Equations for loci of commuting nilpotent matrices

Mats Boij, Anthony Iarrobino*, Leila Khatami, Bart Van Steirteghem, Rui Zhao

KTH Stockholm
Northeastern University
Union College
Medgar Evers College, CUNY
U. Missouri

CAAC: Combinatorial Algebra and Algebraic Combinatorics
January 23, 2016, Western University, London, Ontario

Abstract

The Jordan type of a nilpotent matrix A is the partition P_A giving the sizes of the Jordan blocks of the Jordan matrix in its conjugacy class. For $Q = (u, u - r)$ with r at least 2, there is a known table $\mathcal{T}(Q)$ of Jordan types P for $n \times n$ matrices whose maximum commuting nilpotent Jordan type $\mathfrak{Q}(P)$ is Q (arXiv math 1409.2192). Let B be the Jordan matrix of partition Q, and consider the affine space $\mathcal{N}_B$ parametrizing nilpotent matrices commuting with B. For a partition P in $\mathcal{T}(Q)$, the locus $\mathfrak{Z}(P)$ of P is the subvariety of $\mathcal{N}_B$ parametrizing matrices A having Jordan type P. In this talk we outline conjectures and results concerning the equations defining $\mathfrak{Z}(P)$. If time permits, we state analogous loci equation conjectures for partitions in the boxes $\mathcal{B}(Q)$ when Q has three or more parts.

Section 0: The map $\mathfrak{Q} : P \to \mathfrak{Q}(P)$

Definition (Nilpotent commutator $\mathcal{N}_B$ and $\mathfrak{Q}(P)$.)

$V \cong \mathsf{k}^n$ vector space over an infinite field k.
$A, B \in \mathrm{Mat}_n(\mathsf{k}) = \mathrm{Hom}_{\mathsf{k}}(V, V)$, nilpotent matrices.

$\mathcal{C}_B \subset \mathrm{Mat}_n(\mathsf{k})$ centralizer of B.

$\mathcal{N}_B \subset \mathcal{C}_B$: the variety of nilpotent elements of $\mathcal{C}_B$.

$P \vdash n$ is a partition of n;
J_P = Jordan block matrix, the sizes of whose blocks is P.

P_A = Jordan type of A – the partition such that $J_{P_A} = CAC^{-1}$ is similar to A.

$\mathfrak{Q}(P)$: the maximum Jordan type in Bruhat order of a nilpotent matrix commuting with J_P.
$r_P = \#$ almost rectangular partitions (parts differ at most by 1) needed to cover P.

Problem 1: Determine the map $\mathfrak{Q} : P \to \mathfrak{Q}(P)$.

Fact (T. Košir, P. Oblak): $\mathfrak{Q}(P)$ is *stable*: parts differ pairwise by at least 2 and $\mathfrak{Q}(\mathfrak{Q}(P)) = \mathfrak{Q}(P)$.

Fact (R. Basili): $\mathfrak{Q}(P)$ has r_P parts.

Partial Answers: Oblak Recursive Conjecture:
$\mathfrak{Q}(P) = Oblak(P)$. Known for $Q = \mathfrak{Q}(P)$ with 2 or 3 parts (P. Oblak, L. Khatami).
Thm: $Oblak(P) = \lambda_U(P) \leq \mathfrak{Q}(P)$ (L. Khatami, L.K. and A.I.).

Problem 2: Given Q determine all P such that $\mathfrak{Q}(P) = Q$.

Partial Answer: i. **Table Theorem** for $Q = (u, u - r), r \geq 2$ (A.I., L.Khatami, B.Van Steirtegham, R. Zhao).

ii. **Equations conjecture** and **Box Conjecture**.

