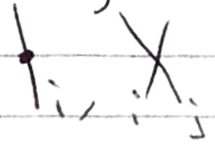
QH: Quiver Hecke Category → type $sl_N$, $I = \{1, \dots, N-1\}$
→ type $sl_\infty$, $I = \mathbb{Z}$

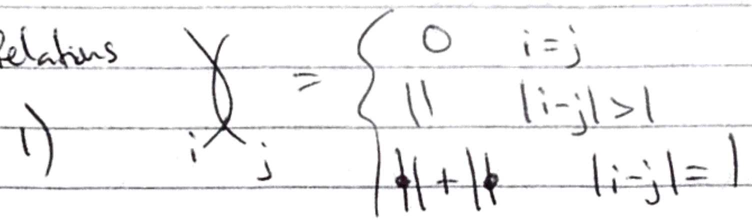
(strict monoidal category)

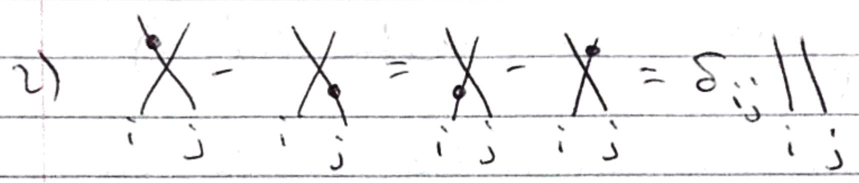
Generators:

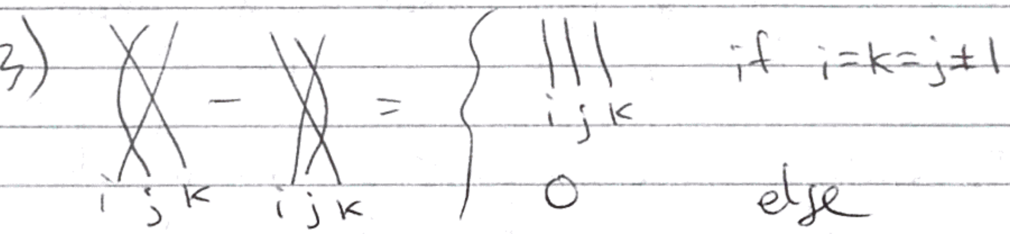
- objects $I$ (so general object is of the form $\underline{i} = i_d \cdots i_1$
- morphisms




w/ Relations




**Def** (Chuang-Rouquier) Let $\mathcal{C}$ be a "nice" abelian category (linear/$\mathbb{C}$) ("nice" = artinian, + other conditions, e.g. modules/a fin dim'l algebra, or even slightly more gen'l)

A categorical $sl_I$-action on $\mathcal{C}$ is data:

- $\mathcal{C} = \bigoplus_{\xi \in P} \mathcal{C}_\xi$ "weight decomp" ($P$ = weight lattice of $sl_I$)
- $\forall \xi \in P$, $\forall i \in I$, $\exists$ functors $\mathcal{C}_\xi \underset{F_i}{\overset{E_i}{\rightleftarrows}} \mathcal{C}_{\xi + \alpha_i}$ biadjoint
- $\exists$ strict monoidal functor $\Phi : QH \longrightarrow End(\mathcal{C})$

with $i \mapsto E_i$,  $\mapsto x$,  $\mapsto \tau$

... with axioms!

The axioms: $e_i := [E_i]$ $f_i := [F_i]$

- on $K_0(\mathcal{C}_\xi)$, need $[e_i, f_j] = \xi(h_i)\delta_{ij}$

- $x$ shall be locally nilpotent, i.e. $\forall$ object $M$, $x_M \circlearrowright M$ is nilpotent.

Next part of the talk will focus on:

<u>Consequences of $\Phi$</u>

- Focus on a single $i \in I$. $i^n$ is object in QH ($n$-fold tensor product), and we define

$$NH_n := \mathrm{End}_{QH}(i^n) \quad \text{(nil-Hecke algebra)}$$

Think about the relation in QH in the one-color case:

- let $x_r$ denote 

(dot on ~~r~~ $r^{th}$ strand) $(1 \le r \le n)$

and $T_r$ similarly the crossing of $r,(r+1)^{st}$ strands, $1 \le r < n$.
The $T_r$ satisfy the braid relations, and $T_r^2 = 1$.
Because the braid relations hold, can define $T_w$ $\forall w \in S_n$.

