EXAMPLES OF HYPERKÄHLER MANIFOLDS AS MODULI SPACES OF SHEAVES ON K3 SURFACES

BARBARA BOLOGNESE

Contents

Introduction

A compact Kähler surface X is a K3 surface if it is simply connected and it carries a global homolorphic symplectic form (i.e. the canonical bundle $K_X \cong \mathcal{O}_X$). An example is given by the *Fermat quartic*: consider the degree four polynomial $P(X_0, ..., X_3) = X_0^4 + X_1^4 + X_2^4 + X_3^4 \in \mathbb{C}[X_0, ..., X_3]$. The vanishing locus $S = V(P)$ is an irreducible quartic hypersurface in $\mathbb{P}^3_{\mathbb{C}}$, which is simply connected by the Lefschetz Hyperplane Theorem[1], and has canonical bundle $K_S = (\mathcal{O}_{\mathbb{P}^3}(-4) \otimes \mathcal{O}_{\mathbb{P}^3}(4))|_S \cong \mathcal{O}_S$ by adjunction. Hence, the surface S is a K3 surface and, by applying the same reasoning verbatim, every irreducible quartic hypersurface in $\mathbb{P}^3_{\mathbb{C}}$ is. K3 surfaces play a fundamental role in the classification of algebraic surfaces, hence it is natural to look for generalizations in higher dimensions. The following (beautiful) classification theorem motivates the definition of a hyperkähler manifold (HK):

Theorem 0.1 (Beauville-Bogomolov decomposition, [Bea83])**.** *Let X be a compact Kähler manifold with $c_1(X) = 0$. There exists an étale finite cover $\prod_{i=1}^d M_i \longrightarrow X$ where each of the factors M_i is either a compact complex torus, a Calabi-Yau variety or a HK.*

Date: 2016-3-26.

[1] More specifically, one can use the Veronese embedding of degree $d = 4$ to see the hypersurface X as a hyperplane in $\mathbb{P}^{34}$ and deduce that $\pi_1(X) = 0$ from $\pi_1(\mathbb{P}^{34}) = 0$.

A *Calabi-Yau* variety (CY) is a compact Kähler manifold M of dimension $n \geq 3$ with trivial canonical bundle and such that the Hodge numbers $h^{p,0}(X)$ vanish for $0 < p < n$. These can very well be considered generalizations of K3 surfaces, which indeed satisfy $h^{1,0}(X) = 0$.[2] Nonetheless, most of the strikingly interesting properties of K3 surfaces come from the existence of a symplectic structure which is compatible with the complex holomorphic structure, hence a more satisfactory generalization can be found in the following definition:

Definition 0.2. A compact Kähler manifold X is *hyperkähler* if it is simply connected and the space of its global holomorphic two-forms is spanned by a symplectic form.

The only varieties which are both CY and HK are K3 surfaces, since the vanishing condition $h^{p,0}(X) = 0$ for $0 < p < \dim X$ is compatible with the existence of a global holomorphic symplectic form only when $\dim X = 2$. In higher dimension, the first two families of examples were produced by Beauville in [Bea83]: the Hilbert scheme of points over a K3 surface, which will be discussed at length in section 2, and a suitable subvariety of the Hilbert scheme of points over an abelian surface, called *generalized Kummer variety.* More families of examples have been constructed since then, but they can all be shown to be deformation equivalent to one of the two families already found by Beauville. Recently, two more sporadic examples were found by O'Grady in [O'G99] and [O'G03] by desingularizing a singular moduli space of sheaves on a K3 (respectively, abelian) surface. The HK manifolds this obtained are ten (respectively, six) dimensional varieties, which we will denote by OG10 (respectively, OG6). The variety OG10 will be the main focus of section 4.

1. Compact hyperkähler manifolds

We will now discuss some of the most interesting properties of compact hyperkähler manifolds, some of which are direct generalizations of the main properties of K3 surfaces.

- **(Hyperkähler geometry)** Complex varieties as in Definition 0.2 are alternatively known as *irreducible holomorphic symplectic* (IHS). The reason behind the name "hyperkähler" lies in the following fact: if X is a IHS manifold, i.e. simply connected Kähler manifold with a global holomorphic symplectic form spanning the space $H^{2,0}(X)$, then by Yau's solution to Calabi's conjecture there exists a Riemannian metric g on X such that the holonomy of (X, g) is isomorphic to the tautological representation of the symplectic group

$$Sp(r) := \{\phi : \mathbb{H}^r \longrightarrow \mathbb{H}^r \mid \phi \text{ is right-linear and } \overline{\phi(v)}^t \cdot \phi(w) = \overline{v}^t \cdot w\}$$

on $\mathbb{H}^r$, where $\dim X = 4r$. This can be interpreted as the existence of a quaternionic structure on X, meaning that there are three distinct complex structures I, J and K satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -1.$$

[2] Indeed, the first homology group $H_1(X, \mathbb{Z})$ is the abelianization of the first homotopy group $\pi_1(X)$, hence the simple connectedness of X implies the vanishing of the first Betti number $b_1(X)$. Now, since X is Kähler, one has that $b_1(X) = h^{1,0} + h^{0,1} = 2h^{1,0}$.

Hence, the manifold (X, g) is "hyperkäher" in the sense of hypercomplex geometry: in other words, it is a Kähler manifold whose Kähler structure is compatible with three complex structures interacting quaternionically. Conversely, any manifold (X, I, J, K), having a quaternionic structure as above can be thought as a complex manifold (X, I) with a global holomorphic symplectic form given by the fact that $Sp(r) = U(\mathbb{H}^r) \cap SO(\mathbb{H}^r)$. It can be also shown that $\pi_1(X) = 0$ (see e.g. [Bea83]), hence X is a IHS manifold.

There are many examples of non-compact hyperkähler manifolds (some of the most interesting both in representation theory and in algebraic geometry being the Nakajima quiver varieties), and some recent results show that the local structure of compact HK manifolds can be understood by means of those (see e.g. [AS15]).