Claim: **These should have been classical problems!** Canonical form is due to C. Jordan, 1870. But the map $P \to \mathfrak{Q}(P)$ was not studied classically.[1]

In 2006, three independent groups began to work on the $P \to \mathfrak{Q}(P)$ problem

P. Oblak and T. Košir (Ljubljana)
D. Panyushev (Moscow)
R. Basili, I.-, and L. Khatami (Perugia, Boston).

Connected to Hilbert scheme work of J. Briançon, M. Granger, R. Basili, V. Baranovsky, A. Premet, N. Ngo and K. Šivic.

Links to work of E. Friedlander, J. Pevtsova, A. Suslin, on representations of Abelian p-groups [FrPS,CFrP].

[1] Instead, I. Schur (1905), N.Jordan, M. Gerstenhaber (1958), E. Wang (1979) studied *maximum dimension* commuting subalgebras/nilpotent subalgebras of matrices.

Section 1: Artinian Gorenstein quotients of $R = \mathsf{k}\{x, y\}$ and Jordan type of multiplication maps.

When the Hilbert function H of an Artinian R- module $\mathfrak{X}$ is fixed, the conjugate partition $H^\vee$ is an upper bound for the partitions that might occur as the Jordan type P_x for the multiplication m_x on $\mathfrak{X}$ by $x \in R$. Given H what are the possible Jordan types $P_y, y \in R$ for m_y on $\mathfrak{X}$? Conversely, let $P = P_A$: what is the maximum Jordan type $\mathfrak{Q}(P)$ in Bruhat order of a nilpotent matrix B commuting with A?

Example

Let $\mathcal{A} = \mathsf{k}\{x, y\}/I, I = (xy, y^2 + x^3) = f^\perp$ where $f = Y^2 - X^3 \in \mathsf{k}_{DP}[X, Y]$. Here $H(\mathcal{A}) = (1, 2, 1, 1)$ and as $\mathsf{k}[x]$ module $\mathcal{A} \cong \langle 1, x, x^2, x^3; y\rangle$, so $P_x = (4, 1) = H^\vee$.

Question. What are the possible Jordan types P_A of $m_A, A \in \mathcal{A}$?
Variation Fix $Q = (4,1)$. Assume Q is the *maximum* Jordan type $Q = \mathfrak{Q}(P)$ (in Bruhat order) of a nilpotent matrix B commuting with a matrix A. What are the possible Jordan types $P = P_A$?

Answer Besides $(4,1)$, here $P = (3,1,1)$ is the only other partition for which $\mathfrak{Q}(P) = (4,1)$.

We say $Q = (u, u-r)$ is *stable* if $u > r \geq 2$ (i.e. if its parts differ pairwise by at least 2). The last four authors show the following in [IKVZ].

Theorem (Table theorem)

Let $Q = (u, u-r)$ be stable. Then there are exactly $(r-1)(u-r)$ partitions $P_{ij}(Q), 1 \leq i \leq r-1, 1 \leq j \leq u-r$ such that $\mathfrak{Q}(P_{ij}) = Q$. These form a table $\mathcal{T}(Q)$ and P_{ij} has $i+j$ parts. The table is comprised of B hooks and A rows or partial rows that fit together as in a puzzle.

An AR (almost rectangular) partition has parts differing pairwise by at most 1. Notation: $[n]^k$ is the AR partition of n into k parts.

Example

Let $Q = (8,3)$. Then

$$\mathcal{T}(8,3) = \begin{pmatrix} (8,3) & (8,[3]^2) & (8,[3]^3) \\ \mathbf{(5,[6]^2)} & ([8]^2,[3]^2) & ([8]^2,[3]^3) \\ \mathbf{(5,[6]^3)} & ([7]^2,[4]^3) & ([7]^2,[4]^4) \\ \mathbf{(5,[6]^4)} & \mathbf{(5,[6]^5)} & \mathbf{(5,[6]^6)} \end{pmatrix}$$

$$= \begin{pmatrix} (8,3) & (8,2,1) & (8,1^3) \\ \mathbf{(5,3,3)} & (4,4,2,1) & (4,4,1^3) \\ \mathbf{(5,2,2,2)} & (4,3,2,1,1) & (4,3,1^4) \\ \mathbf{(5,2,2,1,1)} & \mathbf{(5,2,1^4)} & \mathbf{(5,1^6)} \end{pmatrix}$$

red – first B hook *blue – second B hook*

Def. The **diagram** of a partition P is a poset whose rows are the parts of P (P. Oblak, L. Khatami)

