Artinian Gorenstein algebras and the Symmetric Decomposition of the Associated Graded Algebra

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Abstract

The Macaulay duality for a (non-homogeneous) Artinian Gorenstein algebra A determines a filtration of the associated graded algebra A^* of socle degree j by ideals

$$A^* = C(0) \supset C(1) \supset \ldots \supset C(j) = 0$$

whose successive quotients $Q(a) = C(a)/C(a+1)$ are reflexive A^* modules centered at $(j-a)/2$. In codimension two, the $Q(a)$ are (shifted) graded complete intersections, and they are completely determined by A^*. However, in higher codimension the structure of $Q(a)$ can be quite subtle and $Q(a)$ may be generated in several degrees. Also, two AG algebras A, B with $A^* = B^*$ may induce different Hilbert function decompositions $H(A) = \sum H(Q(a))$, so the filtration of A^* by $\{C(a)\}$ gives additional structure on A^*.

We outline some examples, some results, and propose some open problems. These concern the structure of $Q(a)$, and the symmetric Hilbert function decomposition of $H(A)$.

Section 1: Symmetric decomposition and its dual.

Let $A = R/I, R = \mathrm{k}\{x_1, \dots, x_r\}$. *Assume A is Artinian Gorenstein!*
$\mathfrak{m} = (x_1, \dots, x_r)$ maximal ideal of R (or A).
$A^* = \oplus A_i^*$ = associated graded algebra.
$\mathrm{Soc}(A) = (0 : \mathfrak{m})$ socle of A:
$j(A)$ = socle degree: $A_j \neq 0, A_{j+1} = 0$.

Definition (Symmetric Decomposition)

We define a filtration of A^* by ideals

$$A^* = C(0) \supset C(1) \supset \cdots \subset C(j-1) = 0:$$

$$C(a)_i = \mathfrak{m}^i \cap (0 : \mathfrak{m}^{j+1-a-i}) / (\mathfrak{m}^{i+1} \cap (0 : \mathfrak{m}^{j+1-a-i}))$$

whose successive quotients $Q(a)$ satisfy

$$Q(a)_i = \frac{C(a)_i}{C(a+1)_i} \cong \frac{\mathfrak{m}^i \cap (0 : \mathfrak{m}^{j+1-a-i})}{\mathfrak{m}^i \cap (0 : \mathfrak{m}^{j-a-i}) + \mathfrak{m}^{i+1} \cap (0 : \mathfrak{m}^{j+1-a-i})}.$$

Macaulay duality theory.

$\mathfrak{D} = \mathrm{k}_{DP}[X_1, \ldots, X_r]$ the ring of divided powers: $X_i^{[k]}$ = divided k-th power; and $X_1^k = k! X_1^{[k]}$. R acts on $\mathfrak{D}$ by contraction:

$$x_i^k \circ X_i^{[K]} = \begin{cases} X_i^{[K-k]} & \text{if } K \geq k, \\ 0 & \text{if } K < k. \end{cases} \tag{0.1}$$

Let $A = R/I$ be an Artinian Gorenstein (AG) quotient, with maximum ideal $\mathfrak{m} = (x_1, \ldots, x_r)$. We have

Lemma

(F.H.S. Macaulay) There is a 1-1 isomorphism of sets

$\{$*AG quotients A of R having socle degree j*$\} \Leftrightarrow$
$\{\mathrm{k}$*-linear homomorphisms* $\phi : R \to \mathrm{k}$, *with* $\phi|_{\mathrm{m}^{j+1}} = 0$ *but* $\phi|_{\mathrm{m}^j} \neq 0\}$.

Here $A = R/I$ *with* $I = \{h \mid \phi(R \cdot h) = 0\}$.

Dual generator of A.

Denote by $\mathrm{Gor}(R,j)$ the family of AG quotients $A = R/I$ of R having socle degree j.