- **(The Beauville-Bogomolov form)** The standard intersection pairing on the middle cohomology group $H^2(S) := H^2(S, \mathbb{C})$ of a K3 surface S can be shown to be even, unimodular and of signature $(3, 19)$, hence isomorphic to the lattice
$$\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$
Similarly, if X is HK and ω is its symplectic form, we have the following:

Theorem 1.1 (Beauville [Bea83] and Fujiki [Fuj87])**.** *There exists a positive rational number c_X (Fujiki's constant) and an integral indivisible non-degenerate symmetric bilinear form $(,)_{BB}$ on $H^2(X)$ (Beauville-Bogomolov's form) of signature $(3, b_2(X) - 3)$ such that the following hold:*
(1) $\int_X \alpha^{2n} = c_X \cdot (\alpha, \alpha)_{BB}^n$ *for* $\alpha \in H^2(X)$,
(2) $(\alpha, \alpha')_{BB} = 0$ *if* $\alpha \in H^{p,2-p}(X),\ \alpha' \in H^{p',2-p'}(X)$ *with* $p + p' \neq 2$.

The Beauville-Bolomolov form endows the cohomology group $H^2(X)$ with the structure of a lattice. Such lattice structure, and the respective Fujiki constant, can be explicitly computed for all the known examples (see e.g. [Rap08]). We will sketch the computation for the Hilbert scheme of point on a K3 surface in section 2, and for O'Grady's ten-dimensional example in section 4.

- **(A global Torelli theorem)** The result which is by many considered as the culmination of the theory of K3 surfaces are the local and global Torelli theorems. They give a positive answer to the question whether a K3 surfaces can be recovered by the Hodge structure on its H^2 lattice. More specifically, we say that a lattice isomorphism (i.e., an isometry) $f : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$, where X and X' are two K3 surfaces, is a *Hodge isometry* if it preserves the Hodge structure, i.e. if $f(H^{2,0}(X)) \subset H^{2,0}(X')$. Then we have the following:

Theorem 1.2 (Classical Global Torelli)**.** *Two K3 surfaces X and X' are isomorphic if and only if there exists a Hodge isometry*
$$f : H^2(X, \mathbb{Z}) \xrightarrow{\cong} H^2(X', \mathbb{Z}).$$

A partial version of the Torelli theorem carries over: in [Ver13] Verbitsky proved that a similar statement involving birational Teichmüller spaces holds for HK manifolds.

- **(Birational projective HK)** A well-known fact about K3 surfaces is that they are all deformation equivalent[3] to each others. This was first proved by Kodaira in [Kod64], and it is a consequence of the fact that every regular surface with trivial canonical bundle can be deformed to the Fermat quartic. Moreover, if any two K3 are birational equivalent, they can be shown to be isomorphic: indeed, the minimal model of a surface of non-negative Kodaira dimension is unique. In higher dimension the situation changes, but the above statement can be replaced by the following:

  Theorem 1.3 (Essentially Theorem 4.6, [Huy99]). *Two birational projective irreducible symplectic manifolds are deformation equivalent and, hence, diffeomorphic.*

  The result was used to show that most of the known examples, with the exception of O'Grady's exceptional examples in dimension six and ten, are deformations of the two standard series provided by Hilbert schemes of points on K3 surfaces and generalized Kummer varieties. It was first shown with projective techniques for projective HK (under additional assumptions), and later extended to the current form using the same techniques employed in the proof of the Torelli theorem for HK.

In virtue of these properties, compact HK manifolds are indeed considered to be the closest higher dimension analogues of K3 surfaces. We will now proceed to discussing the main examples we are interested in.

2. The Hilbert scheme of points on a K3 surface

Let X be a complex K3 surface. We want to construct and study the moduli space parametrizing tuples of (not necessarily distinct) points on X. Such moduli space is known as the *Hilbert scheme of points* on X, and it will turn out to be a smooth, fine moduli space with a IHS structure. First, let us recall that if $\mathfrak{X}$ is a smooth complex projective variety, the *Hilbert functor*

$$\underline{\mathcal{H}ilb}^n : \mathrm{Sch}/\mathbb{C} \to \mathrm{Set} \tag{1}$$

is defined by

$$S \mapsto \{\text{closed subschemes } Z \subset \mathfrak{X} \times_{\mathbb{C}} S \text{ flat over } S \text{ with Hilbert polynomial } n\}.$$

We would like to find a *fine moduli space* for this moduli problem, i.e., find a complex scheme representing the moduli functor above. It is known that such moduli functor is representable, and we will denote by $X^{[n]}$ the scheme representing it. In the next subsections, we are going to describe such a complex scheme in a more direct way and study its geometric properties.

---

[3]Any two compact complex surfaces S_0 and S_1 are deformation equivalent if there are $\mathcal{S}$, $\mathcal{T}$ compact complex manifolds, $\pi : \mathcal{S} \to \mathcal{T}$ holomorphic map and$t_0, t_1 \in \mathcal{T}$ such that s_i is isomorphic to $\pi^{-1}(t_i)$ for $i = 1, 2$.

2.1. **The construction.** We are going to construct the Hilbert scheme of n points on X by first taking the n-th symmetric product of X, and by then resolving its singularities. More precisely, let us denote by $X^n := X \times \cdots \times X$ then n-fold product of X. The variety X^n parametrizes ordered n-tuples of not necessarily distinct points on X. The n-th symmetric group $\mathfrak{S}_n$ acts naturally on the product variety X^n by simply permuting the factors. The variety $X^{(n)} := X^n/\mathfrak{S}_n$ obtained by taking the quotient with respect to this action, often known as the n-th *symmetric product* of X, clearly parametrizes unordered n-tuples of points in X. It has quotient singularities along the images of the loci

$$\Delta_{ij} = \{(x_1, ..., x_n) \in X^n \text{ s.t. } x_i = x_j\},$$

which are precisely the fixed loci of the action of $\mathfrak{S}_n$. We denote by