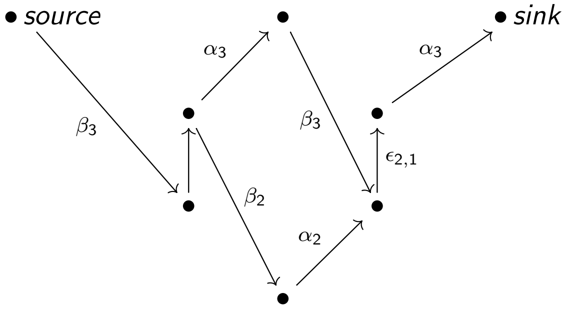


Figure: $\mathrm{Diag}(\mathcal{D}_P)$ for $P = (3, 2, 2, 1)$.

Def: U-chain in $\mathcal{D}_P$ determined by an AR $P_c \subset P$: a chain that includes all vertices of $\mathcal{D}_P$ from an AR subpartition P_c, + two tails.

The first tail descends from the source of $\mathcal{D}_P$ to the AR chain of P_c, and the second tail ascends from the AR chain to the sink of $\mathcal{D}_P$.

Oblak Recursive Conjecture

One obtains $\mathfrak{Q}(P)$ from $\mathcal{D}_P$:

(i) Let C be a longest U-chain of $\mathcal{D}_P$. Then $|C| = q_1$, the biggest part of $\mathfrak{Q}(P)$.

(ii) Remove the vertices of C from $\mathcal{D}_P$, giving a partition $P' = P - C$. If $P' \neq \emptyset$ then $\mathfrak{Q}(P) = (q_1, \mathfrak{Q}(P'))$ (Go to (i).).

Known for Q stable with two or three parts (P. Oblak determines the largest part, and L. Khatami the smallest part of $\mathfrak{Q}(P)$).

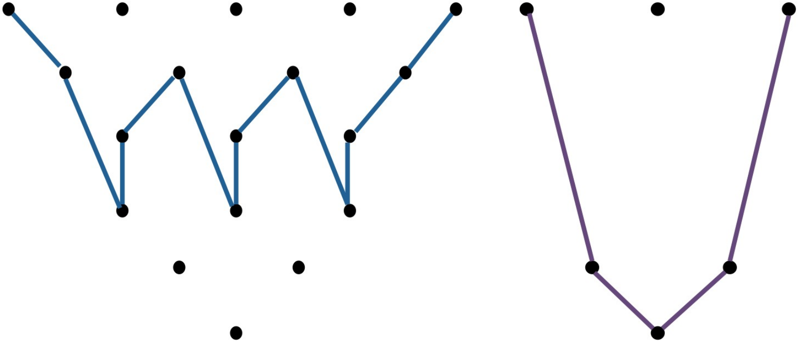


Figure: U-chain C_4: $P = (5,4,3,3,2,1)$ and new U-chain of $P' = P - C_4 = (3,2,1)$.
$\mathfrak{Q}(P) = (12,5,1)$ [figure from LK NU GASC talk 2013]

Section 2: Table Loci

Assume that $Q = (u, u - r)$ is stable. Recall that $B = J_Q$, the nilpotent Jordan block matrix of a partition Q above, and $\mathcal{N}_B =$ family of nilpotent matrices commuting with B.

Def. Let $P_{ij} \in \mathcal{T}(Q)$. Then the *locus* $\mathfrak{Z}(P_{ij})$ is the subvariety of $\mathcal{N}_B$ parametrizing matrices A such that $P_A = P_{ij}(Q)$.

