Combinatorics of two commuting matrices

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Abstract

The Jordan type of a nilpotent matrix is the partition given by the sizes of its Jordan blocks. We consider pairs of partitions (P, Q) where $Q = \mathfrak{Q}(P)$ is the Jordan type of a generic nilpotent matrix A commuting with a given nilpotent matrix B of type P. P. Oblak, L. Khatami and others related the problem of determining $\mathfrak{Q}(P)$ to the study of a certain poset $\mathcal{D}_P$.

An almost rectangular partition is one whose largest part minus smallest part is at most one. R. Basili showed that $Q(P)$ has r_P parts, where r_P is the minimum number of almost rectangular partitions whose union is P. P. Oblak determined the largest part of $Q(P)$, and L. Khatami determined the smallest part, using $\mathcal{D}_P$.

P. Oblak and T. Košir showed that $Q(P)$ has parts that differ pairwise by at least two (is "stable").

In Part I we define the poset $\mathcal{D}_P$, and discuss these results and P. Oblak's recursive conjecture for $Q(P)$.

Recently P. Oblak and R. Zhao made a conjecture about the set $\mathfrak{Q}^{-1}(Q)$ when Q has two parts. In Part II we report progress on proving their conjecture for $Q = (u, u - r)$; and we extend it to all stable Q. A consequence would be a mysterious bijection between $\mathfrak{Q}^{-1}(Q)$ and the set of partitions having diagonal hook lengths Q. We also ask further questions about partitions motivated by these results.

Part I reports on results of many. Part II is joint work with Leila Khatami, Bart Van Steirteghem and Rui Zhao.

Section 1: The Map P to $Q(P)$

Definition (Nilpotent commutator $\mathcal{N}_B$)

Let $P \vdash n$ a partition of n, A, B nilpotent matrices over an infinite field k:

J_P = nilpotent matrix of Jordan type P
(the Jordan blocks of J_P are given by P)

$\mathcal{C}_B$ = the centralizer in $\mathrm{Mat}_n(\mathsf{k})$ of B.

$\mathcal{N}_B \subset \mathcal{C}_B, \mathcal{N}_B = \{$nilpotent elements of $\mathcal{C}_B\}$.

P_A = Jordan type partition of A.

Fact: $\mathcal{N}_B$ is an irreducible variety [Bas, BI].

Def: $Q(P) =$ generic Jordan type for $A \in \mathcal{N}_B$, $B = J_P$.

Question. Can we explicitly determine $Q(P)$?

A classic, but not a classical problem! So far as we can tell, no one worked on it until 2006, then three groups independently began studying it.

Def. Let $B = J_{(n)}$, and denote by $[n]^k = P_{B^k}$.

Fact: $[n]^k$ is *almost rectangular*.

For $n = kq$, $[n]^k = (q^k) = (q, q, \ldots, q)$.
For $n = kq + r, 0 < r < k$,

$$[n]^k = ((\lceil n/k \rceil)^r, (\lfloor n/k \rfloor)^{k-r})$$

Here $[n]^k$ has parts that differ at most by 1: we term $[n]^k$ *almost rectangular (AR)*.

Ex. $n = 5$, $[5]^2 = (3, 2), [5]^3 = (2, 2, 1), [5]^4 = (2, 1, 1, 1), [5]^5 = (1, 1, 1, 1, 1)$.

Definition (The map $\mathfrak{Q}$.)

Def. Denote by $\mathfrak{Q} : P \to Q(P)$.

Theorem ((R. Basili) $\mathfrak{Q}$ for $r_P = 1$)

For $P = [n]^k, Q(P) = [n]$ and
$\mathfrak{Q}^{-1}([n]) = \{[n]^k, 1 \leq k \leq n\}$

Proof. $A = J_{[n]}$ commutes with A^k. So $A \in \mathcal{N}_B$ for $B = A^k$ and B is similar to $J_{[n]^k}$.

Example ($P = (3, 1)$ does not commute with $[4]$.)

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ & & A & & \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ & & A^2 & & \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ & & J_P & & \end{matrix}$$

Figure : $A = J_{[5]}, B = A^2$ and $J_P, P = [5]^2 = (3, 2)$. Here $A^2 \cong J_P$.

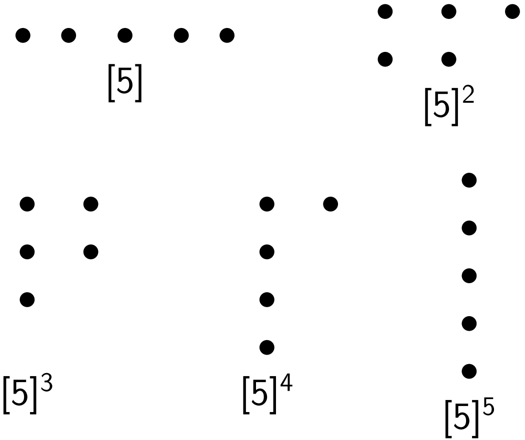


Figure : The AR partitions of 5.

Example (Table $\mathcal{T}(5) = \mathfrak{Q}^{-1}(5)$)

$[5]$	$[5]^2$	$[5]^3$	$[5]^4$	$[5]^5$
5	$(3,2)$	$(2,2,1)$	$(2,1^3)$	(1^5)

Figure : Partitions in $\mathfrak{Q}^{-1}(Q)$, $Q = [5]$

Theorem (R. Basili [Bas])

$Q(P)$ has r_P parts

Proof. Show rank $A = n - r_P$, A generic in $\mathcal{N}_{J_P}$ (see [Bas, Prop. 2.4] or [BIK, Thm. 2.17]).