To understand $NH_n$, we give it a polynomial representation:

$$NH_n \circlearrowright \mathrm{Pol}_n := \mathbb{C}[x_1, \dots, x_n].$$

$$x_r \cdot f = x_r f,$$

and $T_r$ acts by <u>Demazure operator</u>:

$$T_r \cdot f = \frac{s_r(f) - f}{x_r - x_{r+1}}, \quad 1 \le r < n.$$

Let $\mathrm{Sym}_n \subset \mathrm{Pol}_n$ be the $S_n$-invariant polys.

Claim $\mathrm{Pol}_n$ is a free $\mathrm{Sym}_n$-module of rank $n!$

note for $f \in Pol_n$, $g \in Sym_n$, $w \in S_n$,
$$T_w \cdot (gf) = g(T_w \cdot f)$$

have grading of $NH_n$ w/ $\deg(x_r) = 2$, $\deg(T_r) = -2$ so that $Pol_n$ is a graded module

In fact, we can give a basis. Consider
$$\{b_w := T_w \cdot x_1^{n-1} x_2^{n-2} \cdots x_{n-1}\}_{w \in S_n}.$$
this gives a basis for $Pol_n$ over $Sym_n$.

Idea of proof use induction, show $b_{w_0} = 1$, show the $b_w$'s are ~~...~~ linearly independent, then compute graded dimension w/ Poincaré polynomial for $S_n$.

So we get an algebra homomorphism
$$(*) \quad NH_n \longrightarrow End_{Sym_n}(Pol_n) \cong Mat_{n!}(Sym_n)$$
You then prove that this map is injective. This is done using the basis $\{b_w\}$ and a triangularity argument.
Even better: it's an isomorphism! This is done by comparing (graded) dimensions (they coincide).

In particular, $Z(NH_n) \cong Sym_n$ (so $NH_n$ is free of finite rank over its center)

Define $\pi_n = x_1^{n-1} \cdots x_{n-1} T_{w_0}$.

Note $\pi_n b_1 = 1$, and $\pi_n b_w = 0$ for $w \in S_n, w \neq 1$, by degree considerations.
This is analogous (w/r/t $(*)$) to matrix unit $\begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$, a primitive idempotent.

We see $NH_n \cong (NH_n \pi_n)^{\oplus n!}$ as left $NH_n$-modules.

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So now, ~~let $\mathcal{C}$ have a categorical action as above.~~

(next page)

Let $\mathcal{C}$ have a categorical action as we've defined it.

- Define $E_i^{(n)} = \Phi(\pi_n)E_i^n$ (gives $\Phi(\pi_n)$ a projection of $E_i^n$ to a direct summand).

So we see (from the $NH_n \cong (NH_n\pi_n)^{\oplus n!}$) that

$E_i^n \cong (E_i^{(n)})^{\oplus n!}$ (get divided power functors)

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- x locally nilpotent $\Rightarrow$ on any $M \in \mathcal{C}_\xi$, $E_i^n = 0$ for $n \gg 0$.
  (similarly for the $F_i$ by adjunction)
  Why is this true? Well, it's enough to show $E_i^{(n)} = 0$ on $\mathcal{C}_\xi$ for sufficiently large $n$ (depending on $\xi$).
  But $E_i^{(n)} = \Phi\Big($  $\Big)E_i^n$, get 0 on M by the $| \cdots \bullet^{n-1}$ part.

integrability of the module

- Cute theorem (K, L, R)
  $e_i$'s satisfy Serre relation of $\mathfrak{sl}_I$
  (and too for $f_i$'s)
  Idea of proof: Now we need to think about the case with $>1$ color. It's basically enough to deal with $|i-j|=1$. There's a (split) short exact sequence
  $$0 \to E_j E_i^{(2)} \to E_i E_j E_i \to E_i^{(2)} E_j \to 0$$
  (that's already enough).

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- Let $\{L(b) : b \in B\}$ be a full set of irreducibles in $\mathcal{C}$.
  $$\mathcal{C} = \bigoplus_{\xi \in P} \mathcal{C}_\xi \Rightarrow B = \coprod_{\xi \in P} B_\xi.$$

For $b \in B$, set $\tilde{e}_i b = 0$ if $E_i L(b) = 0$,

some formal symbol

and otherwise it turns out (theorem) $E_i L(b)$ has a simple socle and head, and isomorphic, say $L(\tilde{e}_i b)$, ~~so~~ and similarly for $\tilde{f}_i$ with $F_i$.