Lemma (Apolarity)

There is an isomorphism β of sets from $\mathrm{Gor}(R,j)$ to the set of principal inverse systems $\{R \circ f, f \in \mathfrak{D}, \deg f = j\}$. Here

$$\beta(A) = \{Q \in \mathfrak{D} \mid I \circ Q = 0\}, \text{ and } \beta^{-1}(R \circ f) = R/\operatorname{Ann} f. \quad (0.2)$$

We call such an f a *dual generator* or *apolar generator* of the AG algebra A. Given $A = R/I$ as a quotient of R, f is unique only up to multiplication by a differential unit: $\operatorname{Ann} f = \operatorname{Ann}(u \circ f)$, u a unit of R. We have $\operatorname{Hom}(A, \mathrm{k}) = \mathrm{A}^\vee \cong \mathrm{R} \circ \mathrm{f}$.

Finding $I = \mathrm{Ann}(f)$ from the dual generator f.

Example

Let $f = X^{[3]}Y^{[3]}$, $R = \mathsf{k}\{x, y\}$. Then $I = \mathrm{Ann}(f) = (x^4, y^4)$ and $H(R/I) = (1, 2, 3, 4, 3, 2, 1)$.

Let $f' = X^{[3]}Y^{[3]} + ZX^{[4]}$, $S = \mathsf{k}\{x, y, z\}$. Then $I' = \mathrm{Ann}(f') = (xz - y^3, yz, z^2, x^5, x^4y)$ and $H(S/I') = (1, 3, 3, 4, 4, 2, 1) = (1, 2, 3, 4, 3, 2, 1) + (0, 1, 0, 0, 1)$. Here $Q^\vee(1) = \langle Z, X^4\rangle \subset S \circ f'$.

Let $g = X^{[3]}Y^{[3]} + Z^{[4]}$. Then $\mathrm{Ann}(g) = (xz, yz, x^4, y^4, z^4 - x^3y^3)$ and for $B = S/\mathrm{Ann}(g)$, $H(B) = (1, 3, 4, 5, 3, 2, 1) = (1, 2, 3, 4, 3, 2, 1) + (0, 1, 1, 1)$. Here $Q^\vee(2) = \langle Z, Z^{[2]}, Z^{[3]}\rangle$. B is a *connected sum* in $\{X, Y; Z\}$.

Fact: $Q(a)$ is a reflexive module over A^*.

Theorem

The exact pairing $A \times A \to \mathsf{k}$ determines an exact pairing

$$\phi_a : Q(a)_i \times Q(a)_{j-a-i} \to \mathsf{k}. \quad (0.3)$$

The Hilbert function $H(A) = \sum_a H(Q(a))$, and each $H(Q(a))$ is symmetric with center of symmetry $(j-a)/2$.

Example

For $f' = X^{[3]}Y^{[3]} + ZX^{[4]}$ we have $Q(0) = S/(z, x^4, y^4)$ and $H(0) = (1, 2, 3, 4, 3, 2, 1)$. Also $Q(1) = (z, x^4, y^4)/(xz, yz, z^2, y^4, x^5) \cong \langle z, x^4 \rangle$ and $H(Q(1)) = (0, 1, 0, 0, 1)$.
Here $Q(0), Q(1)$ are the symmetric subquotient strata of $A'^* = S/(xz, yz, z^2, y^4, x^5)$.

The $Q(a)$ decomposition is additional structure!

Example (r=4)

Here are two height four AG algebras A, B having different Hilbert function decompositions but the same associated graded algebra[1]:
$A^* = B^* = R/\cdot(x^2, y^2, w^3, z^4)V, V = R_1$:

$A = R/\operatorname{Ann} F,\ F = X^2YZ^3W^2 + Y^2Z^4 + X^3W^3$ and
$B = R/\operatorname{Ann} G,\ G = X^2YZ^3W^2 + X^3Z^4 + Y^2W^3$.

Here $Q_A(0) = Q_B(0) = R/\operatorname{Ann} X^2YZ^3W^2 = R/(x^3, y^2, z^4, w^3)$.

$H_A(1) = 0$ but $Q_A^\vee(2) = \langle Y^2, X^3, W^3, Z^4\rangle,\ H_A(2) = (0,0,1,2,1)$.

