Jon Brundan's lectures

Today: Baby example: categorification of the $\mathfrak{sl}_N$-module $\Lambda^m\mathbb{C}^N \otimes \Lambda^n\mathbb{C}^n$

<u>Notation</u> $\mathfrak{t}$ = diagonal mat ~~in~~ ~~$\mathfrak{sl}_N$~~

$\varepsilon_i \in \mathfrak{t}^*$, $\quad \alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad P = \bigoplus_{i=1}^{N} \mathbb{Z}\varepsilon_i$ weight lattice

$\mathbb{C}^N$ - basis $\quad v_1, \dots, v_N \qquad v_i \underset{e_i}{\overset{f_i}{\rightleftarrows}} v_{i+1} \quad wt(v_i) = \varepsilon_i$

$\Lambda^{m,n} := \Lambda^m\mathbb{C}^N \otimes \Lambda^n\mathbb{C}^N$ has the <u>monomial basis</u>

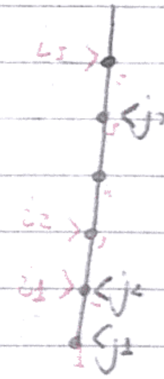
$$(v_{i_1} \wedge \dots \wedge v_{i_m}) \otimes (v_{j_1} \wedge \dots \wedge v_{j_n})$$
$$i_1 > \dots > i_m \qquad j_1 > \dots > j_n$$

<u>Notational gimmick</u> Index the monomial basis by

$$B = \{\text{markers}\}$$
$$a, b, c, \dots$$

e.g. $v_a = (v_2 \wedge v_3 \wedge v_6) \otimes (v_1 \wedge v_2 \wedge v_5) \rightsquigarrow$

$(N=6)$



So we get a line decorated with $\times, \circ, >, <$

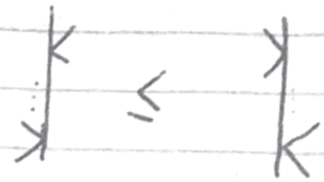
$wt(v_a) = \varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 + \varepsilon_5 + \varepsilon_6$

coeff. 1 $\rightsquigarrow$ $>$ or $<$ $\qquad$ coeff. 2 $\longleftrightarrow \times$ $\qquad$ coeff. 0 $\rightsquigarrow \circ$

So $wt(v_a) = wt(v_b) \iff \circ, \times$ are in the same positions (the "core" of the marker)
$a \sim b$ (linkage relation)

②

- Bruhat order on markers

$a \leq b$ is generated by 

Goal Define an ab cat-y $\mathcal{C}$, with endofunctors $E_i, F_i : \mathcal{C} \circlearrowleft$ biadjoint $i = 1, \dots, N-1$

So that

$$K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C} \equiv \Lambda^{m,n}$$

$$[E_i], [F_i] \longleftrightarrow e_i, f_i$$

(this is essentially due to Khovanov)

For this, we'll define a f.d algebra K

$$\mathcal{C} = K\text{-mod}_{f.d}$$

Actually, $\mathcal{C} = \bigoplus_{\lambda \in P} \mathcal{C}_\lambda \qquad \mathcal{C}_\lambda \underset{[F_i]}{\overset{[E_i]}{\rightleftarrows}} \mathcal{C}_{\lambda + \alpha_i}$

$$K = \bigoplus_{\lambda \in P} K_\lambda$$

Khovanov algebra K

Take $a \in B$ → Ca left arc diagram
→ $a)$ right arc diagram

"close with counterclockwise arcs"

NO CROSSINGS!