Theorem (Chuang–Rouquier)

$(B = \coprod B_\xi, \tilde{e}_i, \tilde{f}_i)$ is a normal crystal.

Theorem For $\xi \in P$, $i \in I$, there is a derived equivalence

$$\Theta_i : D^b(\mathcal{C}_\xi) \longrightarrow D^b(\mathcal{C}_{s_i(\xi)})$$ where $s_i \in W = S_N$ ($i^{th}$ simple reflection)

$\Rightarrow$ W-orbit of $\mathcal{C}_\xi$'s are all derived equivalent

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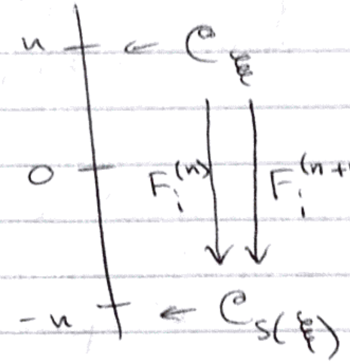
Let $n = \xi(h_i) \geq 0$. In mid 90's, it was shown $\exists$ complex




$$0 \to F_i^{(n)} \to F_i^{(n+1)}E_i \to F_i^{(n+2)}E_i^{(2)} \to \cdots$$

(this is finite on any given $M$)

(Rickard complex)

This complex of functors defines ~~the~~ a functor between the derived cats, and this was a candidate for $\Theta_i$. To show this defines an equivalence, the full power of Chuang–Rouquier's work was needed.

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Example $\mathfrak{gl}_{m|n}(\mathbb{C}) =: \mathfrak{g}$. This is a Lie superalgebra ($\mathbb{Z}/2$-graded),

$(m+n)\times(m+n)$-matrices, $\begin{array}{c|cc} & m & n \\ \hline m & \text{even} & \text{odd} \\ n & \text{odd} & \text{even} \end{array}$, supercommutator (on homogeneous elements) is $[x,y] = xy - (-1)^{|x||y|}yx$.

$\rightsquigarrow$ get Cartan subalgebra $\mathfrak{t} = \begin{pmatrix} \diagdown & 0 \\ 0 & \diagdown \end{pmatrix} = $ diags, weights $\mathfrak{t}^*$

Borel subalg $\mathfrak{b} = \begin{pmatrix} \diagdown & * \\ 0 & \diagdown \end{pmatrix}$

special weights $\delta_i \in \mathfrak{t}^*$ coordinate of $i^{th}$ diag entry,

get a pairing $(\delta_i, \delta_j) = (-1)^{|i|}\delta_{ij}$, with $|i| = \begin{cases} \bar{0} & i = 1, \ldots, m \\ \bar{1} & i = m+1, \ldots, m+n \end{cases}$

(this is just the super-trace form).

Have $\rho = 0\delta_1 - \delta_2 - 2\delta_3 - \cdots + (1-m)\delta_m + (m-1)\delta_{m+1} + \cdots + (m-n)\delta_{m+n}$.

Define cat $\mathcal{O}$ to be (full subcat of supermodules) cat of finitely generated $\mathfrak{g}$-supermodules which are locally finite over $\mathfrak{b}$ and semisimple over $\mathfrak{t}$ with $M = \bigoplus_{\lambda \in \mathfrak{t}^*_{\mathbb{Z}}} M_\lambda$

$\mathfrak{t}^*_{\mathbb{Z}} = \bigoplus \mathbb{Z}\delta_i$

Assume
- all weights are integral
- $\lambda$-weight space is in parity $(\lambda, \delta_{m+1} + \cdots + \delta_{m+n})$.

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Construct categorical $\mathfrak{sl}_\infty$-action on $\mathcal{O}$ ($\mathcal{O}$ is a highest weight cat, so its "supernice" $\Rightarrow$ "nice" :) )

Let $B = \left\{ b = \begin{array}{|l|}\hline b_1 \cdots b_m \\ \hline b_{m+1} \cdots \cdots b_{m+n} \\ \hline \end{array} \right\}$ (2-row tableaux, entries in $\mathbb{Z}$)

Given $b \in B$, define a Verma module in the usual way:

$M(b) = U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda$, with $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$ given by $(\lambda + \rho, \delta_i) = b_i$

(put highest wt space in the right parity).