Theorem (R. Basili and I.- [BI])

$Q(P) = P$ if and only if P has parts differing pairwise by at least two.

Def. We call such P with $Q(P) = P$ " stable."

Also known as "super-distinct" or "Rogers-Ramanujan" partitions [AlBe, An].

Theorem (P. Oblak and T.Košir [KO])

For A generic in $\mathcal{N}_B$, the Artin algebra $\mathrm{k}[A, B]$ is Gorenstein, so a complete Intersection (CI).

Proof. Exploits an involution in the poset $\mathcal{D}_P$, See also [BIK, Theorem 2.20].

Corollary (ibid. with F.H.S. Macaulay [Mac])

$Q(P)$ is stable!

Proof: Macaulay characterizes HF of such CI.

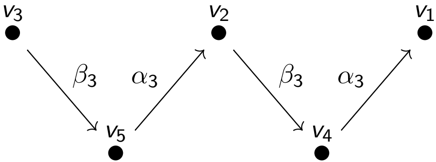
Definition (Poset $\mathcal{D}_P$ [Obl1, KO, BIK, Kh1])

Let $P \vdash n, P = (\ldots i^{n_i} \ldots), S_P = \{i \mid n_i > 0\}$. The poset $\mathcal{D}_P$ has rows corresponding to the Ferrer's graph of P, but with each row centered on the y axis.

Label the vertices in the n_i rows of length i

$$(u, i, k), 1 \leq u \leq i, 1 \leq k \leq n_i.$$

Let i^-, i^+ be the next lower, higher elements of S_P, if they exist. The edges in the diagram of $\mathcal{D}_P$ correspond to the *elementary maps:*



$$A = \left(\begin{array}{ccc|cc} 0 & x_{\alpha_3\beta_3} & x_{(\alpha_3\beta_3)^2} & x_{\alpha_3} & x_{\alpha_3\beta_3\alpha_3} \\ 0 & 0 & x_{\alpha_3\beta_3} & 0 & x_{\alpha_3} \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & x_{\beta_3} & x_{\beta_3\alpha_3\beta_3} & 0 & x_{\beta_3\alpha_3} \\ 0 & 0 & x_{\beta_3} & 0 & 0 \end{array}\right), v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix}$$

Figure : Generic element A of $\mathcal{N}_B$, $B = J_P$, $P = (3, 2)$.

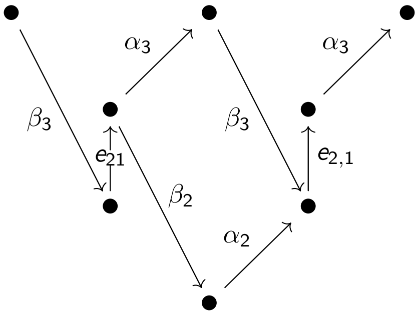


Figure : $\mathrm{Diag}(\mathcal{D}_P)$ for $P = (3, 2, 2, 1)$.

$$
\left(\begin{array}{ccc|cc|cc|c}
0 & x_{c_3} & x_{(c_3)^2} & x_{\alpha_3} & x_{\alpha_3 c_2} & x_{\alpha_3 e_{21}} & x_{\alpha_3 c_2'} & x_{\alpha_3 e_{21} \alpha_2} \\
0 & 0 & x_{c_3} & 0 & x_{\alpha_3} & 0 & x_{\alpha_3 e_{21}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & x_{e_{21} \beta_3} & x_{43} & 0 & x_{c_2} & x_{e_{21}} & x_{c_2'} & x_{\alpha_3 e_{21}} \\
0 & 0 & x_{\beta_3 e_{21}} & 0 & 0 & 0 & x_{e_{21}} & 0 \\
\hline
0 & 0 & x_{63} & 0 & x_{\alpha_2 \beta_2} & 0 & x_{\alpha_2 \beta_2 e_{21}} & x_{\alpha_2 \beta_2} \\
0 & 0 & x_{\beta_3} & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & x_{\beta_2 e_{21} \beta_3} & 0 & x_{\beta_2} & 0 & x_{\beta_2 e_{21}} & 0
\end{array}\right)
$$

$$x_{63} = x_{\alpha_2 \beta_2 e_{21} \alpha_3}$$

Figure : Generic element A of $\mathcal{N}_B$ for $P = (3, 2, 2, 1)$.

Elementary maps and edges in the diagram of $\mathcal{D}_P$

(i) $\beta_i = \beta_{i,i^-} : (u, i, n_i) \to (u, i^-, 1)$ for $u \leq i^-$.

(ii) $\alpha_i = \alpha_{i^-,i} : (u, i^-, n_{i^-}) \to (u + i - i^-, i, 1)$.

(iii) $e_{i,k} : (u, i, k) \to (u, k, k+1), 1 \leq u_i \leq i, 1 \leq k < n_i$.

(iv) When both $i - 1 \notin S_P, i + 1 \notin S_P$ (i is isolated),
$\omega_i : (u, i, n_i) \to (u + 1, i, n_i)$ when $1 \leq u < i$.