⇝ left arc diagram

right arc diagram is constructed similarly and it is the mirror image of the left arc diagram

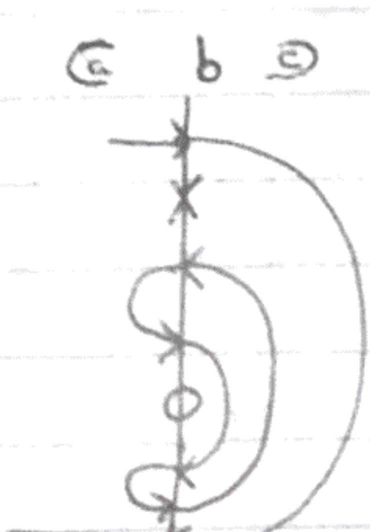
Given $a \sim b \sim c$, consider $(a\ b)$ (ie vertices are marked with ( ))

We say this is consistently oriented if every arc is ↺ or ↻ and all the "left" rays are below the "right" rays

Similarly we have $(b\ c)$

And we can glue $(a\ b\ c)$ (so b gives orientation of circles)

circle diagram

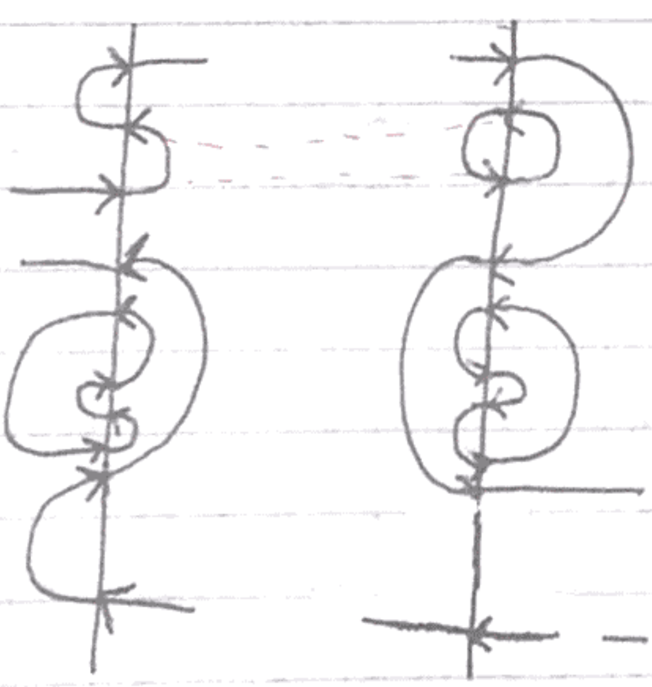


Def K has basis (a b c) $\forall$ $a \sim b \sim c$ in B consistently oriented.

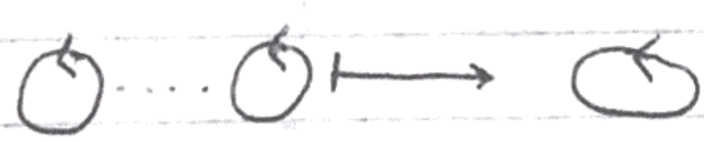
Multiplication (a b c)(d e f)

$$= \begin{cases} 0, \text{ if } c \neq d \\ (a\,b\,c \frown c\,e\,f) & \text{if } c = d \end{cases}$$

(So the claim is that after "surgery" this is a sum of basic diagrams)



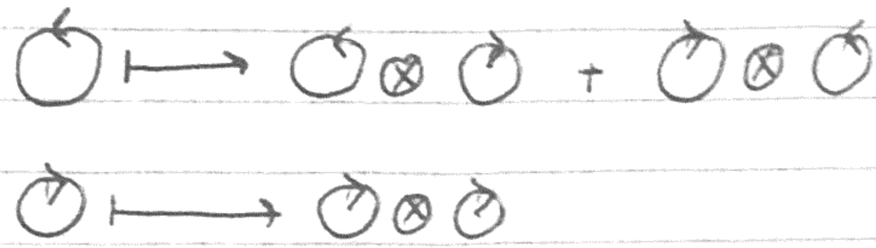
Surgery rules



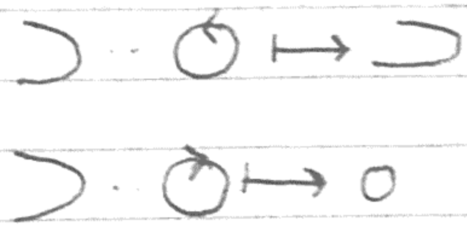
 0 (zero)