Here $Q_B^\vee(1) = \langle X^3, Z^4\rangle$ and $H_B(1) = (0,0,0,1,1)$
$Q_B^\vee(3) = \langle Y^2, W^3\rangle$ and $H_B(3) = (0,0,1,1)$.

Note: For B we have $\left\langle X^3 = z^4 \circ G, Z^4 = x^3 \circ G\right\rangle = Q^\vee(1)$.

$\left\langle Y^2 = w^3 \circ G,\ W^3 = y^2 \circ G\right\rangle = Q^\vee(3)$ * see duality here..

When $r = 2$ the $Q(a)$ decomposition is determined by A^*: each $Q(a)$ is a shifted graded CI.

Example (r=2)

Take $F = X^{[5]} + Y^{[5]} + (X+Y)^{[4]}$. Then
$I = \mathrm{Ann}(F) = (xy(x-y), xy(x+y) - 2(x^4+y^4) + 4x^5)$ and
$I^* = (xy(x-y), xy(x+y), x^5 - y^5)$.
$H(A) = (1,2,3,2,2,1), H(0) = (1,2,2,2,2,1), H(1) = (0,0,1)$.
$C(0) = R/I^*, C(1) = (xy, x^5-y^5)/I^*, C(2) = 0$

$Q(0) = R/(xy, x^5-y^5), Q(1) = (xy)/I^* \cong \langle \overline{xy} \rangle$

$Q(0)^\vee = R \circ F_5 = R \circ (X^{[5]} + Y^{[5]})$

$= \langle F_5, \{X^{[i]}, Y^{[i]}, 1 \leq i \leq 5\}, 1 \rangle,$

$Q(1)^\vee = \langle (X+Y)^{[2]} \rangle = xy(x+y) \circ F.$

When $r = 3$: *There are height three examples of the same phenomenon* $A^* = B^*$ *as in height four*

Finding the $Q^\vee(a)$ decomposition from $f \in \mathfrak{D}$.

Idea In the previous example $j = 8$ and
$G = X^2YZ^3W^2 + X^3Z^4 + Y^2W^3$:

When $h \circ G \in \mathfrak{D}_{\leq 8-\deg h - a)}$ and has a term of that degree then the class of $h \circ G \in Q^\vee(a)$.

Ex. $z^4 \circ G = X^3$ of degree $(8-4-1)$ so $X^3 \in Q_B^\vee(1)$
$w^3 \circ G = Y^2$ of degree $(8-3-3)$, so $Y^2 \in Q_B^\vee(3)$

Example (Connected sum)

Let $F = X^{[5]} + Y^{[4]} + Z^{[2]}$. Then $I = (xy, xz, yz, z^2 - y^4, y^4 - x^5)$,
$Q(0) = R/\operatorname{Ann} X^5 = R/(y, z, x^6)$, $H_F(0) = (1,1,1,1,1,1)$ so
$Q^\vee(0) = \langle 1, X, X^2, X^3, X^4, X^5 \rangle \cong Q(0)^\vee$
$Q^\vee(1) = \langle Y, Y^2, Y^3 \rangle, H(1) = (0,1,1,1)$
$Q^\vee(3) = \langle Z \rangle$ and $H(3) = (0,1,0)$.

Gorenstein sequences, I.

I. Graded algebras: Two variables: Gorenstein $\Rightarrow$ C.I. (complete int.). $A = R/I, I = (g, h), \deg g = a, \deg h = b, a + b = j + 2$. $H(A) = (1, 2, 3, 4, \ldots, a - 1, a_{a-1}, \ldots, a_{b-1}, a - 1, \ldots, 2, 1)$. (classical)

Three variables: (Buchsbaum-Eisenbud structure theorem, R. Stanley, S.J. Diesel): H is a G.Seq iff $\Delta H_{\leq j/2}$ is an O-sequence. Example: $H = (1, 3, 5, 6, 7, 6, 5, 3, 1_8), \Delta H_{\leq 4} = (1, 2, 2, 1, 1)$.