$$\Delta = \bigcup_{i,j} \Delta_{ij}.$$

We can consider the structure morphism

$$X^{[n]} \xrightarrow{h} X^{(n)},$$

which can be seen explicitly as the map sending a zero-dimensional scheme $Z \subset X$ to its associated zero-cycle $|Z|$, and it is referred to as *Hilbert-Chow morphism.* It is possible to show that the morphism h is a resolution of singularities: the variety $X^{[n]}$ is hence smooth, and it parametrized zero dimensional subschemes of length n on the K3 surface X. Note that h is an isomorphism over $sm(X^{(n)})$, the smooth locus of $X^{(n)}$, i.e. the subset parametrizing cycles $x_0 + ... + x_n$ with pairwise distinct $x_i's$. The fibers of h over the singular locus $sing(X^{(n)})$ are positive dimensional. In order to have a better idea of what the singular locus might look like, let us take a closer look at $X^{[2]}$. Since the singular locus of $X^{(2)}$ consists of double points, the map h becomes the blow-up along the diagonal $sing(X^{(2)}) = \Delta_2 = \{(x, x) \mid x \in X\}$, and one can define

$$X^{[2]} := \mathrm{Bl}_{\Delta_2}(X^{(2)}).$$

The variety $X^{[2]}$ is stratified according to the dimensions of the fibers of h. There are two strata: an open stratum isomorphic to $sm(X^{(2)})$, and a closed stratum isomorphic to the projectivization of the tangent bundle of X. Any two points in the two strata look like figures 1a and 1b below:

(A) A reduced, length 2 subscheme

(B) A nonreduced, length 2 subscheme

By construction, the variety $X^{[n]}$ represent the Hilbert functor (1), and we will show in the next subsection that it is a compact hyperkähler manifold.

Remark 2.1. The Hilbert scheme of points $X^{[n]}$ can be alternatively seen as a moduli space of sheaves over X. Indeed, by taking a zero-dimensional subscheme $Z \subset X$ of length n to its ideal sheaf $\mathcal{I}_Z$, one has a morphism

$$X^{[n]} \xrightarrow{\cong} \mathcal{M}_H(1,0,-n),$$

where H is any polarization on X and $\mathcal{M}_H(1,0,-n)$ is the moduli space parametrizing torsion-free sheaves E on X of rank $r(E)=1$, first Chern class $c_1(E)=0$ and second Chern class $c_2(E)=n$. On the other side, given a semistable torsion free sheaf E of rank $r(E)=1$, first Chern class $c_1(E)=0$ and second Chern class $c_2(E)=n$, we can realize E as a subsheaf of $\mathcal{O}_X$: indeed, any torsion-free sheaf includes into its double dual $E \hookrightarrow E^{**}$. The double dual is a reflexive sheaf, so any singularities occur in codimension 3. In the surface case, we conclude E^{**} is a line bundle with trivial determinant, so must be $\mathcal{O}_X$.

2.2. $X^{[n]}$ **is compact HK.** In this section we are going to prove that the Hilbert scheme of points on a K3 surface is a compact hyperkähler manifold. We will construct a holomorphic symplectic form on the Hilbert scheme, then we will show that $h^{2,0}(X^{[n]})=1$ and, finally, that $X^{[n]}$ is simply connected.

- *The symplectic form:*[4]

$$\begin{array}{ccc} \mathrm{Bl}_\Delta X^n_* & \xrightarrow{\eta} & X^n_* \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ X^{[n]}_* & \xrightarrow{\rho} & X^{(n)}_* \end{array}$$

  If ω is a holomorphic symplectic form on X, one can define a two-form on the product X^n by

  $$\tilde{\omega} := \sum_{i=1}^{n} pr_i^* \omega,$$

  where $pr_i : X^n \to X$ is the projection to the i-th factor. Such two-form is clearly invariant under the action of the symmetric group on X^n, hence its pullback $\eta^*\tilde{\omega}$ also is, which implies that it comes then from a form τ on $X^{[n]}_*$ such that $\rho^*\tau = \eta^*\tilde{\omega}$. Now since the map ρ is a covering map ramified along the exceptional divisor E of η, one has

  $$\mathrm{div}(\rho^*\tau^n) = \rho^*\mathrm{div}(\tau^n) + E$$

  and since $\rho^*\tau = \eta^*\tilde{\omega}$ and the form $\tilde{\omega}$ is closed, the left hand side is equal to

  $$\mathrm{div}(\eta^*\tilde{\omega}^n) = \eta^*\mathrm{div}(\tilde{\omega}^n) + E = E.$$

  This implies that $\rho^*\mathrm{div}(\tau^n)=0$, so the form τ is a holomorphic symplectic form on $X^{[n]}_*$, which extends to $X^{[n]}$ by Hartog's theorem.
- $X^{[n]}$ *is simply connected*: We will now study the topology of $X^{[n]}$. We have the following:

  Proposition 2.2 ([Bea83], Lemma 1)**.** *Let $r \geq 2$.*
  (1) *The group homomorphism $h_* : \pi_1(X^{[n]}) \to \pi_1(X^{(n)})$ is bijective.*

[4] To see that $X^{[n]}$ admits a symplectic form, one could alternatively use the same argument as Example 2.1.2 in Sveta's notes, [Mak].

(2) *The group homomorphism* $\pi_* : \pi_1(X^n) \to \pi_1(X^{(n)})$ *is surjective, and its kernel is the subgroup of* $\pi_1(X^n)$ *generated by elements of the form* $(\sigma \cdot \gamma)\gamma^{-1}$, *with* $\sigma \in \mathfrak{S}_n$ *and* $\gamma \in \pi_1(X^n)$.

Indeed, by the short exact sequence for the fundamental group of a branched cover associated to the action of a group (cf. e.g. [Noo07], Theorem 9.1), one gets that

$$\pi_1(X^{(n)}) = (\mathfrak{S}_n \ltimes \pi(X)^r)/\text{stabilizers},$$

hence assertion (2) follows. For what concerns assertion (1), one can apply the same reasoning to $X_*^{(n)} = X_*^n/\mathfrak{S}_n$. Since the injection $X_*^n \hookrightarrow X^n$ induces an isomorphism of fundamental groups and so does $X_*^{(n)} \hookrightarrow X^{(n)}$. Also, neither the blow-up $X_*^{[n]} \to X_*^{(n)}$ nor the injection $X_*^{[n]} \hookrightarrow X^{[n]}$ change the fundamental group. In our case, since X is a K3 surface, $\pi_1(X) = 0$ implies $\pi_1(X^{[n]}) = 0$. Hence $X^{[n]}$ is simply connected.