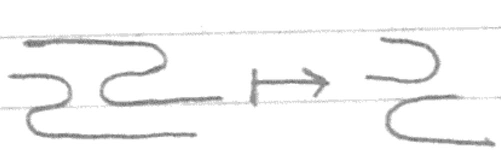
think of $\mathbb{C}[x]/(x^2)$, $x =$  (motivation comes from TQFT)

For splitting circles (comultiplication $1 \mapsto 1 \otimes x + x \otimes 1$, $x \mapsto x \otimes x$)



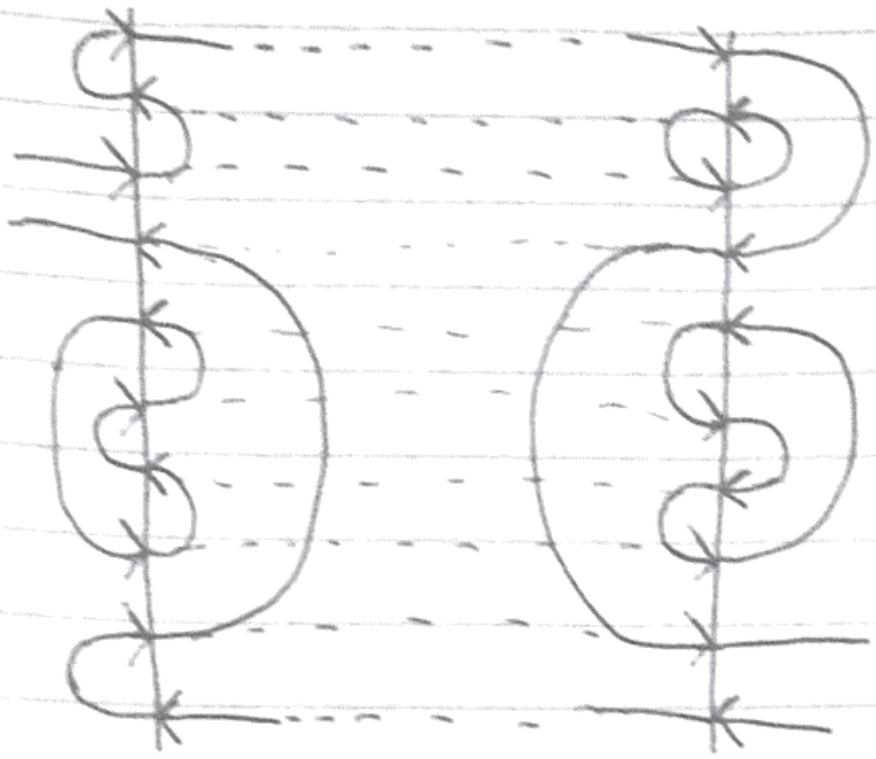
Surgeries involving rays

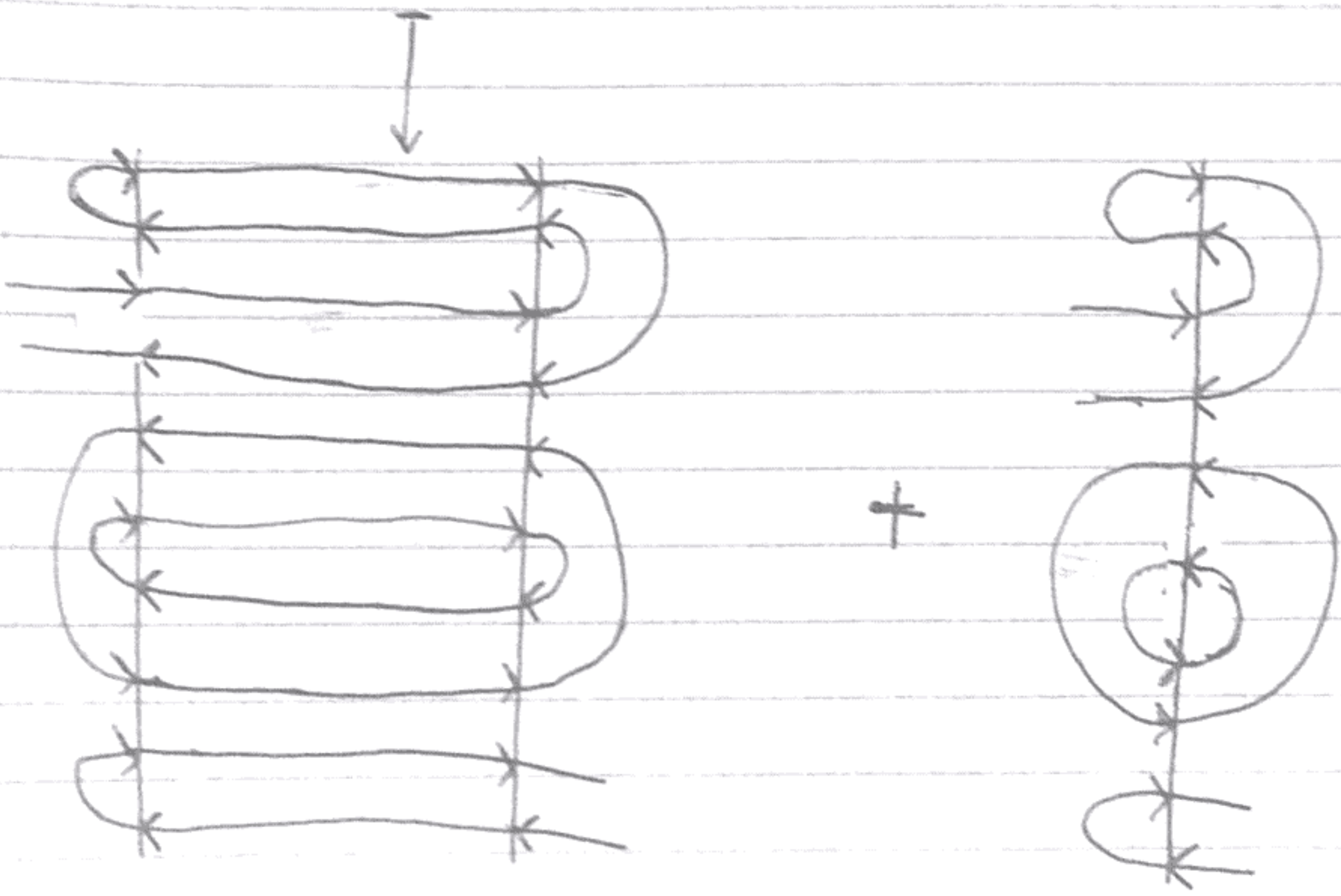


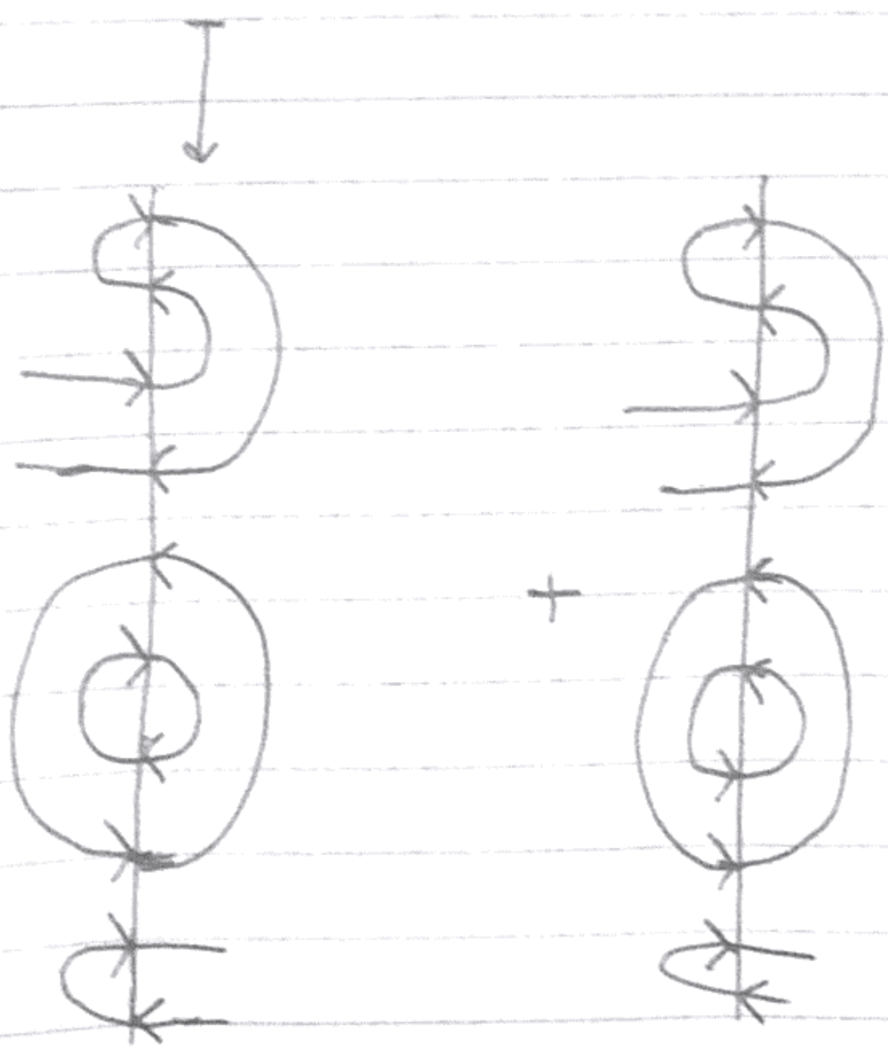


let's go back to the example

(Next page!)







* As opposed to other diagrammatic algebras, multiplication here is not local.

i.e. indep. of the order of surgery

It's not obvious that this is well-defined and associative

Remarks about K:

1. K is positively graded, deg (abc) = # of clockwise arcs

deg ⟲ = 0, deg ⟳ = 2

(this is actually a Koszul grading)

2. $K_0 = \langle (b\,b\,b) \mid b \in B \rangle$

$1_b := (b\,b\,b)$ - mutually orthogonal idempotents

