P. Achar's lectures. 02/21

Affine Grassmannian

Plan: 1.- What is it? (4 or 5 definitions, from elementary to fancy)
2.-(Thursday) Geometric Satake
3.-(Friday) Refinements, application to rep. th of quantum gps & red-ve groups in positive char.

Exercises: In Ivan's webpage!

I- Lattices

Notation: $\mathbb{K} := \mathbb{C}((t))$ (the field of formal Laurent series)

$\cup\!\!|$

$\mathbb{O} := \mathbb{C}[[t]]$ (the ring of formal power series)

Rmks 1.- $\mathbb{O}^\times$ = power series $\sum_{i\geq 0} a_i t^i$ w/ $a_0 \neq 0$

2.- $\mathbb{K}$ is a field

Def-n A lattice in $\mathbb{K}^n$ is a free $\mathbb{O}$-submodule of rank n. We typically denote this by $\mathcal{L}$.

Examples $\mathcal{L}^0 := \mathbb{O}^n$, the standard lattice

More generally, take the $\mathbb{O}$-span of any basis.

e.g. $n=2$

$$\mathcal{L}_1 = \mathrm{span}_{\mathbb{O}} \left\{ \begin{bmatrix} t^{-5} + e^t \\ t\sin t \end{bmatrix}, \begin{bmatrix} 1 \\ t^3 \end{bmatrix} \right\}$$

Def-n 1a The affine Grassmannian for GL_n is the set of lattices in $\mathbb{K}^n$. We denote it by Gr or Gr_{GL_n}

Goal Equip Gr w/ a topology

<u>Tools</u> 1) <u>Valuation</u> Suppose $\mathcal{L} = \mathrm{span}_{\mathcal{O}}\{\vec{v}_1, \ldots, \vec{v}_n\}$. Take

$$\det[\vec{v}_1 | \cdots | \vec{v}_n] \in \mathbb{K}^\times$$

This is not uniquely determined by $\mathcal{L}$ (ie depends on the chosen basis)
BUT the smallest power of t in it is well-defined

$$\nu(\mathcal{L}) := \min\{n \mid t^n \text{ occurs in } \det(\text{basis})\}$$

e.g.

$$\nu(\mathcal{L}^\circ) = 0$$
$$\nu(\text{the other example}) = -2$$

2) <u>Comparison w/ standard lattice</u>. In the example we have

$$\underset{\substack{\uparrow \\ \text{tiny bit} \\ \text{of work}}}{t^3\mathcal{L}^\circ \subseteq} \mathcal{L}_1 \underset{\substack{\uparrow \\ \text{obvious}}}{\subseteq} t^{-5}\mathcal{L}^\circ$$

<u>Lemma</u> $\forall \mathcal{L}\ \exists\ a \le b$ s.t. $t^b\mathcal{L}^\circ \subseteq \mathcal{L} \subseteq t^a\mathcal{L}^\circ$

Now define

$$Gr^{k[a,b]} := \left\{\text{lattices } \mathcal{L} \;\middle|\; \begin{array}{c}\nu(\mathcal{L}) = k \\ t^b\mathcal{L}^\circ \subseteq \mathcal{L} \subseteq t^a\mathcal{L}^\circ\end{array}\right\}$$

<u>WARNING</u> $Gr^{k[a,b]}$ might be empty

Obviously,

$$Gr = \bigcup_{k,a,b} Gr^{k[a,b]}$$

Note that $Gr^{k[a,b]} \hookrightarrow Gr^{k[a',b']}$ if $a' \le a \le b \le b'$.

Also note that

$$Gr^{k[a,b]} \hookrightarrow \left\{ \begin{array}{l} \text{subspaces of } t^a L^0/t^b L^0 \ (= \mathbb{C}^{(b-a)n}) \\ \text{of } \dim = nb-k \end{array} \right\}$$

exercise

$$L \mapsto L/t^b L^0$$

This map is not surjective, in general. The image consists of t-stable subspaces.

