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P. Achar - Lecture 2

Plan: 1. Convolution
2. Lusztig's q-analogue
3. Geometric Satake
4. MV cycles

During the 1st hour, we'll use the top. version of Gr, i.e. def'n ⑲ (for GL_n) or ⑮ from previous lecture, that is

$$Gr = G(K)/G(\mathcal{O}), \quad G \text{ conn. red've gp.}/\mathbb{C}.$$

<u>Fact</u> $G(\mathcal{O})$-orbits on $Gr \longleftrightarrow X_*^+$ = dominant coweights
$$Gr_\lambda \longleftarrow\!\shortmid \lambda$$

I - Convolution

Start with $\mathcal{F}_1, \mathcal{F}_2$, $G(\mathcal{O})$-equivariant sheaves or functions (constant on orbits and nonzero on finitely many orbits)

<u>Convolution diagram</u>

$$Gr \tilde{\times} Gr \;\; \| \;\; \text{"convolution space"}$$

$$\begin{tikzcd}
Gr \times Gr & G(\mathbb{K}) \times Gr \arrow[l, "p"'] \arrow[r, "q"] & G(\mathbb{K}) \times^{G(\mathcal{O})} Gr \arrow[r, "m"] \arrow[d, "pr_1"] & Gr \\
 & & Gr &
\end{tikzcd}$$

p: quotient map on 1st factor

$G(\mathcal{O})$ ↻ "middle" action on $G(\mathbb{K}) \times Gr$: $g\cdot(x,y) = (xg^{-1}, gy)$. Note that it's free!

$(x,y) \sim (xg^{-1}, gy)$ $\qquad m(x,y) = xy$

$pr_1 : (x,y) \mapsto xG(\mathcal{O})$

<u>Rmk</u> 1) pr_1 makes $Gr \tilde{\times} Gr$ into a Gr-bundle over Gr, 2) $pr_1 \times m : Gr \tilde{\times} Gr \to Gr \times Gr$ is iso!!!

"Sheaf" = constructible sheaf

Definition of convolution

Step 0 Form the sheaf/function $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ on $Gr \times Gr$

Step 1 Pullback along p

$$p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$$

(for functions, this is just $(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \circ p$)

Step 2 Descend along q.

Function version: $\exists!$ function $\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2 : Gr \tilde{\times} Gr \to \mathbb{C}$ s.t. $q^*(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2) = p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$

This is so because $p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ is constant along middle $G(\mathcal{O})$-orbits.

Sheaf version

Thm $q^* : Sh(Gr \tilde{\times} Gr) \xrightarrow{\sim} Sh_{G(\mathcal{O})}(G(\mathbb{K}) \times Gr)$

equiv for the middle action

So we define

$$\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2 := (q^*)^{-1} p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2).$$

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Step 3: Pushforward/integration along m.

Sheaf version: $\mathcal{F}_1 * \mathcal{F}_2 := m_*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$

derived functor

Function version

$$(\mathcal{F}_1 * \mathcal{F}_2)(x) = \int_{m^{-1}(x)} \mathcal{F}_1 \boxtimes \mathcal{F}_2$$

Need a bit more information to make sense of the integral

Easiest way: Replace Gr with an $\mathbb{F}_q$-version, $Gr = G(\mathbb{F}_q((t)))/G(\mathbb{F}_q[[t]])$

In this setting, $\mathcal{F}_1, \mathcal{F}_2$ compactly supported $\Rightarrow$ supp $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ is a finite set, and $\int$ means Σ.

Exercise The convolution product is associative, but it is not if we use pr_1 instead of m.

in the sheaf version this satisfies the pentagon axiom.

Conclusion 1) $D^b_{G(\mathcal{O})}(Gr) =$ ~~[illegible]~~ $G(\mathcal{O})$-eq. derived cat-y of sheaves is a monoidal cat-y under $*$.