- *Cohomology of* $X^{[n]}$: Let us now proceed to studying the Hodge structure on the cohomology of $X^{[n]}$. One has the following result:

Proposition 2.3 ([Bea83], Proposition 6). *Let X be a K3 surface. Then there exists an injective homomorphism* $i : H^2(X,\mathbb{C}) \to H^2(X^{[n]},\mathbb{C})$, *compatible with the Hodge structures, and one has*

$$H^2(X^{[n]},\mathbb{C}) = i(H^2(X,\mathbb{C})) \oplus \mathbb{C}[E].$$

Since the exceptional divisor class E is an algebraic class, hence of type $(1,1)$, it follows that $h^{1,1}(X^{[n]}) = h^{1,1}(X) + 1$ and that $h^{2,0}(X^{[n]}) = h^{2,0}(X) = 1$.

Remark 2.4. From the above results, one can deduce the second Betti number of the Hilbert scheme of points on a K3 surface X. One has, in fact:

$$b_2(X^{[n]}) = b_2(X) + 1 = 23.$$

The computation of the second Betti number for hyperkähler manifolds is crucial: since any two birational equivalent compact HK are also deformation equivalent by Theorem 1.3, one immediately has that any two HK are not birational if their second Betti numbers are different. This was originally used by O'Grady to show that his example OG10 was not birational to any of the known compact HK manifolds.

3. Rank two moduli spaces

In this section we want to give an example of a rank two moduli space which is naturally associated to a K3 surface of degree 8 in $\mathbb{P}^5$.

Let X be a degree 8 hypersurface in $\mathbb{P}^5$, given by the complete intersection of three quadrics Q_0, Q_1 and Q_2. Let H denote its hyperplane class in $\mathcal{O}_X(1)$. One can consider the *net of quadrics*[5] generated by Q_0, Q_1 and Q_2, i.e. the locus

$$\lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2 = 0$$

[5] It can be shown that, in fact, a net of quadrics in X is the locus of quadric passing through seven given points in X.

for $[\lambda_0, \lambda_1, \lambda_2] \in \mathbb{P}^2$. If we denote by $[X_0, ..., X_5]$ the coordinates of $\mathbb{P}^5$, and suppose that the equations of the quadrics are given by

$$Q_i = \sum_{j,k=1}^{5} a_i^{jk} X_j X_k \ , \ i = 1, 2, 3$$

then the zero locus of the determinant of the matrix

$$A = \begin{pmatrix} & \vdots & \\ \cdots & \lambda_1 a_1^{jk} + \lambda_2 a_2^{jk} + \lambda_3 a_3^{jk} & \cdots \\ & \vdots & \end{pmatrix}$$

is the vanishing locus of a degree six polynomial in $\lambda_1, \lambda_2, \lambda_3$, hence it is a sextic curve $C = V(\det A) \subset \mathbb{P}^2$ and it parametrize the degenerate quadrics in the net. If $rkA = 5$, moreover, such sextic is smooth. One denote by $\phi : M \to \mathbb{P}^2$ the degree two branched cover of $\mathbb{P}^2$ ramified along C: by the general theory of K3 surfaces (cf., e.g., Sveta's notes) M is a K3 surface. Our goal is to show that, under additional assumptions on X, the K3 surface M is in fact naturally isomorphic to a moduli space $\mathcal{M}$ of degree two sheaves on X.

3.1. **The correspondence between M and $\mathcal{M}$.** Let us first give a couple of generalities about moduli spaces of sheaves on K3 surfaces. if v is a primitive Mukai vector, and H is a polarization which is generic with respect to v, then the moduli space $\mathcal{M}_v(H)$ of H-stable sheaves with Mukai vector v is smooth, projective and of dimension $\langle v, v\rangle+2$. Mukai showed that if v is isotropic, i.e. $\langle v, v\rangle = 0$, then $\mathcal{M}_v(H)$ is in fact a K3 surface. We also have an explicit symplectic form on $\mathcal{M}_v(H)$. Recall that for any stable sheaf E in $\mathcal{M}_v(H)$, the tangent space $T_{[E]}\mathcal{M}_v(H)$ is isomorphic to $\mathrm{Ext}^1(E, E)$. Then the standard symplectic form is given by the pairing:

$$\mathrm{Ext}^1(E, E) \times \mathrm{Ext}^1(E, E) \longrightarrow \mathrm{Ext}^2(E, E) \overset{trace}{\longrightarrow} H^2(X, \mathcal{O}_X) \cong \mathbb{C},$$

where the first morphism is the *Yoneda product*, i.e. the morphism

$$(\{0 \to E \to F_1 \to E \to 0\} \ , \ \{0 \to E \to F_2 \to E \to 0\}) \mapsto \{0 \to E \to F_1 \to F_2 \to E \to 0\}.$$

In this special case, we focus on the moduli space $\mathcal{M} = \mathcal{M}(2, H, 2)$. The main result we are interested in is the following:

Theorem 3.1 ([IK08], Theorem 5.4 et al.)**.** *Let X be the complete intersection of three quadrics Q_0, Q_1 and Q_2 in $\mathbb{P}^4$ containing a line and such that the rank of the generic quadric in the net is 5, and let $\phi : M \to \mathbb{P}^2$ the degree two branched cover of $\mathbb{P}^2$ ramified along the smooth sextic parametrizing degenerate quadrics in the quadric net. Then the moduli space $\mathcal{M}(2, H, 2) \cong M$ is a fine moduli space and one has $\mathcal{M}(2, H, 2) \cong M \cong X$;*

The proof of this result is articulated in the following steps:

(1) First, we construct an explicit map

$$M \to \mathcal{M}.$$

We do this by exploiting the fact that a smooth quadric Q in the net is isomorphic to the Grassmannian $\mathrm{Gr}(2, 4)$ of two planes in $\mathbb{C}^4$ via the Plücker embedding. Therefore, one can consider the tautological sequence

$$0 \to S \to \mathcal{O}_Q^4 \to F \to 0$$

obtained by choosing an isomorphism $Q \cong \mathrm{Gr}(2,4)$. The correspondence relies on the fact that there is a point $x \in Q$ such that the bundle $I_Q = \mathbb{P}(S_x^\vee) \amalg \mathbb{P}(F_x)$ is isomorphic to *conic bundle*. One can show that if X contains a line and Q is a singular quadric, then $I_Q = \mathbb{P}(E_Q)$, where E_Q is a stable, rank two vector bundle on X with $c_1 = H$ and $c_2 = 2$. Therefore the correspondence is given by

$$M \to \mathcal{M}\ ,\ Q \mapsto E_Q.$$

Now, such correspondence can be shown to be injective. Since by the general theory both M and $\mathcal{M}$ are irreducible K3 surfaces, and since they inject into one another, one has $M \cong \mathcal{M}$

(2) The reason why X is isomorphic to M comes from Mukai's work. Indeed, if $\mathcal{M}$ is a fine moduli space, the universal family $\mathcal{E}$ on $X \times \mathcal{M}$ induces a cohomological Fourier-Mukai transform. Let π_X and $\pi_{\mathcal{M}}$ be the projections of $X \times \mathcal{M}$ onto the two factors. One can define the cycle

$$Z_{\mathcal{E}} = \left(\pi_X^* \sqrt{\mathrm{Td}_X} \cdot \mathrm{ch}(E^\vee) \cdot \pi_{\mathcal{M}}^* \sqrt{\mathrm{Td}_{\mathcal{M}}}\right).$$

Mukai showed that such a cycle induces an isometry

$$f_{\mathcal{E}} : H^*(X, \mathbb{Q}) \to H^*(\mathcal{M}, \mathbb{Q})\ ,\ w \mapsto \pi_{\mathcal{M}*}(Z_{\mathcal{E}} \cdot \pi_X^*(w))$$

which preserves the Hodge structures. Hence, by Torelli Theorem for K3 surfaces, $X \cong \mathcal{M}$.

4. O'Grady's example

Let X be a projective K3 surface and $\mathcal{M}(r, c_1, c_2)$ be the moduli space of semistable (with respect to the ample line bundle $\mathcal{O}_X(1)$) torsion-free sheaves on X of rank r, with first Chern class c_1 and second Chern class c_2. It is possible to show that if every semistable sheaf is actually stable, the moduli space $\mathcal{M}(r, c_1, c_2)$ is smooth and a compact HK manifold which is birational to $X^{[n]}$, the example we treated in the previous section. Here we want to treat the following case: take

$$(r, c_1, c_2) = (2, 0, c)$$

where c is even. Then there are sheaves which are strictly semistable: take any two zero dimensional subschemes Z and W of X with lenght $\ell(Z) = \ell(W) = \frac{c}{2}$, and consider the sheaf $I_Z \oplus I_W$. Such sheaf is always semistable, and it is possible to show that, if the polarization is generic, the only strictly semistable sheaves take this form. The moduli space $\mathcal{M}_c = \mathcal{M}(2, 0, c)$ is then singular, but when $c = 4$ we will show that there exists a symplectic resolution which is a compact HK manifold, not birational to any of the other known examples.

We will now give a sketch of the main ideas that go into the argument.

4.1. **Kirwan's desingularization.** We will first recall briefly the construction of the moduli space $\mathcal{M}_c$, which is going to be broken in a number of steps:

(1) **(Boundedness)**.Let P be the Hilbert polynomial of a sheaf with numerical invariants $v = (2, 0, c)$. The first important result is the following (see e.g. [HL97, Theorem 3.3.7]):

Theorem 4.1. *The family of torsion-free sheaves on X with Hilbert polynomial P is bounded, i.e. there exists a scheme of finite type S and a coherent sheaf of $\mathcal{O}_{S\times X}$-modules $\mathcal{F}$ such that the given family is contained in the set*

$$\{\mathcal{F}|_{Spec(k(s))\times X} \mid s \textit{ is a closed point in } S\}.$$

This is equivalent to say that there exists a coherent sheaf $\mathcal{H}$ on X such that every sheaf in the family can be realized as a quotient of $\mathcal{H}$, and it relies heavily by a theorem of Grauert-Mülich, which gives a uniform bound for the number of sections of a slope semistable sheaf, and by a result of Le Potier and Simpson, which gives an estimate of the depth of a slope semistable sheaf as an $\mathcal{O}_X$ module.

(2) **(Realize the semistable locus as an open subset of a Quot scheme)**. Given our boundedness result, we can realize each H-semistable sheaf F with numerical invariants v as a quotient of a common sheaf. Boundedness and Serre's vanishing imply the existence of an integer m such that F is m-regular, hence $F(m)$ is globally generated and $N+1 := h^0(F(m)) = P(m)$. Therefore, if we denote by $\mathcal{H} := H^0(F(m)) \otimes_{\mathbb{C}} \mathcal{O}_X(-m)$, then there is a surjection
$$\rho : \mathcal{H} \longrightarrow F.$$
Hence each F defines a closed point in the Quot scheme
$$[\rho : \mathcal{H} \longrightarrow F] \in \mathrm{Quot}(\mathcal{H}, P).$$
It can be shown that the locus Q_c of such points is open.

(3) **(Take a GIT quotient)**. The group $PGL(N) = \mathrm{Aut}(\mathcal{H})$ acts on $\mathrm{Quot}(\mathcal{H}, P)$ from the right by composition. Hence one can take the categorical quotient
$$\mathcal{M}_c = Q_c /\!\!/ PGL(N).$$
It can be shown that stability and semistability coincide with GIT stability and semistability with respect to the inearized line bundle. What we are really doing is then taking is a GIT quotient.