(Each map is 0 on basis vectors of $\mathcal{D}_P$ not listed for it.)

The diagram of a poset is comprised of its covering edges.
The poset is related to a maximum nilpotent subalgebra
$\mathcal{U}_B \subset N_B, B = J_P$ by $v < v'$ if $\exists A \in \mathcal{U}_B \mid A_{v,v'} \neq 0$.

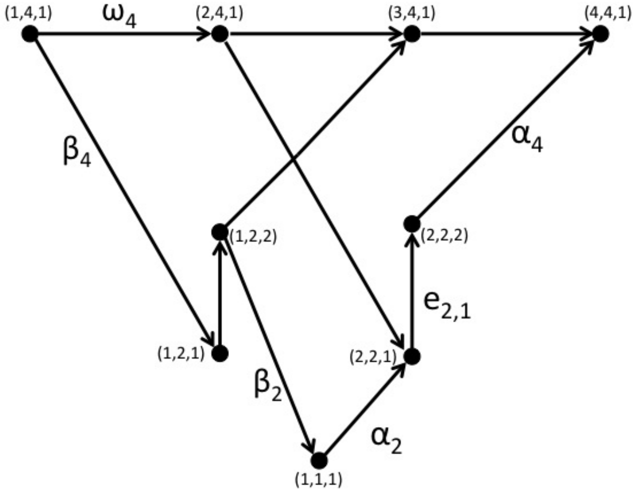


Diagram of the poset $\mathcal{D}_P$ and maps for $P = (4, 2, 2, 1)$.

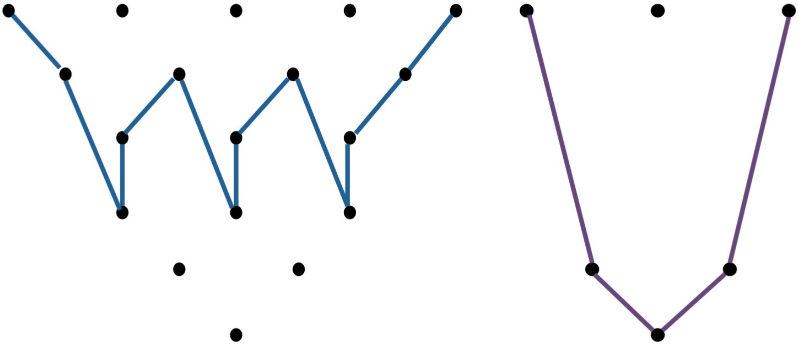


Figure : U-chain for $P = (5, 4, 3, 3, 2, 1)$ and new U-chain for $P' = (3, 2, 1)$.[Source: LK NU GASC talk 2013]

Def. (U-chain)

A U-chain C_i in $\mathcal{D}_P$ is the saturated (maximal) chain through the union of three subsets of vertices:

(i) All rows of length $i, i-1$, corresponding to an AR subpartition of P.

(ii) A descending chain from the source $(1, p_1, 1)$ of $\mathcal{D}_P$ to $(1, i, 1)$

(iii) An ascending chain from (i, i, n_i) to the sink (p_1, p_1, n_{p_1}) of $\mathcal{D}_P$.

Oblak Recursive Conjecture

We obtain $Q(P)$ as follows from $\mathcal{D}_P$:

(i) Choose a maximum length U-chain in $\mathcal{D}_P$. Its length is q_1, the largest part of $Q(P)$.

(ii) Remove the vertices in the chain from $\mathcal{D}(P)$, obtaining a smaller partition P'. If $P' \neq \emptyset$ then $Q(P) = (q_1, Q(P'))$ (go to (i).).

Warning. The poset $\mathcal{D}_{P'}$ in the Oblak recursion is *not* in general a subposet of $\mathcal{D}_P$.

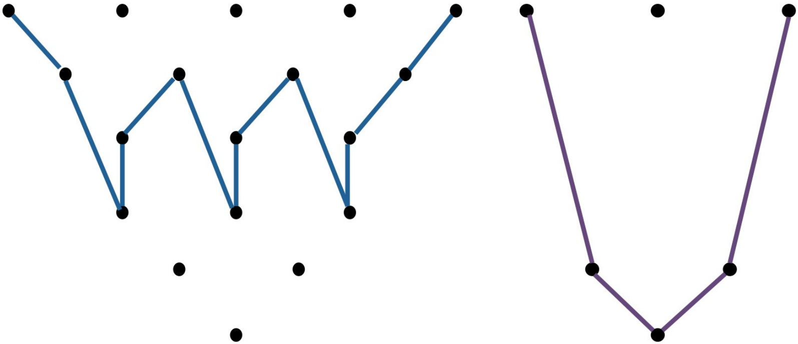


Figure : U-chain for $P = (5,4,3,3,2,1)$ and new U-chain for $P' = (3,2,1)$.

Theorem (P. Oblak[Obl1] – Index of $Q(P)$)

The index (largest part) of Q_P is the length of the longest U-chain in $\mathcal{D}_P$.

Theorem (L. Khatami [Kh1] – $Ob(P) = \lambda_U(\mathcal{D}_P)$)

The partition $Ob(P)$ obtained by the Oblak recursive process is independent of the choices of AR subpartitions; and $Ob(P) = \lambda_U(\mathcal{D}_P)$, obtained as $\lambda(\mathcal{D}_P)$ below by restricting to sets of i U-chains.