these are the primitive idempotents in K

So $P(b) = K 1_b \longrightarrow L(b)$

(proj-ve ind modules) (irreducibles)

3. $K = \bigoplus_{\lambda \in P} K(\lambda)$ where $K(\lambda)$ has basis $(a\,b\,c)$ $a \vee b \vee c$

a, b, c of weight λ

$K_\lambda \neq 0$ iff λ is a weight of $\Lambda^{m,n}$

4. $(a\,b\,c)(d\,e\,f) = \sum_{b \leq g \geq e} {}^{0 \text{ or } 1} (a\,g\,f)$

$\Downarrow$

Cellular basis $\Rightarrow$ Quasi-hereditary algebra

Standard modules $V(b)$, $b \in B$

$$K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \Lambda^{m,n}$$

$$[V(b)] \longmapsto v_b$$

Explicitly, one can compute that

$[P(b)] \longmapsto$ Lusztig's canonical basis

5 If λ is regular, $\lambda = \varepsilon_1 + \cdots + \varepsilon_{m+n}$ (empty core) then

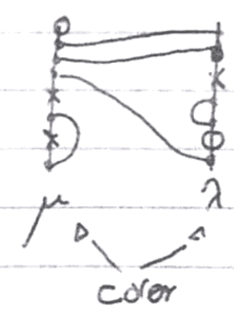
$\mathcal{C}_\lambda = K_\lambda\text{-mod}_{f.d} \simeq Perv(Gr_{m,m+n})$ (w.r.t Schubert strat)

Still need to get $E_i, F_i \ldots$

<u>Geometric bimodules</u>

K_t a (K,K)-bimodule $\longleftrightarrow$ $K_t \otimes_K \cdot : \mathcal{C}$ ⟲

t here is a <u>crossingless matching</u> $\longrightarrow$



Fix λ, μ weights of $\Lambda^{m,n}$

K_t has basis $(a\,b \mid t \mid c\,d)$ with μ under $a\,b \mid t$ and λ under $t \mid c\,d$; $a \sim b$ of weight μ, $c \sim d$ of weight λ

s.t this is consistently oriented. The multiplication on the left/right is again by surgery. So K_t is a bimodule

Thm $K_s \otimes K_t \cong K_{st}^{\oplus(2^{\#\text{ of circles removed}})}$

~~What to do with [illegible]~~

"Circles removed" means that concatenating st we may get floating circles that we need to remove.

