathfrak{g} \) gives

\( \) our categorical action.
Here $C_n$ is the Serre subring generated by those $L(b)$'s for those $b \in B$, where that means $w_i(b) = e_i$, where that means
$$
\sum_{r=1}^{n} (-1)^{r} e_{br}.
$$

How about the endofunctors $E, F$? Well, let $V$ be the natural vector $g$-supmodule (rep on column vectors), and let $V^{\text{op}}$ be its dual. Define functors
$$
F = V \otimes \text{op}, \quad E = V^{\otimes \text{op}}.
$$

We can get natural transformations
$$
F \Rightarrow F, \quad F^2 \Rightarrow F^2.
$$

For this, note $FM = V \otimes V$. Let $\gamma = \sum_{i=1}^{n} (-1)^{i+1} e_{ij} \otimes e_{ij}$ (Cassini term), get $\sigma \in V \otimes V$ as some $xM$, the $M \to \pi$ defines $x$. For $F^2$, definitely have
$$
\xi : V \otimes V \otimes V \to V \otimes V \otimes V
$$

coming from $V \otimes V \to V \otimes V$
$$
\sigma \circ \pi = (-1)^{n+1} \sigma \otimes \pi.
$$

Then $M \to \xi$ defines $\xi$.

Bad news! We don't get the relation of $Q1$. Instead, we get the relation for $A1$, the affine Hecke category.

This is a monoidal cat

- generated by objects $1$ (so all objects $\to 1N$)
- morphisms $\bullet, X$
- relations $X \bullet X = 11$, $X = 11$, $X = X$.

We get a monoidal functor $\bar{F} : A1 \to \text{End}(O)$.
Sending \( H \mapsto F \)
\[ + \mapsto x, \]
\[ x \mapsto 2. \]

Magic: turns out (small lie) \( \hat{AH} \cong \hat{QH} \).

What's actually true is \( \hat{AH} \cong \hat{QH} \)
(appropriate notion of completion)

We define \( F_i = \) generalized \( i \)-eigenspace of \( x \),
and we get \( F = \bigoplus_{i \in \mathbb{Z}} F_i \),
and biradical gives \( E = \bigoplus_{i \in \mathbb{Z}} E_i \).

General theory \( \Rightarrow \) \( H \)-action \( \mathfrak{s} \mathfrak{l}_\mathfrak{o} \hookrightarrow \mathfrak{k}_0(\mathfrak{O}) \otimes \mathbb{C} \).

So what module do we get? We can consider \( \mathfrak{O}^{\infty} \), the
set of objects of \( \mathfrak{O} \)-Venus filtrations.

\( \mathfrak{s} \mathfrak{l}_\mathfrak{o} \hookrightarrow \mathfrak{k}_0(\mathfrak{O}) \otimes \mathbb{C} \) have basis \( \left\{ \mathfrak{P}(\mathfrak{B}) : b \in \mathfrak{B} \right\} \).

\[ \mathfrak{s} \mathfrak{l}_\mathfrak{o} \cong \mathfrak{k}_0(\mathfrak{O}^{\infty}) \otimes \mathbb{C} \cong \mathfrak{m}^\mathfrak{m}(\mathfrak{C}^{\infty}) \otimes \mathfrak{m}^\mathfrak{n}(\mathfrak{C}^{\infty}) \otimes \mathfrak{m}^\mathfrak{s} \]

this has a basis \( \left\{ \mathfrak{m}(\mathfrak{B}) : b \in \mathfrak{B} \right\} \)
(\( \mathfrak{C} = \) module of \( \infty \) column
vector \( \mathfrak{C}^{\infty} \) \( \otimes \) its
restricted dual!)

Have basis \( \mathfrak{nil}_{\mathfrak{o}} \) for \( \mathfrak{C}^{\infty} \), and \( \mathfrak{nil}_{\mathfrak{o}} \) for \( \mathfrak{C}^{\infty} \).

Thus \( \left\{ \mathfrak{P}(\mathfrak{B}) : b \in \mathfrak{B} \right\} \) is the canonical basis of \( \mathfrak{C}^{\infty} \! \).
(due to Cheng-Lam-Wang, Brundan-Losev-Webster.)