Example ($Ob(P)$ independent of the choice of C_i)

$P = (5, 4, 3, 3, 2, 1)$. Taking out the maximum-length chain C_4 of length 12 we have left $P' = (3, 2, 1)$, and $Q(P') = (5, 1)$, so $Q(P) = (12, 5, 1)$.

Taking instead the maximum-length chain C_3 we again have $P' = (3, 2, 1)$. (The next steps need not be the same partition).

Def: $P \geq P'$ in the orbit closure (Bruhat) order if $\forall i \ \sum_{u=1}^{i} p_u \geq \sum_{u=1}^{i} p'_u$.

Theorem (I,L.Khatami [IKh])

$Q(P) \geq Ob(P)$.

Proof idea. For each maximal-length set of s U-chains, we specify an $A \in \mathcal{N}_B$ such that $\dim_k k[A] \circ \{v_1, \dots, v_s\}$ where the v_i are initial elements, agrees with the sum of the first s parts of $Ob(P)$. This involves an analysis of the sets of chains from the v_i to all the vertices covered by the s U-chains.

Def. (C. Greene et al, see[BrFo])

Let $\mathcal{D}$ be a poset without loops. Define $c_i = \max\#$ vertices covered by i chains. Set

$$\lambda(\mathcal{D}) = (c_1, c_2 - c_1, c_3 - c_2, \ldots).$$

Theorem (C. Greene, S. Poljak, E.R. Gansner, see [BrFo])

Let $\mathcal{D}$ be any finite poset without loops, and let A be a generic nilpotent matrix in the incidence algebra $\mathfrak{I}(\mathcal{D}_P)$. Then the Jordan type $P_A = \lambda(\mathcal{D})$.

Question: Combinatorial Oblak conjecture

Is $Q(P) = \lambda(\mathcal{D}_P)$? Since $Ob(P) = \lambda_U(\mathcal{D}_P) \geq Q(P)$ this is equivalent to "Is $\lambda_U(\mathcal{D}_P) = \lambda(\mathcal{D}_P)$?"

The key issue is that $A \in \mathcal{N}_B$ commutes with B, that acts by moving vertices of $\mathcal{D}_P$ to the right: this greatly restricts $A \in \mathfrak{I}(\mathcal{D}_P)$. Does it matter for the Jordan type P_A?[1]

[1] R. Basili has posted a preprint in June 2012 that asserts a proof of the Oblak conjecture. It appears to assume $\mathrm{char}\, k = 0$; if correct, by [IKh] it would imply the Oblak conjecture over any infinite field k.

Theorem (L. Khatami [Kh2] – Minimum part)

The minimum part of $Q(P)$ is a specified combinatorial invariant $\mu(P)$. Also

$$Ob(P)_{\min} = Q(P)_{\min} = \lambda(\mathcal{D}_P)_{\min} = \mu(P) \qquad *$$

Proof idea. First show $\mu(P)$ is the minimum part of $\lambda_U(P) = Ob(P)$. Then an intensive study of the antichains of $\mathcal{D}_P$ shows $\lambda(\mathcal{D}_P)_{\min} = \mu(P)$. By [IKh], $Ob(P) \leq Q(P) \leq \lambda(\mathcal{D}_P)$, showing $(*)$.

Corollary ([Obl1, KO, Kh2])

$Q(P)$ is explicitly known for $r_P \leq 3$, over any infinite field k.

The invariant $\mu(P)$ for a spread.

Let $P = ((p+s-1)^{n_s}, \ldots, p^{n_1})$ be an s-spread, $n_i > 0$ for $1 \leq i \leq s$. Set

$$\mu(P) = \min\{pn_{2i-1} + (p+1)n_{2j} | 1 \leq i \leq j \leq r_P\}$$

Note: For s odd $r_P = (s+1)/2$ so $n_{2r_P} = 0$ and $\mu(P) = p \cdot \min\{n_{2i-1} \mid 1 \leq i \leq r_P\}$.

Theorem ([Kh2])

For P a spread, $\mu(P)$ is the $\#$ of disjoint length-r_P antichains in $\mathcal{D}_P$.

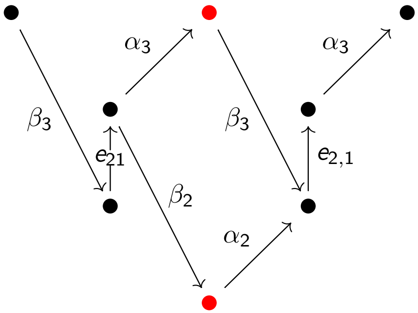


Figure : $\mu(P) = 1$ for $P = (3, 2, 2, 1)$, $Q(P) = (7, 1)$

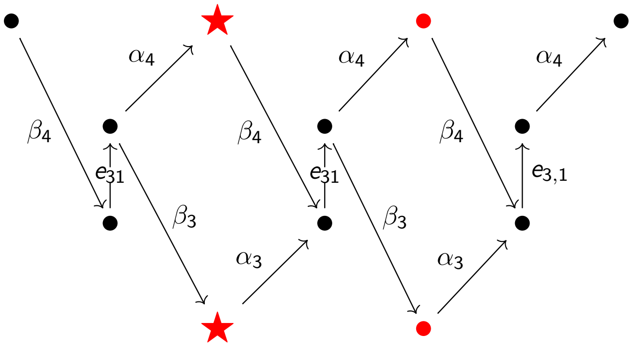


Figure : $\mu(P) = 2$ for $P = (4, 3, 3, 2)$, $Q(P) = (10, 2)$

Section 2: The inverse $\mathfrak{D}^{-1}(Q)$, Q stable.