So, in particular, the K_t categorify the Temperley-Lieb algebra

for the principal block, i.e. empty case

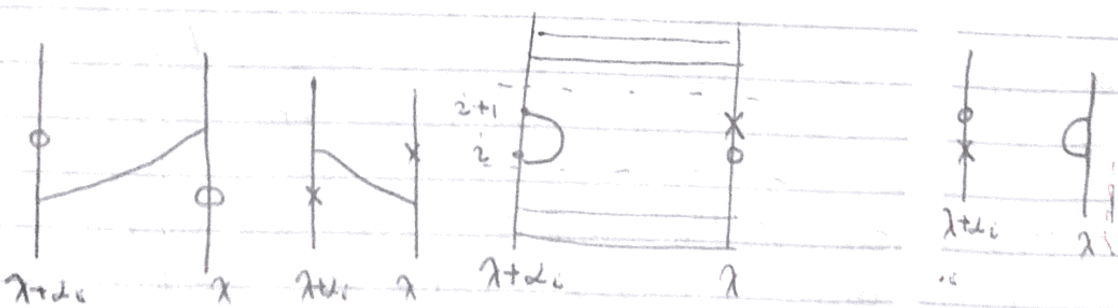
Note ① K_t is proj-ve both as left & right module

② $K_t \otimes_K \cdot$, $K_{t^*} \otimes_K \cdot$ are biadjoint

So we can now define E_i, F_i:

$$E_i = \bigoplus_{\substack{\lambda \in P \\ \text{s.t. both} \\ \lambda,\, \lambda+\alpha_i \text{ are} \\ \text{wts of } \Lambda^{m,n}}} K_{t_i(\lambda)} \otimes_K \cdot$$

where $t_i(\lambda)$ is



An explicit computation shows that E_i, F_i do categorify e_i, f_i.

So we have a "weak" categorification. To have an honest categorification, we need natural transformations

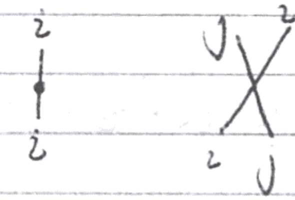
$$E_i \overset{x}{\Longrightarrow} E_i \qquad E_i E_j \overset{\tau}{\Longrightarrow} E_j E_i$$

★ The quiver Hecke category

QH - strict monoidal cat-y (so we write $\circ$ for vertical composition, $\otimes$ for horizontal comp)

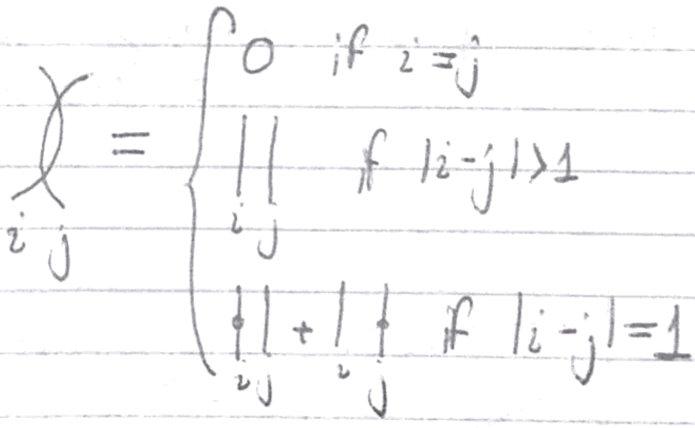
Generators objects $I = \{1, \dots, N-1\}$ ← simple roots