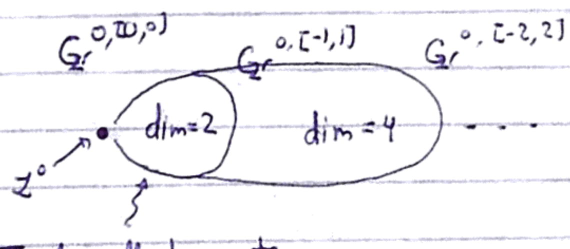
So $Gr^{k[a,b]}$ is embedded in the ordinary (ie not affine) Grassmannian $Gr(nb-k, (b-a)n)$ which is familiar from alg. top. (it's a compact mfld) or from alg. geom. (it's a proj-ve vty). It sits via the Plücker equations in $\mathbb{P}^{\dim}$

Equip $Gr^{k[a,b]}$ w/ subspace topology

Thm This data equips Gr w/ the structure of an ind-(proj-ve variety)

Part of the content of the thm says that $Gr^{k[a,b]} \hookrightarrow Gr$.

Example
$n=2$, $k=0$



Topologically homeo to $(S^2 \times S^2)/(pt \times S^2)$

In general, $Gr^{0[-a,a]}$ is a singular proj-ve vty of dim = $2a$

④

II. Orbits $GL_n(\mathbb{K}) \curvearrowright \mathbb{K}^n$ sends lattices to lattices and so it acts transitively on $\underline{Gr}$.

The stabilizer of $\mathcal{L}^\circ$ is $GL_n(\mathbb{O})$.

Thus, we get a bijection

$$\underline{Gr} \longleftrightarrow GL_n(\mathbb{K})/GL_n(\mathbb{O})$$

Now look at $GL_n(\mathbb{O}) \curvearrowright \underline{Gr}$. This is not transitive, and it is easy to see that

- preserves valuations
- preserves comparison w/ std. lattice

So $\underline{Gr}^{k,[a,b]}$ is a union of $GL_n(\mathbb{O})$-orbits.

Thm (Exercise)

$$\begin{matrix} GL_n(\mathbb{O})\text{-orbits} \\ \text{on } \underline{Gr} \end{matrix} \longleftrightarrow \left\{ \begin{matrix} (a_1, \ldots, a_n) \in \mathbb{Z}^n \\ a_1 \geq \cdots \geq a_n \end{matrix} \right\}$$

orbit containing

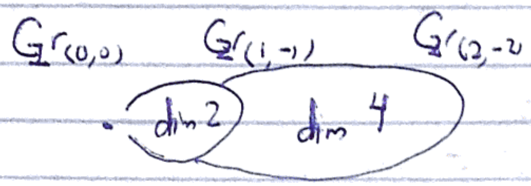
$$\text{span}_{\mathbb{O}} \left\{ \begin{bmatrix} t^{a_1} \\ \\ \\ \end{bmatrix}, \begin{bmatrix} \\ t^{a_2} \\ \\ \end{bmatrix}, \ldots \begin{bmatrix} \\ \\ t^{a_n} \end{bmatrix} \right\} \longleftarrow\!\shortmid (a_1, \ldots, a_n)$$

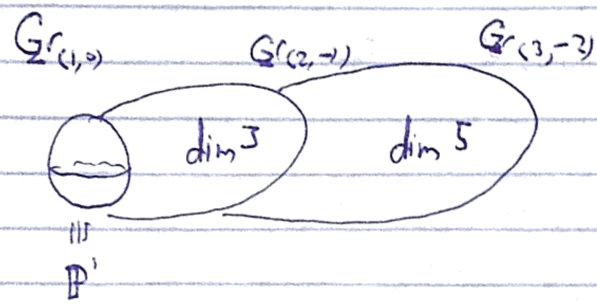
The proof is a version of Gauss-Jordan elimination

$\underline{Gr}_\lambda$ = the orbit associated to $\lambda = (a_1, \ldots, a_n)$

⑤

$n=2$





<u>Total picture for GL_2</u>

- Valuation is constant on each connected comp.
  - Conn. comp. $\longleftrightarrow \mathbb{Z}$
  - All the "even" components look like (= are $GL_n(\mathcal{O})$-eq. iso. to) the 0^{th} component
  - All the "odd" components look like the 1^{st} comp.

<u>For GL_n</u>

We still have

$$\text{conn. comp} \xleftrightarrow{\text{valuation}} \mathbb{Z}$$

But

$$\text{Isoclasses of conn. comp.} \longleftrightarrow \mathbb{Z}/n\mathbb{Z}$$

⑥

<u>Closed orbits</u> $\longleftrightarrow$ miniscule coweights, ie
$(a_1, \ldots, a_n)$ s.t $|a_i - a_j| \leq 1$.