The inverse image $\mathfrak{Q}^{-1}(Q)$ has been quite mysterious, even for $Q = (u, u - r), r \geq 2$. However, P. Oblak in 2012 made another beautiful conjecture, later refined by R. Zhao.

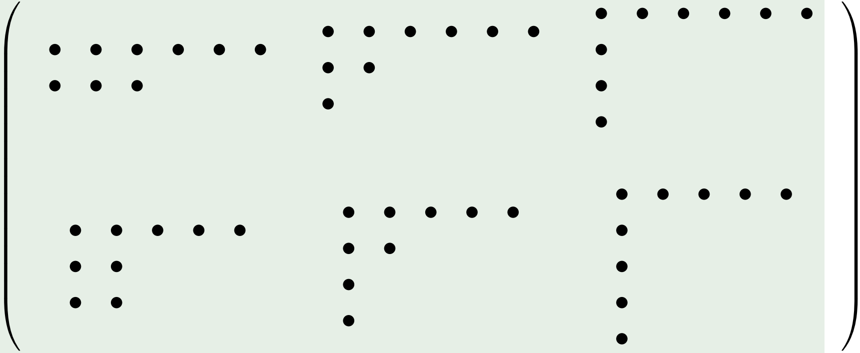
Conjecture for $\mathfrak{Q}^{-1}(Q)$ (P. Oblak, R. Zhao)

The elements of $\mathfrak{Q}^{-1}(Q), Q = (u, u - r), r \geq 2$ may be arranged in a $(r - 1) \times (u - r)$ table $\mathcal{T}(Q)$ such that $\mathcal{T}(Q)_{i,j}$ is a partition having $i + j$ parts.

Example (Table $\mathcal{T}(Q)$ for $Q = (6,3)$)

Let $Q = (6,3)$. Then $\mathcal{T}(Q)$ satisfies

$$\mathcal{T}(Q) = \begin{pmatrix} (6,3) & (6,[3]^2) & (6,[3]^3) \\ (5,[4]^2) & (5,[4]^3) & (5,[4[^4) \end{pmatrix} =$$



Definition (Type A,B,C partitions in $\mathfrak{Q}^{-1}(Q)$)

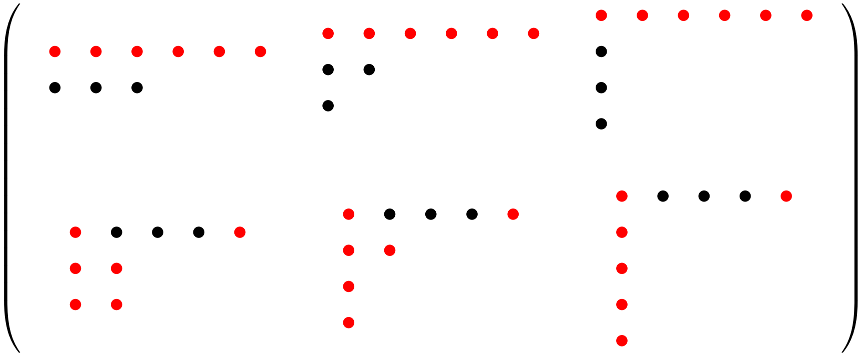
Here $Q = (u, u - r), r \geq 2$, so $r_P = 2$ and $S_P = (a, a - 1, b, b - 1)$ or $(a, a - 1, a - 2)$. We label $P \in \mathfrak{Q}^{-1}(Q)$ according to whether the largest part u of Q comes from a U-path C_a (type A) or C_b (type B) or C_{a-1} (type C).

Theorem ([Obl2, Z] Special $\mathfrak{Q}^{-1}(u, u - r)$)

The $\mathfrak{Q}^{-1}$ table conjecture is true for $2 \leq r \leq 4$ and all u (P. Oblak), and for $u >> r$ – the "normal pattern" case where A and B rows strictly alternate (R.Zhao in progress).

Example (Normal pattern)

The table $\mathcal{T}(Q)$ for $Q = (6,3)$ has normal pattern: the first row $(6,3), (6,[3]^2), (6,[3]^3)$ is type A, and the second $(5,[4]^2), (5,[4]^3), (5,[4]^4)$ is type B.



Theorem ([IKSZ] Table $\mathcal{T}^{-1}(Q)$.)

Let $Q = (u, u - r)$. We can fill the $(r-1) \times (u-r)$ table $\mathcal{T}(Q)$ with specific partitions that are in the inverse image $\mathfrak{Q}^{-1}(Q)$. These are arranged in type A rows or partial rows that fit with hooks of type B or C,B. [Conj: this is all $\mathfrak{Q}^{-1}(Q)$.]