morphisms



$(i \otimes j \to j \otimes i)$

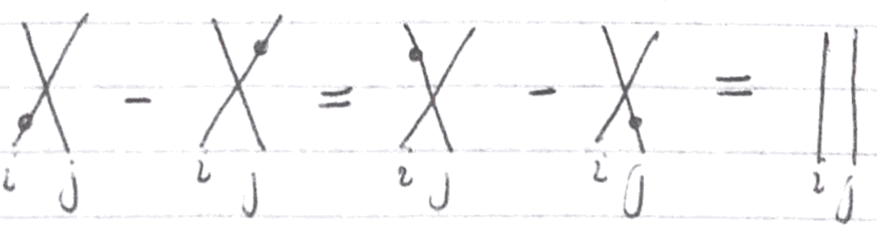
Relations



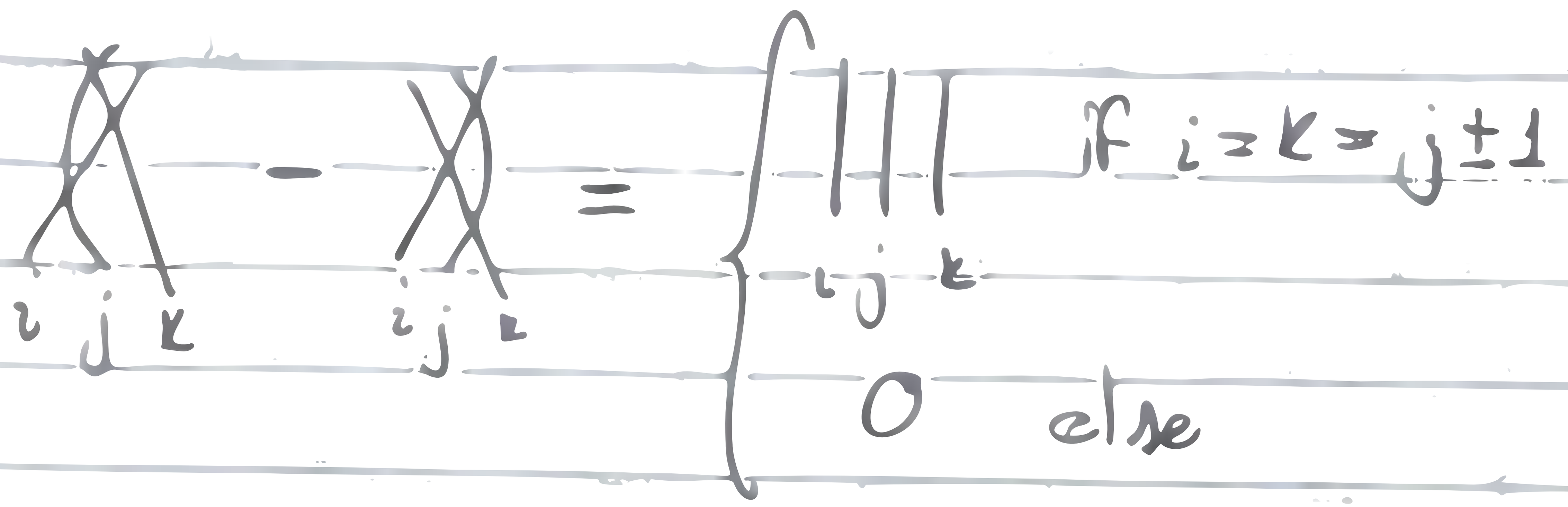
0 if $i=j$

if $|i-j|>1$

if $|i-j|=1$



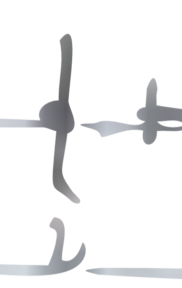
δ_{ij}



Claim We have a morphism (= strict monoidal functor)

$$Q\mathcal{H} \xrightarrow{\Phi} \mathrm{End}(\mathcal{C})$$

$$i \longmapsto E_i$$



$\longmapsto x$

$\longmapsto \tau$

Where x, τ are defined as follows

x : defined by surgery: change clockwise circles to counterclockwise and so on

τ