Table Loci Conjecture for stable Q with two parts The locus $\mathfrak{Z}(P_{ij})$ in $\mathcal{N}_B$, is a complete intersection (CI) defined by a specified set of $i + j$ linear and quadratic equations.

Degenerate case, when $Q = (5)$ has a single part

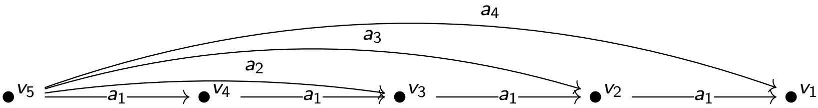


Figure: Diagram of $\mathcal{D}_Q$ and maps for $Q = (5)$.

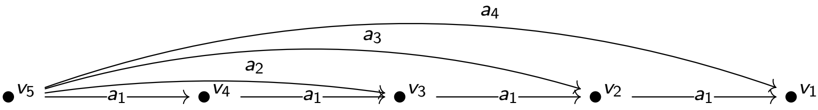
Example (Diagram and equations of column loci for $Q = (5)$.)

When[2] $a_1 = 0, a_2 \neq 0$ then we have strings (cyclic modules) $v_5 \to v_3 \to v_1$ and $v_4 \to v_2$ so $P_A = (3, 2) = [5]^2$.

When $a_1 = a_2 = 0, a_3 \neq 0$ then we have strings $v_5 \to v_2$ and $v_4 \to v_1$ and v_3, so $P_A = (2, 2, 1) = [5]^3$.

[2] We write a_1 for x_{a_1}, ...

Table and table equations - single columns - for $Q = (5)$



$\mathcal{T}(Q)$ and $\mathcal{E}(Q)$ for $Q = (5)$.
Here $B = J_Q : a_1 = 1, a_2 = a_3 = a_4 = 0$.

$\mathcal{T}(Q)$	$\mathcal{E}(Q)$
(5)	$-$
$[5]^2 = (3,2)$	a_1
$[5]^3 = (2,2,1)$	a_1, a_2
$[5]^4 = (2,1^3)$	a_1, a_2, a_3
$[5]^5 = (1^5)$	a_1, a_2, a_3, a_4

$A \in \mathcal{N}_B$

$$\begin{matrix} 0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

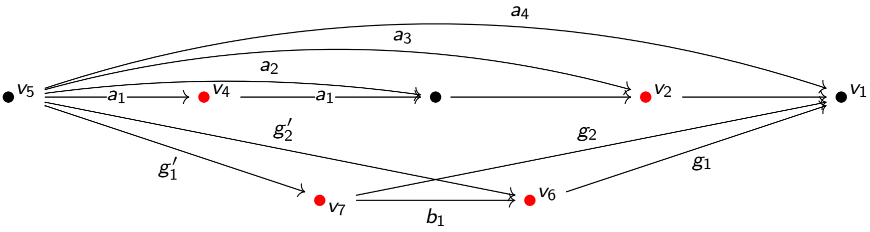


Figure: Diagram of $\mathcal{D}_Q$ and maps for $Q = (5, 2)$.

Example (Equations for table loci: $\mathcal{T}(Q)$, $Q = (5, 2)$)

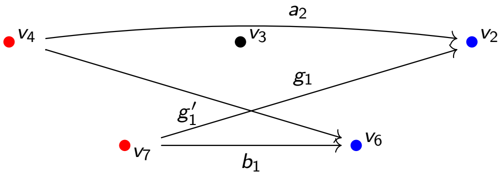
$$\mathcal{T} = \begin{pmatrix} (5,2) & (5,[2]^2) \\ (4,[3]^2) & (4,[3]^3) \end{pmatrix}; \qquad \mathcal{E} = \begin{pmatrix} - & b_1 \\ a_1 & a_1, \mathrm{Q} \end{pmatrix}$$

$$\text{where } \mathrm{Q} = \begin{vmatrix} a_2 & g_1 \\ g_1' & b_1 \end{vmatrix}.$$

When $a_1 = 0$, rest general, then $P_A = (4, 2, 1)$.