Proof idea

(i) Identify the candidate entries of the table, and show that each entry is in $\mathfrak{Q}^{-1}(Q)$. (Complete)

(ii) Verify using GF that $\#\{P \mid r_P = 2, P \vdash n\}$ agrees with the sum $\sum |\mathcal{T}(Q)| = \sum (r-1)(u-r) \mid Q = (u, u-r), 2u - r = n.$

Example ($\mathcal{T}(Q)$ for $Q = (8, 3)$.)

$$\mathfrak{Q}^{-1}(8,3) = \begin{pmatrix} (8,3) & (8,[3]^2) & (8,[3]^3) \\ (\mathbf{5},[\mathbf{6}]^{\mathbf{2}}) & (\mathbf{5},[\mathbf{6}]^{\mathbf{3}}) & (\mathbf{5},[\mathbf{6}]^{\mathbf{4}}) \\ ([8]^2,[3]^2) & ([8]^2,[3]^3) & (\mathbf{5},[\mathbf{6}]^{\mathbf{5}}) \\ ([7]^2,[4]^3) & ([7]^2,[4]^4) & (\mathbf{5},[\mathbf{6}]^{\mathbf{6}}) \end{pmatrix}$$

$$= \begin{pmatrix} A & A & A \\ B & B & B \\ A & A & B \\ B' & B' & B \end{pmatrix}.$$

Note two B hooks.

$\mathcal{T}(Q)$	3	$[3]^2$	$[3]^3$
8	$(12,3)$	$(12,[3]^2)$	$(12,[3]^3)$
$[8]^2$	$([12]^2,3)$	$[12]^2,[3]^2)$	$([12]^2,[3]^3)$
$[8]^3$	$(\mathbf{5},[\mathbf{10}]^{\mathbf{3}})$	$(\mathbf{5},[\mathbf{10}]^{\mathbf{4}})$	$(\mathbf{5},[\mathbf{10}]^{\mathbf{5}})$
$[8]^4$	$([12]^3,[3]^2)$	$([12]^3,[3]^3)$	$(\mathbf{5},[\mathbf{10}]^{\mathbf{6}})$
$[8]^5$	$(4,[10]^4,1)^C$	$([7]^2,[8]^5)$	$(\mathbf{5},[\mathbf{10}]^{\mathbf{7}})$
$[8]^6$	$([12]^4,[3]^3)$	$([7]^2,[8]^6)$	$(\mathbf{5},[\mathbf{10}]^{\mathbf{8}})$
$[8]^7$	$([\mathbf{9}]^{\mathbf{3}},[\mathbf{6}]^{\mathbf{5}})$	$([7]^2,[8]^7)$	$(\mathbf{5},[\mathbf{10}]^{\mathbf{9}})$
$[8]^8$	$([\mathbf{9}]^{\mathbf{3}},[\mathbf{6}]^{\mathbf{6}})$	$([7]^2,[8]^8)$	$(\mathbf{5},[\mathbf{10}]^{\mathbf{10}})$

.

$\mathcal{T}(Q)$, $Q=(12,3)$. First $C\backslash A\cup B$ case [Z].

Combinatorial Relation $\mathcal{T}(Q)$ and Durfee squares

Definition ($Durf(Q)$, Q stable)

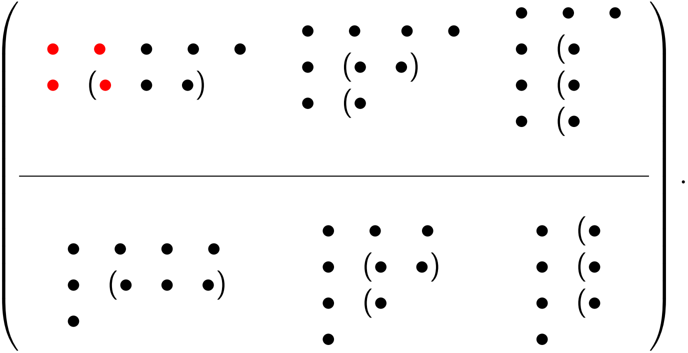
Let Q be a stable partition. Denote by $Durf(Q)$ the set of all partitions having diagonal hook lengths Q.

Example ($Durf(Q)$ for $Q = (6, 3)$)

The inside diagonal hook h_{22} has length 3 so can be

$$P' = (3)\ \begin{matrix}\bullet & \bullet & \bullet\end{matrix},\ (2,1)\ \begin{matrix}\bullet & \bullet \\ \bullet & \end{matrix},\ \text{or } (1,1,1)\ \begin{matrix}\bullet \\ \bullet \\ \bullet\end{matrix}.$$

A diagonal hook h_{11} of length 6 is folded around P'; in each case there are two positions: adding one, or two parts to P'. So $Durf(Q)$, $Q = (6, 3)$ is



$$= \begin{pmatrix} (5,4) & (4,3,2) & (3,2,2,2) \\ (4,4,1) & (3,3,2,1) & (2,2,2,1) \end{pmatrix}.$$

Corollary (Bijection $\mathcal{T}(Q)$ and $Durf(Q)$.)

Let $Q = (u, u - r)$. There is a bijection $\theta : \mathcal{T}(Q) \to Durf(Q)$ that preserves the number of parts of P.

Proof idea. The table $Durf(Q)$ is simply found recursively by adding diagonal hooks (as above), so the parts-generating function is easily written down. The result agrees with the parts-generating function for $\mathcal{T}^{-1}(Q)$ from the above Theorem.