Kirwan's partial desingularization will be the GIT quotient of a variety obtained by successively blowing up points in the semistable locus: the idea is that properly semistable points will gradually disappear leaving only stable points. Since we want to take the GIT quotient of a blow-up, we will need to relate GIT stability on the blow-up to GIT stability on the variety itself. Crucial to this description will be the following result.

Let Y be a complex projective scheme, and let:

- G be a reductive group acting linearly on Y (i.e., the G-action has been lifted to an action on the line bundle $\mathcal{O}_Y(1)$),
- $V \subset Y$ a G-invariant closed subscheme,
- $\pi : \tilde{Y} := \mathrm{Bl}_V Y \longrightarrow Y$ be the blow-up of V,
- E the exceptional divisor of π.

Then G acts on $\tilde{Y}$ and also on the line bundle
$$D_\ell = \pi^*\mathcal{O}_Y(\ell) \otimes \mathcal{O}_{\tilde{Y}}(-E).$$
Moreover, one has the following

Theorem 4.2 (Kirwan, [Kir85]). *For $\ell \gg 0$ the semistable (resp. stable) locus $\tilde{Y}^{ss}(\ell)$ (resp. $\tilde{Y}^{s}(\ell)$) is independent of ℓ: we will denote it by $\tilde{Y}^{ss}$ (resp. $\tilde{Y}^{s}$). Then one has*

$$\pi(\tilde{Y}^{ss}) \subset Y^{ss} \text{ and } \pi^{-1}(Y^{s}) \subset \tilde{Y}^{s},$$

Hence π induces a morphism

$$\tilde{\pi} : \tilde{Y}/\!\!/G \to Y/\!\!/G.$$

If ℓ is also sufficiently divisible, then

$$\tilde{\pi} : \tilde{Y}/\!\!/G \cong \mathrm{Bl}_{V/\!\!/G}(Y/\!\!/G) \longrightarrow Y/\!\!/G$$

is the blow-up morphism.

We will now need a characterization of the semistable locus Q_c^{ss}, which will be key to the argument. To start with, we have the following:

Lemma 4.3 ([O'G99], Lemma 1.1.5). *A points $x = [F_x] \in Q_c$ is strictly semistable if and only if F_x fits into an exact sequence*

$$0 \to I_Z \to F_x \to I_W \to 0$$

where Z and W are zero-dimensional subschemes of length $\ell(Z) = \ell(W) = \frac{c}{2}$, and I_Z, I_W are their ideal sheaves. Furthermore, the orbit $PGL(N)x$ is closed in Q_c^{ss} if and only if the exact sequence above is split.

If one looks at the short exact sequence above, there are two possibilities: either the extension is split, or it is not split. Also, in both cases, one has that either Z and W coincide, or they do not coincide. Following these four cases, we will now define four loci in Q which will decompose the strictly semistable locus. Let

$$\Omega_Q^0 := \{x \in Q_c \mid F_x \cong I_Z \oplus I_Z,\ [Z] \in X^{[n]}\},$$
$$\Gamma_Q^0 := \{x \in Q_c \mid F_x \text{ is a nontrivial extension of } I_Z \text{ by } I_Z,\ [Z] \in X^{[n]}\},$$
$$\Sigma_Q^0 := \{x \in Q_c \mid F_x \cong I_Z \oplus I_W,\ [Z],[W] \in X^{[n]}, [Z] \neq [W]\},$$
$$\Lambda_Q^0 := \{x \in Q_c \mid F_x \text{ is a nontrivial extension of } I_Z \text{ by } I_W,\ [Z],[W] \in X^{[n]}, [Z] \neq [W]\},$$

and let $\Omega_Q, \Gamma_Q, \Sigma_Q, \Lambda_Q$ denote their respective closures in Q. Then, by Lemma 4.3, one has that

$$Q_c^{ss} \setminus Q_c^{s} = \Omega_Q^0 \amalg \Gamma_Q^0 \amalg \Sigma_Q^0 \amalg \Lambda_Q^0.$$

One then constructs Kirwan's desingularization in the following two steps:

- First, we take the blow-up:
$$\pi_R : R_c := \mathrm{Bl}_{\Omega_Q} Q_c \longrightarrow Q_c.$$
- Then we denote by $\Sigma_R \subset R$ the strict transform of Σ_Q, and we notice that $\Omega_Q \subset \Sigma_Q$. Then we take a second blow-up:
$$\pi_S : S_c := \mathrm{Bl}_{\Sigma_R} R_c \longrightarrow R_c.$$

The action of $PGL(N)$ lifts to actions on R and S respectively. Then, applying Theorem 4.2, we get a morphism:

$$\widehat{\mathcal{M}}_c := S_c/\!\!/PGL(N) \longrightarrow Q_c/\!\!/PGL(N) = \mathcal{M}_c$$

. The key result lies in the following:

Proposition 4.4 ([O'G99], Proposition 1.8.3). *When $c = 4$, $\widehat{\mathcal{M}}_c =: \widehat{\mathcal{M}}_4$ is a smooth desingularization of $\mathcal{M}_4$.*

It is possible to show that when $c \geq 6$, $\widehat{\mathcal{M}}_c$ has quotient singularities, but it is possible to produce a smooth desingularization of $\mathcal{M}_c$ by taking a third blow-up.

We will now show that there exists a two form on $\widehat{\mathcal{M}}_c$ which extends the standard Mukai's form on the smooth locus of $\mathcal{M}_c$. We will then show that one can suitably modify $\widehat{\mathcal{M}}_c$ to obtain a variety on which such two-form is symplectic.