2) The vect. sp of $G(\mathcal{O})$-eq. compactly supp. functions on Gr is a ring under $*$, called the spherical Hecke algebra. $\mathcal{H}_{sph}$

II Satake isomorphism

Recall X_* = set of coweights, a free ab' group

For GL_n, $X_* = \mathbb{Z}^n$

Let $\mathbb{C}[X_*]$ be the group ring, with elements written as

$$\sum_{\lambda \in X_*} c_\lambda e^\lambda$$

<u>Thm</u> (Satake, 1963)

$$\mathcal{H}_{sph} \xrightarrow{\sim} \mathbb{C}[X_*]^W \quad (W = \text{Weyl group})$$

<u>Note</u> True for Euler char version of $\int_{m^{-1}(x)}$ as well

Corollary/Surprise: $\mathcal{H}_{sph}$ is commutative!

Some bases for $\mathcal{H}_{sph}$:

1) Let $\lambda \in X_*^+$, $c_\lambda : Gr \to \mathbb{C}$ - the indicator function of Gr_λ

$\{c_\lambda \mid \lambda \in X_*^+\}$ is a basis for $\mathcal{H}_{sph}$.

2) Let $a_\lambda := \sum_{\mu \in W\lambda} e^\mu$. Then $\{a_\lambda \mid \lambda \in X_*^+\}$ is a basis for $\mathbb{C}[X_*]^W$

Emotional obs: These bases don't match under Satake!

Actually, they don't match under <u>any</u> isomorphism! (the structure constants don't match), even for $\mathfrak{sl}_2$.

In type A, the c_λ correspond to Hall-Littlewood polynomials

<u>3rd basis</u>:

$$X_*^+ = \text{dom. coweights for } G = \underbrace{\text{dom. weights for } G^\vee}_{} \longleftrightarrow \mathrm{Irr}(G^\vee)$$

Langlands dual

$$\lambda \longleftrightarrow L(\lambda)$$

So we may form the character, $\mathrm{ch}\,L(\lambda) := \sum_\mu \dim L(\lambda)_\mu e^\mu$

$$= a_\lambda + \sum_{\substack{\mu<\lambda \\ \mu \in X_*^+}} \dim L(\lambda)_\mu a_\mu :$$

So $\{\mathrm{ch}\,L(\lambda) \mid \lambda \in X_*^+\}$ forms a basis of $\mathbb{C}[X_*]^W$. What does this correspond to in $\mathcal{H}_{sph}$?

III. Lusztig's q-analogue of the weight multiplicity

To answer this question, we need to let q vary. $\lambda \in X_*^+$, $\mu \in X_*$

<u>Lusztig</u>: defined $M_\lambda^\mu(q) \in \mathbb{Z}[q]$ · a q-deformation of Kostant's multiplicity formula

$$M_\lambda^\mu(1) = \dim L(\lambda)_\mu.$$

The definition is combinatorial, see the exercises.

Now let $IC_\lambda :=$ simple perverse sheaf supported on $\overline{Gr_\lambda}$.

$IC_\lambda|_x$ is a chain cplx of vect sp, its cohomology only depends on the $G(\mathcal{O})$-orbit of x

<u>Thm</u> (Lusztig, 1983) If $\lambda, \mu \in X_*^+$, then

$$M_\lambda^\mu(q) = \sum_{i \geq 0} \dim H^{\dim Gr_\mu - \dim Gr_\lambda + i}(IC_\lambda|_x)\, q^{i/2}$$

$x \in Gr_\mu$

<u>Consequences of Lusztig's paper</u>

1) $H^i(IC_\lambda|_x)$ obeys a parity-vanishing condition.

2) $M_\lambda^\mu(q)$ has non-negative coeffs. if <u>both</u> $\lambda, \mu \in X_*^+$

3) Note

$IC_\lambda * IC_\mu$ corresponds to $\left(\sum_\nu M_\lambda^\nu(q) e^\nu\right) * \left(\sum_\nu M_\mu^\nu(q) e^\nu\right)$
Lusztig computed this, and the answer is a sum of expressions of the same form. In fact, suppose

$$L(\lambda) \otimes L(\mu) = \bigoplus L(\nu_i).$$

Then (Lusztig)

$$IC_\lambda * IC_\mu = \bigoplus IC_{\nu_i}$$

(Note: no shifts!!)
$\Rightarrow$ $*$ is an exact functor for perverse sheaves

hypercohomology

<u>Thm</u> (Lusztig, 1983) $\dim \mathbb{H}^\bullet(IC_\lambda) = \dim L(\lambda)$