When $a_1 = 0$ but $Q \neq 0$ then $\ker(A) = \langle v_1, v_2, g_1 v_3 - a_2 v_6 \rangle$.

Equations a_1, Q for $P_{2,2}((5, 2)) = (4, 1, 1, 1)$



$$Q_{matrix} = \begin{pmatrix} * & v_4 & v_7 \\ \hline v_2 & a_2 & g_1 \\ v_6 & g_1' & b_1 \end{pmatrix}.$$

$\ker(A) = \langle v_1, v_2, g_1 v_3 - a_2 v_6, g_1 v_4 - a_2 v_7 + \cdots \rangle$, so $k_A = 4$.

Example (Table $\mathcal{T}(Q)$ and Table Loci $\mathcal{E}(Q)$ for $Q=(6,3)$)

Let $Q=(6,3)$.

$$\mathcal{T}(Q)=\begin{pmatrix} (6,3) & (6,[3]^2) & (6,[3]^3) \\ (5,[4]^2 & (5,[4]^3) & (5,[4[^4) \end{pmatrix}$$

$$\mathcal{E}(Q)=\begin{pmatrix} - & b_1 & b_1, \mathrm{Q}' \\ a_1 & a_1, \mathrm{Q}_1 & a_1, \mathrm{Q}_1, \mathrm{Q}_2 \end{pmatrix}$$

where $\mathrm{Q}'=\begin{vmatrix} a_1 & g_1 \\ g_1' & b_2 \end{vmatrix}$, $\quad \mathrm{Q}_1=\begin{vmatrix} a_2 & g_1 \\ g_1' & b_1 \end{vmatrix}$,

$$\mathrm{Q}_2=\begin{vmatrix} a_2 & g_1 \\ g_2' & b_2 \end{vmatrix}+\begin{vmatrix} a_3 & g_2 \\ g_1' & b_1 \end{vmatrix}$$

The matrix $\mathfrak{M}_A$ over $R = \mathsf{k}[t]/(t^u)$ and Diagonalization Loci.

We encode the matrix A for $Q = (u, u - r)$ in the following 2×2 matrix $\mathfrak{M}_A$ over $R = \mathsf{k}[t]/(t^u)$, after the Ljubljana school (T. Košir and B. Sethuranam) variation á la M. Boij et al.

$$\mathfrak{M}_A = \begin{pmatrix} a & g \\ ht^r & b \end{pmatrix}$$

Here, generically, $a = a_1 t + a_2 t^2 + \cdots$, $b = b_1 t + b_2 t^2 + \cdots$ $h = h_0 + h_1 t + \cdots$, and $g = g_0 + g_1 t + \cdots$.
We have $\dim_{\mathsf{k}} Ker(\mathfrak{M}_A) = k_A$. For a particular choice of coefficients, we may try to row-reduce $\mathfrak{M}_A$ over R to echelon form

$$\mathfrak{M}'_A = \begin{pmatrix} \mathrm{u}t^{s_1} & g \\ 0 & \mathrm{u}'t^{s_2} \end{pmatrix} \text{ where } \mathrm{u}, \mathrm{u}' \text{ are units.} \qquad (1)$$

Definition

Fix a stable $Q = (u, u - r)$. Let (s_1, s_2) satisfy

$$1 \leq s_1 \leq r - 1 \text{ and } 1 \leq s_2 \leq u - r. \qquad (2)$$

The *diagonalization locus* $\mathfrak{K}_{s_1,s_2}(Q) = \{A \in \mathcal{N}_B, B = J_Q$ s.t. $\mathfrak{M}_A \cong \begin{pmatrix} \mathrm{u}t^{s_1} & g \\ 0 & \mathrm{u}'t^{s_2} \end{pmatrix}\}$ with u, u' units in R.

Lemma

A general enough $A \in \mathfrak{K}_{s_1,s_2}(Q)$ satisfies $k_A = s_1 + s_2$, so P_A has $s_1 + s_2$ parts.

