Jon Bundon's lectures

Today: Baby example: categorification of the \( \mathbb{Q}_N \)-module \( \Lambda^m \mathbb{C}^N \otimes \Lambda^m \mathbb{C}^N \)

**Notation** \( t = \text{diagonal matrix in } \mathbb{Q}_N \)

\[ E_i \in t^*, \quad \alpha_i = E_i - E_{i+1} \quad \mathcal{P} = \bigoplus_{i=1}^N \mathbb{Z}E_i \text{ weight lattice} \]

\( \mathbb{C}^N \)-basis \( v_1, \ldots, v_N \)

\[ v_i \xrightarrow{f_i} v_{i+1} \quad \text{wt}(v_i) = E_i \]

\( \Lambda^m = \Lambda^m \mathbb{C}^N \otimes \Lambda^m \mathbb{C}^N \) has the *monomial basis*

\[ (v_{i_1} \wedge \ldots \wedge v_{i_m}) \otimes (v_{j_1} \wedge \ldots \wedge v_{j_n}) \]

\[ i_1 \geq \ldots \geq i_m \quad j_1 \geq \ldots \geq j_n \]

**Notational gimmick** Index the monomial basis by \( \beta = \text{markers} \)

\[ \alpha, \beta, \ldots \]

e.g. \( \psi_a = (v_2 \wedge v_7 \wedge v_8) \otimes (v_4 \wedge v_2 \wedge v_3) \)

So we get a line decorated with \( X, 0, >, < \)

\( \text{wt}(\psi_a) = E_4 + 2E_2 + E_7 + E_5 + E_6 \)

coef 1 \( > \) or \( < \) coef 2 \( \leftrightarrow X \) coef 0 \( \longleftrightarrow 0 \)

So \( \text{wt}(\psi_a) = \text{wt}(\psi_b) \iff 0, X \) are in the same positions (the "core" of the marker) \( \alpha \sim \beta \) (lineage relation)
2.

- What order on markers

\[ a \leq b \] is generated by \[ \mathbf{1} \leq \mathbf{1} \]

- Given: Define or de cat-

\[ E_{i, F_u} : G^0 \]

\[ \text{biadjoint} \quad i = 1, \ldots, N - 1 \]

So that

\[ \xi (G) \otimes G \equiv \Lambda^{m,n} \]

\[ \langle E_{i, j}, [F_u] \rangle \rightarrow e_{i,j} \]

(this is essentially due to Khovanov)

- For this, we'll define a f.d. algebra \( K \)

\[ \mathcal{E} = K - \text{mod f.d.} \]

- Actually, \( \mathcal{E} = \bigoplus_{\mathbf{rep}} \mathcal{E}_\alpha \)

\[ \xi \alpha \xrightarrow{[E_{i, j}]} \mathcal{E}_{\alpha + \alpha_{i, j}} \]

\[ K = \bigoplus_{\mathbf{rep}} K_\alpha \]
Khorosov algebra $K$

Take $a \in \mathcal{B}$

Ca left arc diagram

- right arc diagram

"clue with counterclockwise arcs"

NO CROSSINGS!

right arc diagram is constructed similarly and it is the mirror image of the left arc diagram.

Given $a \sim b \sim c$, consider $\alpha_b$ (the vertex arc marked with $C$)

We say this is consistently oriented if every arc is $C$ or $\sim C$ and all the "left" rays are below the "right" rays.

Similarly we have $\beta_c$

And we can give $\alpha_b \beta_c$ (so $b$ gives orientation of circle)

\[ \alpha_b \beta_c \]
Def $K$ has basis $abc$ $\nRightarrow$ $a \sim b \sim c$ in $\beta$ consistently oriented.

**Multiplication**

$$abc \times def = \begin{cases} 
0, & \text{if } c \neq d \\
abc \cdot cef, & \text{if } c = d
\end{cases}$$

(See the claim is that after "surgery" this is a sum of basic diagrams.)

Surgery rules:

- $\bigcirc \rightarrow \bigcirc$
- $\bigcirc \rightarrow \bigcirc$
- $\bigcirc \rightarrow 0$ (zero)

Think of $\mathbb{C}[x]/(x^2), \ x = \bigcirc$ (motivation comes from TOFT)
For splitting circle \( (\text{comultiplication } 1 \mapsto 1 \otimes x + x \otimes 1, \ x \mapsto x \otimes x) \)

\[
\begin{align*}
\circ & \rightarrow \circ \otimes \circ + \circ \otimes \circ \\
\circ & \rightarrow \circ \otimes \circ \\
\circ & \rightarrow \circ \otimes \circ
\end{align*}
\]

Surgeries involving rays

\[
\begin{align*}
\circ & \rightarrow \circ \\
\circ & \rightarrow \circ
\end{align*}
\]

\[
\begin{align*}
\circ & \rightarrow \circ \otimes \circ
\end{align*}
\]

\[
\begin{align*}
\circ & \rightarrow \circ
\end{align*}
\]

Let's go back to the example
As opposed to other diagrammatic algebras, multiplication here is not local.
It's not obvious that this is well-defined and associative.