So we start hoping :

$$\begin{tikzcd}
(\mathrm{Perv}_{G(\mathcal{O})} Gr, *) \arrow[rr, "\sim"] \arrow[dr, "\mathbb{H}^\bullet"'] & & (\mathrm{Rep}(G^\vee), \otimes_{\mathbb{C}}) \arrow[dl, "\text{forget}"] \\
& \mathrm{Vect}_{\mathbb{C}} &
\end{tikzcd}$$

Note that the formulas in 3) imply

$$IC_\lambda * IC_\mu \simeq IC_\mu * IC_\lambda$$

this exists, but is it natural?

In fact, this naturality is the main problem. If we knew this, then Tannakian formalism tells us that $\mathrm{Perv}_{G(\mathcal{O})}(Gr)$ with $\mathrm{Rep}(G')$ for some alg. gp. G'. After this, it takes a bit more work to show $G' \cong G^\vee$. This is now a thm of Mirkovic-Vilonen.

IV Fusion product

Recall from last time

Thm/Defn 2c: Gr_G represents the functor

$$R \mapsto \left\{ (\mathcal{L}, \beta) \;\middle|\; \begin{array}{l} \mathcal{L} \text{ is an } R\text{-fam of } G\text{-bundles on } \mathbb{A}^1 \\ \beta : \mathcal{L}|_{\mathbb{A}^1 \setminus x} \xrightarrow{\sim} \mathcal{L}^\circ|_{\mathbb{A}^1 \setminus x} \end{array} \right.$$

Now we can take 2 pts in $\mathbb{A}^1$, and let them vary.

The fusion space, Fus, is the ind-scheme representing the functor

$$R \mapsto \left\{ (x_1, x_2, \mathcal{L}, \beta) \;\middle|\; \begin{array}{l} x_1, x_2 \in \mathbb{A}^1 \\ \mathcal{L} \text{ - } R\text{-family of } G\text{-bundles on } \mathbb{A}^1 \\ \beta : \mathcal{L}|_{\mathbb{A}^1 \setminus \{x_1, x_2\}} \xrightarrow{\sim} \mathcal{L}^0|_{\mathbb{A}^1 \setminus \{x_1, x_2\}} \end{array} \right\}$$

The fusion diagram

$$\begin{tikzcd} Gr \times \mathbb{A}^1 \arrow[r, hook, "i"] \arrow[d, "\pi"] & Fus \arrow[d, "\pi"] & Gr \times Gr \times U \arrow[l, hook', "j"'] \arrow[d] \\ \mathbb{A}^1 \arrow[r, hook, "\text{diagonal}"] & \mathbb{A}^2 & U = \mathbb{A}^2 \setminus \text{diagonal} \arrow[l, hook'] \end{tikzcd}$$

$$(x_1, x_2, \mathcal{L}, \beta) \mapsto (x_1, x_2)$$

Thm (Mirković-Vilonen) Let $\mathcal{F}_1, \mathcal{F}_2 \in Perv_{G(\mathcal{O})} Gr$. Then

$$\mathcal{F}_1 * \mathcal{F}_2 \simeq i^* j_{!*} (\underbrace{\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathbb{C}_U}_{\text{perv. sheaf on } Gr \times Gr \times U}) \Big|_{Gr \times \text{your fav pt}}$$

Note $(\mathcal{F}_1 * \mathcal{F}_2) \boxtimes \mathbb{C}_{\mathbb{A}^1} = i^* j_{!*} (\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathbb{C}_U)$

Now, $\mathbb{A}^2$ has an automorphism, just swap x_1, x_2

On the right side of the fusion diagram induces swapping of $\mathcal{F}_1, \mathcal{F}_2$

On the left side it induces the identity

So we get a natural iso $\mathcal{F}_1 * \mathcal{F}_2 \simeq \mathcal{F}_2 * \mathcal{F}_1$.

<u>Note</u> No easy function analogue of this thm! In particular, this does not give a proof of commutativity of $\mathcal{H}_{sph}$