Conjecture.

- i For pairs (s_1, s_2) satisfying (2) the rank of each power A^i of a general enough $A \in \mathfrak{K}_{s_1,s_2}$ is determined by the pair (s_1, s_2).
- ii The table locus $\mathfrak{Z}(P_{ij})$ is just the diagonalization locus $\mathfrak{X}_{s_1,s_2}$.

We use $\mathfrak{X}_{2,1}$ in a special case to determine $P_{2,1}$.

Example

The locus $\mathfrak{Z}(P_{2,1})$ for $Q = (6,3)$ and $\mathfrak{K}_{2,1}(Q)$.
When $a_1 = 0$ and a_2, b_1, g, h are generic we have
$\mathfrak{M}_A = \begin{pmatrix} \mathrm{u}t^2 & g \\ ht^3 & \mathrm{u}'t \end{pmatrix}$ with u, u' units. Then we have
$\mathfrak{M}_A \cong \begin{pmatrix} \mathrm{u}t^2 & g \\ 0 & \mathrm{u}''t \end{pmatrix}$, so this is in the locus $\mathfrak{K}_{2,1}$, where
$k_A = 2 + 1 = 3$. Now consider the dimension k_{A^2} of the kernel of A^2. For $P_{2,1}(Q) = (5,2,2)$ we expect $k_{A^2} = 6, k_{A^3} = 7, k_{A^4} = 8$ and $A^5 = 0$.
We have $\mathfrak{M}_A^2 = \begin{pmatrix} ght^3 + \mathrm{u}^2t^4 & g\mathrm{u}'t + g\mathrm{u}t^2 \\ h\mathrm{u}'t^4 + h\mathrm{u}t^5 & \mathrm{u}'^2t^2 + ght^3 \end{pmatrix}$.
The determinant formed from the lowest order terms is zero, so in row-reducing $\mathfrak{M}_{A^2}$ we obtain orders $\begin{pmatrix} 3 & 1 \\ -\infty & 3 \end{pmatrix}$, giving $k_{A^2} = 6$. It is easy to check $k_{A^3} = 7, k_{A^4} = 8, A^5 = 0$.

Set up for the Box equation conjecture.

Let $Q = (q_1, q_2, q_3)$ be a stable partition: so $q_1 > q_2 + 1, q_2 > q_3 + 1$.

Set $\delta_1 = q_1 - q_2, \delta_2 = q_2 - q_3, \delta_3 = q_3 - (-1)$.

The box entries P_{s_1,s_2,s_3} (whatever they may be) are labelled by the triples (s_1, s_2, s_3) satisfying

$$1 \leq s_1 \leq \delta_1 - 1, 1 \leq s_2 \leq \delta_2 - 1, 1 \leq s_3 \leq \delta_3 - 1. \tag{3}$$

We denote by $\mathfrak{U}_{s_1,s_2,s_3}(Q)$ the locus in $\mathcal{U}_B, B = J_Q$ of matrices A having Jordan type $P_A = P_{s_1,s_2,s_3}$, and by $\mathfrak{Z}_{s_1,s_2,s_3}(Q)$ its closure in $\mathcal{U}_Q$. Let $R = \mathsf{k}[t]/(t^{q_1})$. The matrix $\mathfrak{M}_A$ is the 3×3 analogue of the 2×2 case, so we define the diagonalization loci $\mathfrak{K}_{s_1,s_2,s_3}$.

We propose

Box Equations Conjecture. Assume that the triple (s_1, s_2, s_3) for $A \in \mathcal{U}_Q$ satisfies (3). Then

i. The locus $\mathfrak{U}_{s_1,s_2,s_3}(Q) = \mathfrak{K}_{s_1,s_2,s_3}(Q)$.
ii. For $A \in \mathfrak{K}_{s_1,s_1,s_3}(Q)$ the triple (s_1, s_2, s_3) determines the rank of each power of A, hence determines a partition $P_A = P\mathfrak{K}_{s_1,s_2,s_3}$.
iii. If $P_A = P\mathfrak{K}_{s_1,s_2,s_3}(Q)$ then $Q(P_A) = Q$.
iv. The set $\{P\mathfrak{K}_{s_1,s_2,s_3}(Q)\}$ such that (s_1, s_2, s_3) satisfies (3) is the complete set of partitions P such that $Q(P) = Q$.