4.2. **The Mukai-Tyurin form.** Recall that the Quot scheme $\mathrm{Quot}(\mathcal{H}, P)$ we considered at the beginning of this section is a fine moduli space, hence there exists a universal family $\mathcal{E}$ on $X \times \mathrm{Quot}(\mathcal{H}, P)$. With a slight abuse of notation, we denote again by $\mathcal{E}$ the restriction of such universal family to the open locus Q_c and by $\widetilde{\mathcal{E}}$ its pullback via $X \times S_c \to X \times Q_c$. We can then give the following definition:

Definition 4.5. The *Mukai-Tyurin form* $\omega_{MT} \in H^0(S_c, \Omega^2_{S_c})$ is defined as follows: if $v, w \in T_pS_c$

$$\langle \omega_{MT}(p), v \wedge w \rangle := \int_X \mathrm{Tr}(\kappa_{\widetilde{\mathcal{E}}}(p)(v) \cup \kappa_{\widetilde{\mathcal{E}}}(p)(w)) \wedge \omega$$

where ω is the symplectic form on X and $\kappa_{\widetilde{\mathcal{E}}}(p) : T_pS_c \longrightarrow \mathrm{Ext}^1(\widetilde{\mathcal{E}}_p, \widetilde{\mathcal{E}}_p)$ be the Kodaira-Spencer map at p.

It is possible to show that:

- the form ω_{MT} is $PGL(N)$-invariant,
- By applying Luna's étale slice theorem, the form ω_{MT} descends to a form $\widehat{\omega}_4$ on $\widehat{\mathcal{M}}_4$,
- the form $\widehat{\omega}_4$ on $\widehat{\mathcal{M}}_4$ is nondegenerate outside an explicit, distinguished locus.

In the next subsection, we will define a smooth contraction of $\widehat{\mathcal{M}}_4$ on which the two-form $\widehat{\omega}_4$ will be symplectic. Such contraction will be our smooth symplectic resolution.

4.3. **The symplectic resolution.** Let $Gr^\omega(2, T_X^{[2]})$ be the relative symplectic Grassmannian over $X^{[2]}$, with fiber $Gr^\omega(2, E_Z)$ over $[Z] \in X^{[2]}$, and let $\mathcal{A}$ be the tautological bundle over $Gr^\omega(2, T_X^{[2]})$. We denote by $\widehat{\Omega}_4$ the GIT quotient

$$\widehat{\Omega}_4 := \Omega_4 /\!\!/ PGL(N)$$

. It is possible to prove the following:

Proposition 4.6. *The locus $\widehat{\Omega}_4$ is isomorphic to $\mathbb{P}(S^2\mathcal{A})$. Under this isomorphism the map*

$$\widehat{\pi}|_{\widehat{\Omega}_4} : \widehat{\Omega}_4 \longrightarrow \Omega_4 \cong X^{[2]}$$

corresponds to the natural projection $\mathbb{P}(S^2\mathcal{A}) \to X^{[2]}$.

Let us look at this map fiberwise. For $[Z] \in X^{[2]}$, we set

$$\widehat{\Omega}_Z := \widehat{\pi}^{-1}([I_Z \oplus I_Z]).$$

By the above proposition, we get an isomorphism $\widehat{\Omega}_Z \cong \mathbb{P}(S^2\mathcal{A}_Z)$, where $\mathcal{A}_Z$ is the restriction of $\mathcal{A}$ to the fiber $Gr^\omega(2, E_Z)$. We define

- The class $D_Z \in N_1(\widehat{\Omega}_Z)$ to be the class of a line in a fiber of $\mathbb{P}(S^2\mathcal{A}_Z) \to Gr^\omega(2, E_Z)$,

- The class $D_4 \in N_1(\widehat{\mathcal{M}}_4)$ as
$$D_4 := \iota^Z_* D_Z,$$
where $\iota^Z : \widehat{\Omega}_Z \hookrightarrow \widehat{\mathcal{M}}_4$ is the inclusion.

One has, then, the following main result:

Proposition 4.7 ([O'G99], Proposition 2.0.2)**.** *The ray $\mathbb{R}^+ D_4$ is an extremal ray in the Mori cone $N_1(\widehat{\mathcal{M}}_4)$. The scheme $\widetilde{\mathcal{M}}_4$ obtained by contracting such ray is a smooth projective symplectic desingularization of $\mathcal{M}_4$.*

We will not prove this theorem (in fact, not even sketch the proof!), as the argument is extremely technical and involved. We will, indeed, give a sketch of the main ideas that go into the proof of it being a compact HK manifold which is not birational to the Hilbert scheme of points $K3^{[n]}$ for any n. We will follow three steps:

- **$\widetilde{\mathcal{M}}_4$ is irreducible and $h^{2,0}(\widetilde{\mathcal{M}}_4) = 1$.** The idea of the proof is as follows. We will use a high power of a naturally induced line bundle on the moduli space $\mathcal{M}_4$ to obtain a morphism $\widetilde{\mathcal{M}}_4 \to \mathbb{P}^N$. We will then show that such morphism is very close to being an embedding in a certain sense, hence the Lefschetz Hyperplane Section (LHS) Theorem can be applied. Let us first describe the naturally induced line bundle by sketching a very general construction.

Let (X, H) be a complex projective polarized curve or surface, $v \in K(X)$ be a class in the topological K-theory $K(X)$, and let us denote by $\mathfrak{M}_v$ the moduli space of Gieseker H-stable sheaves on X with Mukai vector v. Consider a Mukai vector w, orthogonal to v with respect to the Euler form $(v, w) := \chi(v \otimes w)$ on $K(X)$. There is a group homomorphism

$$\Theta : v^\perp \longrightarrow \mathrm{Pic}(\mathfrak{M}_v)\ ,\ w \mapsto \Theta_w \tag{2}$$

considered by Le Potier and Li. The *Theta line bundle* $\Theta_w \to \mathfrak{M}_v$ is obtained by a standard determinantal construction:

$$\Theta_w := \det \mathbf{R} p_*(\mathcal{E} \otimes q^* F)^{-1},$$

where $\mathcal{E}$ is the universal family of the moduli space $\mathfrak{M}_v$ (although the same construction works even if the moduli space is not fine), p and q are the two projections from $\mathfrak{M}_v \times X$ to the first, respectively the second factor and F is a sheaf on X of class v. In some cases of importance, the group homomorphism (2) is an isomorphism, hence the Picard group of the moduli space can be completely described in terms of Theta divisors. Le Potier and Li have also produced a Mukai vector a, dependent on an integer m, such that the corresponding Theta line bundle $\mathcal{L}_m$ is always base-point free for $m \gg 0$. Consider now the line bundle $\mathcal{L}_m$ on $\mathcal{M}_4$. Since $\mathcal{L}_m$ is bpf, there is a morphism