Remarks about $K$:

1. $K$ is positively graded, $\deg(abc) = \# f$ clockwise over $\deg(C) = 0$, $\deg(C) = 2$
   (this is actually a Koszul grading)

2. $K_0 = \langle \overline{bb} \mid b \in \Sigma \rangle$

   $\overline{bb} = \overline{b} \overline{b}$ - mutually orthogonal idempotents.
   These are the primitive idempotents in $K$.

   So $P(b) = Kbb \rightarrow L(b)$
   (projective module)
   (irreducible)

3. $K = \bigoplus V(\lambda)$ where
   $V(\lambda)$ has basis $\{abc, a\overline{b} \overline{c}, a, b, c \text{ of weight } \lambda\}$

4. $\overline{abc} \overline{def} = \sum_{b \leq g \leq f} \sigma_1 \overline{agf}$
   $K_2 \neq 0$ iff $\lambda$ in a weight of $\Lambda^{m,n}$

   Cellular basis $\Rightarrow$ Quasi-hereditary algebra
   Standard module $\overline{V}(b), b \in \Sigma$
\[ \mathcal{K}_0(\mathfrak{g}) \otimes_{\mathbb{C}} \mathbb{C} \cong \Lambda^{\mathfrak{m}} \]

\[ \mathcal{V}(b) \mapsto v_b \]

Explicitly, one can compute that

\[ \mathcal{L}(b) \mapsto \text{Lusztig canonical basis} \]

5. If \( \mathfrak{g} \) is regular, \( \mathfrak{g} = E_1 + \cdots + E_{m+1} \) (empty case) then

\[ \mathfrak{g}_\mathfrak{a} = \mathfrak{k}_\mathfrak{a} \cdot \text{mod}_{\mathfrak{a}} \cong \text{Perv}(Gr_{m,m}) \text{ (with Schubert struc)} \]

Still need to get \( E_1 F_1 \)

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**Geometric bimodules**

\[ k_t \text{ is a } (k,k)-\text{bimodule} \quad \mapsto \quad k_t \otimes_k k \]

\( t \) here is a crossing in a matching

Fix \( \rho \), weights of \( \Lambda^\mathfrak{m} \)

\( k_t \) has basis

\[ \begin{array}{cccc}
\mu & a & b & t \\
\nu & c & d & e \\
\end{array} \]

\( a \sim b \) of weight \( \mu \)

\( c \sim d \) of weight \( \nu \)

\[ \text{st this is consistently oriented. The multiplication on the left/right is again by surgery. So } k_t \text{ is a bimodule.} \]
Thm \quad K_s \otimes K_t \cong K_{st}^{\oplus (2^\# \text{ of circles removed})}

What \textit{happens with}\, K_{st}^{\oplus (2^\# \text{ of circles removed})}\? 

"Circle removed" means that concatenating it we may get floating circles that we need to remove.

So, in particular, the $K_t$ category the Temperley-Lieb algebra for the principal where $\lambda$ empty can.

\textbf{Note} \#1 $K_t$ is projective both as left & right module.
\textbf{Note} \#2 $K_t \otimes K^\ast, K_t^+ \otimes K$ are biadjoint.

So we can now define $E_i, F_i$.

$$E_i := \bigoplus_{\gamma \in P} K_t(\gamma \otimes K)$$

s.t. both $\gamma, \gamma + \Delta$ are with of $\Delta_{mn}$.

Where $t_\alpha(\lambda)$ is:
An explicit computation shows that $E_i, E_j$ do categorify $e_i, e_j$.

So we have a "weak" categorification. To have an honest categorification, we need natural transformations

$$E_i \Rightarrow E_i \quad E_i E_j \Rightarrow E_j E_i$$

* The quiver Hecke category

$\mathcal{H}_A$ — strict monoidal cat-y (so we write $\otimes$ for vertical composition and $\circ$ for horizontal comp)

Object

Generator $I = 2, \ldots, N - 1$ — simple roots

Morphism

$$\begin{array}{c}
\mathcal{H} - \mathcal{H} = \\
\text{Relation}
\end{array}$$

\[ x_{ij} = \begin{cases} 
0 & \text{if } i = j \\
1 & \text{if } |i - j| > 1 \\
1 + 1 & \text{if } |i - j| = 1 
\end{cases} \]

\[ X - X = X - X = \sum_{i,j} d_{ij} \]
\[
X - X = \begin{cases} 
1 & \text{if } i = k = j + 1 \\
0 & \text{else}
\end{cases}
\]

Claim: We have a morphism (= strict monoidal maps)

\[
Q^H \xrightarrow{\Phi} \text{End}(G)
\]

Where \(x, I\) are defined as follows:

\(x\): defined by surgery: change clockwise circle to counterclockwise odd 2 str.