Here (i.) and (ii.) together assert that $P_{s_1,s_2,s_3} = P\mathfrak{K}_{s_1,s_2,s_3}$. This conjecture is analogous to the Table Conjecture and the Ranks of Powers Conjecture for stable partitions Q having two parts.

Lemma (in process)

The diagonalization loci $\mathfrak{K}_{s_1,s_2}$, and for Q with three parts, $\mathfrak{K}_{s_1,s_2,s_3}$ for permissible triples (s_1, s_2, s_3) are complete intersections defined by specific irreducible equations, whose format and degrees are known.

Conclusion There are many algebraic and combinatorial problems arising from considering two commuting nilpotent matrices!

Thank you!

- K. Alladi, A. Berkovish: *New weighted Rogers-Ramanjan partition theorems and their implications*, Trans. Amer. Math. Soc. 354 (2002) no. 7, 2557–2577.
- G. Andrews: *The Theory of Partitions*, Cambridge University Press, 1984, paper 1988 ISBN 0-521-63766-X.
- V. Baranovsky: *The variety of pairs of commuting nilpotent matrices is irreducible*, Transform. Groups 6 (2001), no. 1, 3–8.
- R. Basili: *On the irreducibility of commuting varieties of nilpotent matrices*. J. Algebra 268 (2003), no. 1, 58–80.
- R. Basili: *On the maximum nilpotent orbit intersecting a centralizer in $M(n, K)$*, preprint, Feb. 2014, arXiv:1202.3369 v.5.

- R. Basili and A. Iarrobino: *Pairs of commuting nilpotent matrices, and Hilbert function*. J. Algebra **320** # 3 (2008), 1235–1254.
- R. Basili, A. Iarrobino and L. Khatami, *Commuting nilpotent matrices and Artinian Algebras*, J. Commutative Algebra (2) #3 (2010) 295–325.
- R. Basili, T. Košir, P. Oblak: *Some ideas from Ljubljana*, (2008), preprint.
- J. Briançon: *Description de* $\mathrm{Hilb}^n C\{x, y\}$, Invent. Math. 41 (1977), no. 1, 45–89.
- J.R. Britnell and M. Wildon: *On types and classes of commuting matrices over finite fields*, J. Lond. Math. Soc. (2) 83 (2011), no. 2, 470–492.

- T. Britz and S. Fomin: *Finite posets and Ferrers shapes*, Advances Math. 158 #1 (2001), 86–127.
- J. Brown and J. Brundan: *Elementary invariants for centralizers of nilpotent matrices*, J. Aust. Math. Soc. 86 (2009), no. 1, 1–15. ArXiv math/0611.5024.
- J. Carlson, E. Friedlander, J. Pevtsova: *Representations of elementary abelian p-groups and bundles on Grassmannians*, Adv. Math. 229 (2012), # 5, 2985–3051.
- D. Collingwood, W. McGovern: *Nilpotent Orbits in Semisimple Lie algebras*, Van Norstrand Reinhold (New York), (1993).
- E. Friedlander, J. Pevtsova, A. Suslin: *Generic and maximal Jordan types*, Invent. Math. 168 (2007), no. 3, 485–522.