Example (θ for $Q = (6,3)$)

The map $\theta(\mathcal{T}(Q)_{ij}) = Durf(Q)_{ij}$. Here

$$\mathcal{T}(Q) = \begin{pmatrix} (6,3) & (6,[3]^2) & (6,[3]^3) \\ (5,[4]^2) & (5,[4]^3) & (5,[4[^4) \end{pmatrix}.$$

$$Durf(Q) = \begin{pmatrix} (5,4) & (4,3,2) & (3,2,2,2) \\ (4,4,1) & (3,3,2,1) & (2,2,2,1) \end{pmatrix}.$$

Example (Basic case $r_P = 1$, Durfee square 1×1.)

Let $n = 5, Q = (5)$.
$\mathcal{T}(Q) = ([5],[5]^2,[5]^3,[5]^4,[5]^5)$
$\mathcal{D}urf(Q) = ((5),(4,1),(3,1^2),(2,1^4),(1^5))$

Definition (Key S_Q of a stable Q)

Let $Q = (q_1, q_2, \ldots, q_k), q_i \geq q_{i+1} + 2, 1 \leq i < k$ be stable. Set the *key* $S_Q = (q_1 - q_2 - 1, q_2 - q_3 - 1, \ldots, q_{k-1} - q_k - 1, q_k)$.

Example

For $Q = (u, u - r)$ the key $S_Q = (r - 1, u - r)$. For $Q = (12, 6, 2)$ the key $S_Q = (5, 3, 2)$

Box conjecture for $\mathfrak{Q}^{-1}(Q)$

Let $Q = (q_1, \ldots, q_k)$ be stable of key S_Q Then

(i) The partitions $\mathfrak{Q}^{-1}(Q)$ can be arranged in a k-way box $\mathcal{T}(Q)$ such that the number of parts in $\mathcal{T}(Q)_I, I = (i_1, \ldots, i_k)$ is $|I|$.

(ii) The codimension of the similarity orbit of $\mathcal{T}(Q)_I$ in $\mathcal{N}_Q$ is $|I| - k$.

Corollary (of box conjecture)

For arbitrary stable Q there is an isomorphism $\theta : \mathfrak{Q}^{-1}(Q) \to Durf(Q)$ preserving the number of parts. *Problem: Determine θ combinatorially!*

Example ($S_Q = (2, 2, 2)$)

Take $Q = (8, 5, 2)$ so $S_Q = (2, 2, 2)$.

The two floors of $\mathcal{T}(Q)$ are

$$\begin{pmatrix} (8,5,2) & (8,5,1^2) \\ (8,4,2,1) & (8,4,1^3) \end{pmatrix}, \begin{pmatrix} (\mathbf{7},\mathbf{4},\mathbf{2^2}) & (7,4,2,1^2) \\ (7,3^2,1^2) & (7,4,1^4) \end{pmatrix}.$$

The corresponding floors of $Durf(Q) = \theta(\mathcal{T}(Q))$ are

$$\begin{pmatrix} (6,5,4) & (5,4,3^2) \\ (5,4^2,2) & (4,3^3,2) \end{pmatrix}, \begin{pmatrix} (\mathbf{5^2},\mathbf{4},\mathbf{1}) & (4^2,3^2,1) \\ (4^3,2,1) & (3^4,2,1) \end{pmatrix}.$$

Question: Can we explain these results?

Lie algebra perspective:

The columns of $\mathcal{D}(P)$ are weight spaces for the sl_2 triple of B. But the S_n irreps for $P \in \mathcal{T}(Q)$ and $\theta(P) \in Durf(Q)$ have different VS dimensions.

Map to the Hilbert scheme:

Let $B = J_Q$. The map

$$\pi : \mathcal{N}_B \to \mathrm{Hilb}^n \mathsf{k}[x, y]: \; A \to \mathsf{k}[A, B]$$

defines an image, whose fixed points under a $\mathbb{C}^*$ action correspond to the monomial ideals of $\mathcal{T}(Q)$, so to homology classes on $\pi(\mathcal{N}_B)$. Will this explain the codimensions in $\mathcal{T}(Q)$?

Combinatorial questions arising from $P \to Q(P)$.

(a) Poset $\mathcal{D}(P)$: Is $\lambda(\mathcal{D}_P) = \lambda_U(\mathcal{D}_P)$?

(b) Verify $\#\{\mathrm{P} \vdash n$ with p parts and $r_P = k\}$.

(c) An a-cluster is a partition $P = (p_1 \geq \ldots \geq p_t)$ with $p_1 - p_t \leq a$.
$r_{a,P} = \min\{\ \#\ a$-clusters needed to cover $P\}$.
$V_{a,k}(n) = \{P \vdash n \mid r_{a,P} = k\}$.
Determine $|V_{a,k}(n)|$.

(d) Consider other posets $\mathcal{P}$ with multiplicities, and a linear action $B \to$ on vertices($\mathcal{P}$). Consider $A \in \mathfrak{I}(\mathcal{P})$ commuting with B. Is $\lambda(\mathcal{P}) = \lambda^B(\mathcal{P})$?

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- K. Alladi, A. Berkovish: *New weighted Rogers-Ramanjan partition theorems and their implications*, Trans. Amer. Math. Soc. 354 (2002) no. 7, 2557–2577.
- G. Andrews: *The Theory of Partitions*, Cambridge University Press, 1984, paper 1988 ISBN 0-521-63766-X.
- V. Baranovsky: *The variety of pairs of commuting nilpotent matrices is irreducible*, Transform. Groups 6 (2001), no. 1, 3–8.
- R. Basili: *On the irreducibility of commuting varieties of nilpotent matrices*. J. Algebra 268 (2003), no. 1, 58–80.
- R. Basili and A. Iarrobino: *Pairs of commuting nilpotent matrices, and Hilbert function*. J. Algebra **320** # 3 (2008), 1235–1254.