As usual, $M(b)$ has a unique irred quotient $L(b)$, $\{L(b) : b \in B\}$ is a complete set of irreds in $\mathcal{O}$.
Also $\exists\, P(b) \to M(b)$ projective cover, these have Verma filtrations, etc all highest weight cat story as usual.

Get $\mathcal{O} = \bigoplus_{\xi \in P} \mathcal{O}_\xi$ (don't confuse the Lie alg $\mathfrak{g} = \mathfrak{gl}_{m|n}$ whose modules we're considering w/ the Lie alg $\mathfrak{sl}_\infty$ giving our categorical action.

Here $\mathcal{O}_\xi$ is the Serre subcat gen'd by those $L(b)$'s for those $b \in B_\xi$, where that means $\mathrm{wt}(b) = \xi$, where that means
$$\sum_{r=1}^{m+n} (-1)^{|r|} \varepsilon_{b_r}.$$

How about the endofunctors $E_i, F_i$? Well, let $V$ be the natural vector $\mathfrak{g}$-supermodule (rep on column vectors), and let $V^*$ be its dual. Define functors

$$F = V \otimes \bullet, \quad E = V^* \otimes \bullet$$

We can get natural transformations

$$F \overset{x}{\Rightarrow} F, \quad F^2 \overset{\tau}{\Rightarrow} F^2$$

For this, note $FM = V \otimes M$. Let $\Omega = \sum_{i,j=1}^{m+n} (-1)^{|j|} e_{ij} \otimes e_{ji}$

(Casimir term), get $\Omega \curvearrowright V \otimes M$ as some $x_M$, the $M \mapsto x_M$ defines $x$. For $F^2$, definitely have

$$\tau_M : V \otimes V \otimes M \longrightarrow V \otimes V \otimes M$$

coming from
$$V \otimes V \longrightarrow V \otimes V$$
$$v \otimes w \longmapsto (-1)^{|v||w|} w \otimes v.$$

Then $M \mapsto \tau_M$ defines $\tau$.

<u>Bad news</u>! We don't get the relation of QH. Instead, we get the relations for AH, the <u>affine Hecke category</u>.
This is a monoidal cat

- generated by objects $|$ (so all objects $\leftrightarrow \mathbb{N}$) and morphisms $\phi$, $X$
- relations 

We get a ~~f~~ monoidal functor $\Phi : AH \longrightarrow \mathrm{End}(\mathcal{O})$

sending $| \longmapsto F$
$\phi \longmapsto x$,
$X \longmapsto \tau$.

<u>Magic</u>: turns out (small lie) $AH \cong QH$.
What's actually true is $\widehat{AH} \cong \widehat{QH}$
(appropriate notion of completion)

We define $F_i$ = generalized $i$-eigenspace of $x$,
and we get $F = \bigoplus_{i \in \mathbb{Z}} F_i$,
and biadjunction gives $E = \bigoplus_{i \in \mathbb{Z}} E_i$.

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General theory $\Rightarrow \exists$ action $\mathfrak{sl}_\infty \curvearrowright K_0(\mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{C}$.

So what module do we get? We can consider $\mathcal{O}^\Delta$, the cat of objects w/ Verma filtrations.

$$\mathfrak{sl}_\infty \curvearrowright K_0(\mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{C}, \text{ have basis } \{[P(b)] : b \in B\}.$$

$$\downarrow$$

$$\mathfrak{sl}_\infty \curvearrowright K_0(\mathcal{O}^\Delta) \otimes_{\mathbb{Z}} \mathbb{C} \;\cong\; T^m(\mathbb{C}^\infty) \otimes T^n((\mathbb{C}^\infty)^\circledast) \curvearrowleft \mathfrak{sl}_\infty$$

$$[M(b)] \longmapsto v_{b_1} \otimes \cdots \otimes v_{b_m} \otimes v^\vee_{b_{m+1}} \otimes \cdots \otimes v^\vee_{b_{m+n}}$$

this has a basis $\{[M(b)] : b \in B\}$

($\mathbb{C}^\infty$ = module of $\infty$ column vectors, $(\mathbb{C}^\infty)^\circledast$ its restricted dual)

Have basis $\{v_i\}_{i \in \mathbb{Z}}$ for $\mathbb{C}^\infty$, and $\{v_i^\vee\}_{i \in \mathbb{Z}}$ for $(\mathbb{C}^\infty)^\circledast$.

<u>Thm</u> $\{[P(b)] : b \in B\}$ is the canonical basis of Lusztig!
(due to Cheng–Lam–Wang, Brundan–Losev–Webster)