- E.R. Gansner: *Acyclic digraphs, Young tableaux and nilpotent matrices*, SIAM Journal of Algebraic Discrete Methods, 2(4) (1981) 429–440.
- M. Gerstenhaber: *On dominance and varieties of commuting matrices*, Ann. of Math. (2) 73 1961 324–348.
- C. Greene: *Some partitions associated with a partially ordered set*, J. Combinatorial Theory Ser A **20** (1976), 69–79.
- R. Guralnick and B.A. Sethuraman: *Commuting pairs and triples of matrices and related varieties*, Linear Algebra Appl. 310 (2000), 139–148.
- C. Gutschwager: *On prinicipal hook length partition and Durfee sizes in skew characters*, Ann. Comb. 15 (2011) 81–94.

T. Harima and J. Watanabe: *The commutator algebra of a nilpotent matrix and an application to the theory of commutative Artinian algebras*, J. Algebra 319 (2008), no. 6, 2545–2570.

A. Iarrobino and L. Khatami: *Bound on the Jordan type of a generic nilpotent matrix commuting with a given matrix*, J. Alg. Combinatorics, 38, #4 (2013), 947–972.

A. Iarrobino, L. Khatami, B. Van Steirteghem, and R. Zhao *Nilpotent matrices having a given Jordan type as maximum commuting nilpotent orbit*, preprint, 2014, ArXiv 1409.2192. v.2 March, 2015.

L. Khatami: *The poset of the nilpotent commutator of a nilpotent matrix*, Linear Algebra and its Applications 439 (2013) 3763–3776.

- L. Khatami: *The smallest part of the generic partition of the nilpotent commutator of a nilpotent matrix*, arXiv:1302.5741, to appear, J. Pure and Applied Algebra.
- T. Košir and P. Oblak: *On pairs of commuting nilpotent matrices*, Transform. Groups 14 (2009), no. 1, 175–182.
- T. Košir and B. Sethuranam: *Determinantal varieties over truncated polynomial rings*, J. Pure Appl. Algebra 195 (2005), no. 1, 75–95.
- F.H.S. Macaulay: *On a method for dealing with the intersection of two plane curves*, Trans. Amer. Math. Soc. 5 (1904), 385–410.
- G. McNinch: *On the centralizer of the sum of commuting nilpotent elements*, J. Pure and Applied Alg. 206 (2006) # 1-2, 123–140.

- P. Oblak: *The upper bound for the index of nilpotency for a matrix commuting with a given nilpotent matrix*, Linear and Multilinear Algebra 56 (2008) no. 6, 701–711. Slightly revised in ArXiv: math.AC/0701561.
- P. Oblak: *On the nilpotent commutator of a nilpotent matrix*, Linear Multilinear Algebra 60 (2012), no. 5, 599–612.
- D. I. Panyushev: *Two results on centralisers of nilpotent elements*, J. Pure and Applied Algebra, 212 no. 4 (2008), 774–779.
- S. Poljak: *Maximum Rank of Powers of a Matrix of Given Pattern*, Proc. A.M.S., 106 #4 (1989), 1137–1144.
- A. Premet: *Nilpotent commuting varieties of reductive Lie algebras*, Invent. Math. 154 (2003), no. 3, 653–683.

- G. de B. Robinson and R.M.Thrall: *The content of a Young diagramme*, Michigan Math.Journal 2 No.2 (1953), 157–167.
- H.W. Turnbull and A.C. Aitken: *An Introduction to the Theory of Canonical Matrices*, Dover, New York, 1961.
- R. Zhao: *Commuting Nilpotent Matrices and Normal Patterns in Oblak's Proposed Formula*, preprint 2014.

DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON MA 02115, USA
E-mail address: a.iarrobino@neu.edu

DEPARTMENT OF MATHEMATICS, UNION COLLEGE, SCHENECTADY, NY 12308, USA
E-mail address: khatamil@union.edu

DEPARTMENT OF MATHEMATICS, MEDGAR EVERS COLLEGE, CITY UNIVERSITY OF NEW YORK, BROOKLYN, NY 11225, USA
E-mail address: bartvs@mec.cuny.edu

MATHEMATICS DEPARTMENT, UNIVERSITY OF MISSOURI, COLUMBIA, MO, 65211, USA
E-mail address: zhaorui0408@gmail.com