Four variables: Conjecture: H is a G.Seq iff $\Delta H_{\leq j/2}$ is an O-sequence. (Yes in special cases: J. Migliore, U. Nagel and F. Zanello, H. Srinivasan and A.I., H. Srinivasan and S. Sumi)

Five and more variables: Non-unimodal examples: Glueing B and $\mathrm{Hom}(B, \mathsf{k})$: R. Stanley $H = (1, 13, 12, 13, 1)$, D. Bernstein and A.I. $(r = 5, j = 16)$. Small socle degrees 4,5 (J. Ahn and Y.S. Shin).

- Non-unimodal homog Gor seq. with $r = h_1 \geq h_i$ for all i: such

Gorenstein sequences, II.

II. Non-graded algebras. Two variables: CI:
$H(A) = (1, 2, 3, 4, \dots, d, h_d, h_{d+1}, \dots, 2, 1)$ with
$\Delta h_{i+1} = h_i - h_{i+1} \leq 1$ for $i \geq d$.(F.H.S. Macaulay, 1904).

Three and more variables: What are the non-homog Gor Seq? Open even for C.I. despite numerous examples, and B-E Pfaffian structure theorem for Gorenstein algebras of height $r = 3$!
A main tool is the symmetric Hilbert function decomposition.

Theorem (Necessary condition for Gorenstein sequence)

For H to be a (nongraded) Gorenstein sequence, we must have

a. *H has a symmetric decomposition $H = \sum_0^{j-2} H(a)$ with $H(a)$ symmetric about $(j-a)/2$*
b. *Each partial sum $\sum_{u=0}^{k} H(u)$ must be an O-sequence.*

But these are not enough to determine possible H when $r > 2$.

Non-ubiquity.

Example ($r = 5$)

Let $R = \mathsf{k}\{x, y\}, \mathfrak{D} = \mathsf{k}_{DP}[X, Y]$ and $S = \mathsf{k}\{x, y, z, w, u\}$, $\mathfrak{E} = \mathsf{k}_{DP}[X, Y, Z, W, U]$. Let $F = f + f_7$ where $f = X^4Y^4$ and $f_7 = ZX^6 + WY^6 + U(X + Y)^{[6]}$, let $A = S/\operatorname{Ann} F$.
$Q_A(0) = R/\operatorname{Ann} f = R/(x^5, y^5)$, $H_A(0) = (1, 2, 3, 4, 5, 4, 3, 2, 1)$.
Since $\operatorname{Ann} f = (x^5, y^5)$ we have $(\operatorname{Ann} f) \circ f_7 \supset \langle x^5, y^5 \rangle \circ f_7$, implying the first half of
$Q(1)^\vee = \langle Z, W, U, x^5 \circ f_7, y^5 \circ f_7, X^5, Y^5, X^6, Y^6, (X + Y)^{[6]} \rangle$ and $H_A(1) = (0, 3, 2, 0, 0, 2, 3)$. Also, we can show $H_A(2) = (0, 0, 3, 3, 3, 0, \ldots)$ and $H(A) = (1, 5, 8, 7, 8, 6, 6, 2, 1)$.

Prop. When an AG algebra B satisfies $H_B(0) = H_A(0), H_B(1) = H_A(1)$ then $H_B(2) \geq (0, 0, 2, 1, 2)$.

Conclude: $H_A(0) + H_A(1)$ is not a Gorenstein sequence.

Dual generator linear in some variables.

We originally tried to understand an apparently strange example of a complete intersection, $r = 3$, where $H(2) = (1, 0, 1)$. We found that such behavior is frequent when F is linear in some variables.

Theorem (Restriction on the decomposition)

Let $f \in \mathfrak{D}_j$, let $k_1, \ldots, k_s$ be integers satisfying $j-2 \geq k_1 \geq \cdots \geq k_s \geq 1$ and for $1 \leq t \leq s$ choose homogeneous polynomials $h_t \in \mathfrak{D}_{k_t}$. Let $a_t = j - (k_t + 1)$ and consider

$$F = f + \sum_{t=1}^{s} h_t \cdot Z_t \in \mathfrak{E}.$$

Then $Q^\vee(u) = 0$ for $u \notin \{0, a_1, \ldots, a_s\} \cup \{a_{t_1} + a_{t_2} \mid 1 \leq t_1 \leq t_2 \leq s\}$.