$$\phi_m : \mathcal{M}_4 \longrightarrow \mathbb{P}(H^0(\mathcal{M}_4, \mathcal{L}_m)^*),$$

and let $\widetilde{\phi} := \phi \circ \widetilde{\pi}$, where $\widetilde{\pi} : \widetilde{\mathcal{M}}_4 \longrightarrow \mathcal{M}_4$ is the desingularization constructed above. It is possible to show that the map $\widetilde{\phi}$ is *semi-small*, i.e., it behaves well with respect to a certain stratification. Hence one has the following result, which is an immediate consequence of the generalized LHS theorem [GM88]:

Proposition 4.8 ([O'G99], Corollary 1.2.6)**.** *Let $\Lambda \subset \mathbb{P}^N$ be a linear subspace of codimension at most c. The map*

$$H^q(\widetilde{\mathcal{M}}_4, \mathbb{Z}) \longrightarrow H^q(\widetilde{\phi}^{-1}\Lambda, \mathbb{Z})$$

induced by the inclusion is an isomorphism for all $q \leq (9 - c)$.

By choosing a special subspace Λ, it is also possible to show that $b_0(\widetilde{\mathcal{M}}_4) = h^{2,0}(\widetilde{\mathcal{M}}_4) = 1$, hence $\widetilde{\mathcal{M}}_4$ is connected.

- $\widetilde{\mathcal{M}}_4$ **is simply connected**. The argument here breaks in two pieces:
  (1) First, one shows that $\widetilde{\mathcal{M}}_4$ is birational to an open subset $\mathcal{J}$ of the Jacibian fibration parametrizing (stable) degree-six line bundles on curves in $|\mathcal{O}_X(2)|$. More specifically, let L be such a line bundle, and i_*L its extension to X. We denote by $\mathcal{J}^0$ the locus of L on smooth curves which are globally generated, hence the evaluation map $H^0(L) \otimes \mathcal{O}_X \longrightarrow i_*L$ is surjective. Let E be the sheaf on X fitting into the short exact sequence

  $$0 \longrightarrow E \longrightarrow H^0(L) \otimes \mathcal{O}_X \longrightarrow i_*L \longrightarrow 0$$

  and set $F := E(1)$. It is possible to show that F is a stable rank two vector bundle with $c_1(F) = 0$ and $c_2(F) = 4$. Then one has a birational isomorphism:

  $$\mathcal{J}^0 \longrightarrow \mathcal{M}_4 \ , \ L \mapsto [F]$$

  which extends to a birational map $\Phi : \mathcal{J} \dashrightarrow \widetilde{\mathcal{M}}_4$. The map Φ, moreover, induce a surjection $\pi_1(\mathcal{J}) \longrightarrow \pi_1(\widetilde{\mathcal{M}}_4)$
  (2) Show that $\mathcal{J}$ is simply connected. this is done by a simple homotopy exact sequence argument which we will not show.
- $b_2(\widetilde{\mathcal{M}}_4) \geq 24$. This is done by producing a 24-dimensional subspace of $H^2(\widetilde{\mathcal{M}}_4, \mathbb{Q})$. Again, we will not show the proof.

As a final remark, it is possible to show that the same method does not work on $\mathcal{M}_c$ when $c \geq 6$: the Mukai-Tyurin form does not extend to give a symplectic resolution.

References

[AS15] E. Arbarello and G. Saccà. Singularities of moduli spaces of sheaves on K3 surfaces and Nakajima quiver varieties. *ArXiv e-prints*, May 2015.

[Bea83] A. Beauville. Variétés kähleriennes dont la première classe de chern est nulle. *J. Differential Geom.*, 18:755–782, 1983.

[Fuj87] Akira Fujiki. On the de Rham cohomology group of a compact Kähler symplectic manifold. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 105–165. North-Holland, Amsterdam, 1987.

[GM88] M. Goresky and R. MacPherson. *Stratified Morse theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3 Folge. Band 10. A series of modern surveys in mathematics. Springer, 1988.

[HL97] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves.* Aspects of Mathematics, E31. Friedr. Vieweg & Sohn, Braunschweig, 1997.

[Huy99] Daniel Huybrechts. Compact hyper-Kähler manifolds: basic results. *Invent. Math.*, 135(1):63–113, 1999.

[IK08] C. Ingalls and M. Khalid. Rank Two Sheaves on K3 Surfaces: A Special Construction. *ArXiv e-prints*, March 2008.

[Kir85] Frances Clare Kirwan. Partial desingularisations of quotients of nonsingular varieties and their betti numbers. *Annals of Mathematics*, 122(1):41–85, 1985.

[Kod64] K. Kodaira. On the structure of compact complex analytic surfaces, i. *American Journal of Mathematics*, 86(4):751–798, 1964.

[Mak] S. Makarova. General introduction to $K3$ surfaces. `http://www.northeastern.edu/iloseu/Sveta_S16.pdf`.

[Noo07] B. Noohi. Fundamental groups of topological stacks with slice property. *ArXiv e-prints*, October 2007.

[O'G99] Kieran G. O'Grady. Desingularized moduli spaces of sheaves on a $K3$. *J. Reine Angew. Math.*, 512:49–117, 1999.

[O'G03] Kieran G. O'Grady. A new six-dimensional irreducible symplectic variety. *J. Algebraic Geom.*, 12(3):435–505, 2003.

[Rap08] Antonio Rapagnetta. On the Beauville form of the known irreducible symplectic varieties. *Math. Ann.*, 340(1):77–95, 2008.

[Ver13] Misha Verbitsky. Mapping class group and a global Torelli theorem for hyperkähler manifolds. *Duke Math. J.*, 162(15):2929–2986, 2013. Appendix A by Eyal Markman.