Example of miniscule coweights

$$(0,0,\ldots,0)$$
$$(1,0,\ldots,0) =: \varpi_1$$
$$(1,1,0,\ldots,0) =: \varpi_2$$
$$\vdots$$
$$(1,1,\ldots,1,0) =: \varpi_{n-1}$$

$$\mathrm{Gr}_{\varpi_k} \overset{\text{orbit of}}{=} \mathrm{span}\left\{\begin{bmatrix} t \\ \\ \end{bmatrix}, \begin{bmatrix} t \end{bmatrix}, \ldots \begin{bmatrix} 1 \end{bmatrix}\right\}$$

$\cup|_0$
$t\mathcal{L}^0$

$\uparrow$ valuation k

$$\begin{tikzcd} \mathrm{Gr}_{\varpi_k} \arrow[d, hook, "\text{Exercise } =" '] \\ \mathrm{Gr}^{k,[0,1]} \arrow[d, hook, "\text{Exercise } =" '] \\ \mathrm{Gr}(n-k, n) \end{tikzcd}$$

So Gr_{ϖ_k} = the ordinary Grassmannian $\mathrm{Gr}(n-k, n)$.

III: General groups

$G :=$ connected reductive group$/\mathbb{C}$ (e.g. GL_n, SL_n, SO_n, Sp_{2n}, ...)

Def-n 16 $Gr_G := G(\mathbb{K})/G(\mathbb{O})$

This is just a set. We need a topology on it

Thm $G(\mathbb{O})$-orbits on $Gr_G \longleftrightarrow X_*^+(T)$ = dominant coweights (max'l torus)

$Gr_\lambda \longleftarrow\!\!\mapsto \lambda$

Defn $\overline{Gr_\lambda} := \bigcup_{\mu \le \lambda} Gr_\mu$

$\mu \le \lambda$: usual partial order on X_*, i.e. $\mu \le \lambda$ if $\lambda - \mu = \mathbb{Z}$ pos. coroots.

Note About Gr_λ, if we have $\lambda: \mathbb{C}^\times \to T$, we may think of this as an element $t^\lambda \in T(\mathbb{K})$, and Gr_λ is the orbit of the coset $t^\lambda G(\mathbb{O})$

Thm Each $\overline{Gr_\lambda}$ admits the structure of a proj-ve vty. Together they equip Gr_G w/ the structure of an ind-(proj-ve vty)

Proof Sketch Embed $\overline{Gr_\lambda} \hookrightarrow \mathbb{P}^{\text{smth}}$, using structure/rep. th. of Kac-Moody groups (see Kumar's book)

For classical gps, there are "lattice"-like descriptions of Gr. Some of these are in the exercises

Example 1) For a torus T, $Gr_T = X_*(T)$ a countable, discrete set

$Gr_{\mathbb{G}_m} = Gr_{GL_1} = \mathbb{Z}$

2) Gr_{SL_n} = lattices of valuation 0 (so this is connected)

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3) Gr_{PGL_2} has 2 connected components (like the isom. classes for GL_2)

<u>Note</u> $SL_2 \twoheadrightarrow PGL_2$, but $Gr_{SL_2} \hookrightarrow Gr_{PGL_2}$.

<u>IV</u> <u>Scheme version for GL_n</u>

<u>Def 2a</u> Gr_{GL_n} is the ind-scheme that represents the functor

$$\underset{\text{a } \mathbb{C}\text{-algebra}}{R} \longmapsto \text{set of } \underbrace{R[[t]]\text{-lattices in } R((t))^n}_{\substack{\text{proj-ve } R[[t]]\text{-submodule} \\ \text{of } R((t))^n \text{ that generates} \\ R((t))^n \text{ when we invert } t}}$$

<u>Thm</u> This functor is represented by an ind. limit of proj-ve schemes / $\mathbb{C}$

This is due to Beauville, Laszlo, Sorger, ...

A reference is lecture notes by Xinwen Zhu.