- R. Basili, A. Iarrobino and L. Khatami, *Commuting nilpotent matrices and Artinian Algebras, J. Commutative Algebra (2) #3 (2010) 295–325*.
- R. Basili, T. Košir, P. Oblak: *Some ideas from Ljubljana*, (2008), preprint.
- J. Briançon: *Description de* $\mathrm{Hilb}^n C\{x, y\}$, Invent. Math. 41 (1977), no. 1, 45–89.
- J.R. Britnell and M. Wildon: *On types and classes of commuting matrices over finite fields*, J. Lond. Math. Soc. (2) 83 (2011), no. 2, 470–492.
- T. Britz and S. Fomin: *Finite posets and Ferrers shapes*, Advances Math. 158 #1 (2001), 86–127.

- J. Brown and J. Brundan: *Elementary invariants for centralizers of nilpotent matrices*, J. Aust. Math. Soc. 86 (2009), no. 1, 1–15. ArXiv math/0611.5024.
- D. Collingwood, W. McGovern: *Nilpotent Orbits in Semisimple Lie algebras*, Van Norstrand Reinhold (New York), (1993).
- E. Friedlander, J. Pevtsova, A. Suslin: *Generic and maximal Jordan types*, Invent. Math. 168 (2007), no. 3, 485–522.
- E.R. Gansner: *Acyclic digraphs, Young tableaux and nilpotent matrices*, SIAM Journal of Algebraic Discrete Methods, 2(4) (1981) 429–440.
- M. Gerstenhaber: *On dominance and varieties of commuting matrices*, Ann. of Math. (2) 73 1961 324–348.

- C. Greene: *Some partitions associated with a partially ordered set*, J. Combinatorial Theory Ser A **20** (1976), 69–79.
- R. Guralnick and B.A. Sethuraman: *Commuting pairs and triples of matrices and related varieties*, Linear Algebra Appl. 310 (2000), 139–148.
- C. Gutschwager: *On prinicipal hook length partition and Durfee sizes in skew characters*, Ann. Comb. 15 (2011) 81–94.
- T. Harima and J. Watanabe: *The commutator algebra of a nilpotent matrix and an application to the theory of commutative Artinian algebras*, J. Algebra 319 (2008), no. 6, 2545–2570.
- A. Iarrobino and L. Khatami: *Bound on the Jordan type of a generic nilpotent matrix commuting with a given matrix*, J. Alg.

Combinatorics, 38, #4 (2013), 947–972. On line DOI: 10.1007/s10801–013–0433–1. ArXiv 1204.4635.

A. Iarrobino, L. Khatami, B. Van Steirteghem, and R. Zhao *Combinatorics of two commuting nilpotent matrices*, preprint in progress, 2013 (52 p.).

L. Khatami: *The poset of the nilpotent commutator of a nilpotent matrix*, to appear, Linear Algebra and its Applications, s arXiv:1202.6089.

L. Khatami: *The smallest part of the generic partition of the nilpotent commutator of a nilpotent matrix*. (2013) to appear, J. Pure and Applied Algebra, arXiv:1302.5741.

- T. Košir and P. Oblak: *On pairs of commuting nilpotent matrices*, Transform. Groups 14 (2009), no. 1, 175–182.
- T. Košir and B. Sethuranam: *Determinantal varieties over truncated polynomial rings*, J. Pure Appl. Algebra 195 (2005), no. 1, 75–95.
- F.H.S. Macaulay: *On a method for dealing with the intersection of two plane curves*, Trans. Amer. Math. Soc. 5 (1904), 385–410.
- G. McNinch: *On the centralizer of the sum of commuting nilpotent elements*, J. Pure and Applied Alg. 206 (2006) # 1-2, 123–140.
- P. Oblak: *The upper bound for the index of nilpotency for a matrix commuting with a given nilpotent matrix*, Linear and Multilinear Algebra 56 (2008) no. 6, 701–711. Slightly revised in ArXiv: math.AC/0701561.

- P. Oblak: *On the nilpotent commutator of a nilpotent matrix*, Linear Multilinear Algebra 60 (2012), no. 5, 599–612.
- D. I. Panyushev: *Two results on centralisers of nilpotent elements*, J. Pure and Applied Algebra, 212 no. 4 (2008), 774–779.
- S. Poljak: *Maximum Rank of Powers of a Matrix of Given Pattern*, Proc. A.M.S., 106 #4 (1989), 1137–1144.
- A. Premet: *Nilpotent commuting varieties of reductive Lie algebras*, Invent. Math. 154 (2003), no. 3, 653–683.
- G. de B. Robinson and R.M.Thrall: *The content of a Young diagram,* Michigan Math.Journal 2 No.2 (1953), 157–167.
- H.W. Turnbull and A.C. Aitken: *An Introduction to the Theory of Canonical Matrices*, Dover, New York, 1961.

R. Zhao: *Commuting nilpotent matrices and Oblak's proposed formula*, preprint in preparation, 2013.

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