<u>Observation</u> $\mathbb{C}$-pts of (2a) = defn (1a)

<u>But</u> Now we can also look at $\underbrace{\mathbb{C}[\varepsilon]/(\varepsilon^2)}_{= R}$-pts of Gr_{GL_n}

$$R\text{-pts of } Gr_{GL_1} = \{ \text{span}_{R[[t]]}([c\varepsilon t^{-1}+1]), c \in \mathbb{C} \}$$

Gr_{GL_1} is <u>not</u> reduced (Exercise)

V. Scheme version for general G

Notation The formal disc $D := \operatorname{Spec} \mathbb{C}[[t]]$
punctured formal disc $D^\times := \operatorname{Spec} \mathbb{C}((t))$.

comm. C-alg   symmetric, rigid, etc

Defn An R-family of G-bundles on D is a $\otimes$-functor

$$\underset{\substack{\text{fin. dim'l} \\ G\text{-rep}}}{\operatorname{Rep}(G)} \xrightarrow{\mathcal{L}} \{\text{f.g. proj-ve } R[[t]]\text{-modules}\}$$

An R-family of G-bundles on $D^\times$ is defined completely analogously

If $\mathcal{L}$ is an R-family of G-bundles on D, we may form

$$\mathcal{L}|_{D^\times} : \underset{\operatorname{Rep} G}{V} \longmapsto R((t)) \otimes_{R[[t]]} \mathcal{L}(V)$$

Defn The standard (or trivial) family

$$\mathcal{L}^\circ : V \longmapsto V \otimes_{\mathbb{C}} R[[t]]$$

Defn (2b) Gr_G is the ind-scheme that represents the functor

$$R \longmapsto \left\{ (\mathcal{L}, \beta) \;\middle|\; \begin{array}{l} \mathcal{L} \text{ is an } R\text{-fam. of } G\text{-bundles on } D \\ \beta : \mathcal{L}|_{D^\times} \xrightarrow{\sim} \mathcal{L}^\circ|_{D^\times} \end{array} \right\} \Big/ \text{iso.}$$

Colloquially: "G-bundles on D, trivialized on $D^\times$"

Thm This functor is represented by an ind-scheme (Ref: Zhu's notes)

<u>Prop-n</u> For GL_n, def-ns 2a & 2b agree

<u>Proof sketch</u> Let's define a map $Gr^{2b} \longrightarrow Gr^{2a}$. Given $(\mathcal{L}, \beta) \in Gr^{2b}$, take $\mathcal{L}(\mathbb{C}^n)$, where $\mathbb{C}^n$ is the defining rep-n of GL_n.

$$\begin{tikzcd}
\mathcal{L}(\mathbb{C}^n) \arrow[d, hook] & \\
R((t)) \underset{R[[t]]}{\otimes} \mathcal{L}(\mathbb{C}^n) \arrow[r, "\cong", "\beta"'] & \mathbb{C}^n \otimes_{\mathbb{C}} R((t)) \arrow[d, equal] \\
& R((t))^n
\end{tikzcd}$$

So the composition $\mathcal{L}(\mathbb{C}^n) \hookrightarrow R((t))^n$ is an $R[[t]]$-lattice

Now let us define a map $Gr^{2a} \longrightarrow Gr^{2b}$. Start w/ a lattice. Define a functor

$$\begin{array}{ccc}
Rep(GL_n) & \longrightarrow & \text{proj } R[[t]]\text{-modules} \\
\text{defining rep} & \longmapsto & \text{lattice}
\end{array}$$

Since we want a symmetric tensor functor, this determines where to send $\otimes^k$, Sym^k, Λ^k and more general Schur functors

<u>Key pt</u> Every irrep of GL_n is a summand of tensor products of $\Lambda^k\mathbb{C}^n$, and $(\Lambda^n\mathbb{C}^n)^*$.

So this actually defines a functor

~~Next~~ **Last thing**

Choose a pt $x \in \mathbb{A}^1$ (or your favorite smooth curve)

<u>Thm/Defn</u> 26. $\underline{Gr}_G$ also represents the functor

$$R \mapsto \left\{ (\mathcal{I}, \beta) \;\middle|\; \begin{array}{l} \mathcal{I} \text{ is an } R\text{-family of } G\text{-bundles on } \mathbb{A}^1 \\ \beta : \mathcal{I}|_{\mathbb{A}^1 \setminus \{x\}} \xrightarrow{\sim} \mathcal{I}^0|_{\mathbb{A}^1 \setminus \{x\}} \end{array} \right\}$$

i.e. we can delete the word "formal" from defn 25.