Why do we get a non-zero $H_A(2u)$?

Example

Let $f = X^{[3]}Y^{[3]}$, $h = X^{[4]} + Y^{[4]}$, $F = f + hZ$.

Then $Q^\vee(2u)_2 = Q^\vee(2)_2$ has two independent elements given by

$-(yz - x^3) \circ F = XZ$,
$-(xz - y^3) \circ F = YZ$.

The key relation $x^3 \circ f = yz \circ hZ = y \circ h \quad (= Y^{[3]})$

allows to "kick out" $x^3 \circ hZ = XZ$ in $-(yz - x^3) \circ F$.

Here $Q^\vee(1) = \langle Z, X^{[4]} + Y^{[4]} \rangle$, $H_F = (1, 3, 5, 4, 4, 2, 1)$.

Note: For a non-maximal case, we can take $h = X^{[4]} + XY^{[3]}$.
Then $Q^\vee(2)_2 = \langle XZ \rangle$.

Parametrizing AG algebras- fix H

r=2. The parametrization of Gor(H), Z_H (non-graded $\mathcal{A}, H(\mathcal{A}) = H$), and G_H (graded $\mathcal{A}$) is understood (J. Briançon, M. Granger, A.I.).
Answer. Z_H has affine (opens in an affine space) fibres over the smooth projective variety G_H.
Questions. Although the cohomology groups of G_J are known (Yaméogo-AI), the cohomology ring structure of G_H is in general open. The closures $\overline{Z_H}$ are not known and puzzled the NIce group.

The smooth projective varieties G_H have large divisor groups, and determining the cohomology ring structure may relate to other problems on weighted projective space, and curves.

r=4. Certain Gor(H) have been parametrized, and connections made to the Hilbert scheme $\mathfrak{H}$ of curves on $\mathbb{P}^3$ as mentioned earlier (J.O. Kleppe, H. Srinivasan-AI et al). There, certain Irreducible components of Gor(H) correspond to components of $\mathfrak{H}$.

Isomorphism classes of AG algebras.

Several groups have studied isomorphism classes of AG algebras, or more general Artin algebras. Early studies in small lengths were by G. Scorza (1935, $n \leq 4$), J. Briançon (some CI height two), G. Mazzola (1979), $n = 5$, and more recently by CTC Wall (cubics), J. Emsalem and AI.
Very recent studies by G. Casnati and R. Notari (especially AG graded of lenghts up to 13), B. Poonen, J. Elias and M. Rossi, J. Elias and G. Valla, J. Jelisiejew, and A. Isaev have brought new methods and sometimes surprising new results.
For example, J. Jelisiejew recently used representation theory to show there are 11 isomorphism types for $H = (1, 3, 3, 3, 1)$, and showed that $\mathcal{A} \cong \mathcal{A}^*$ when $H = (1, r, r, 1)$ (reproving result of J. Elias- M. Rossi), and similarly for socle degree 4. A. Isaev presents a new criterion for isomorphism of Artinians.

Deforming graded AG algebras.

I. **Graded case**: must deform within Hilbert function class!
$r = 2$: Gor(H) is irreducible, an open dense in $G(H)$ (all Artins).
$r = 3$: Gor(H) is irreducible, with those having the smallest number of generators $2d(I) + 1$ specializing to the others.

$r \geq 4$: Gor(H) in general has several irreducible components, related to Hilbert schemes parametrizing $I_{\leq j/2}$. (M. Boij, A.I and V. Kanev, H.Srinivasan, J.O.Kleppe).

Example.[2] $H = (1, r, 2r-1, 2r, \ldots, 2r-1, r, 1)$, $r = 7$ or $r \geq 9$:the first irreducible component is related to $2r$ self-associated points on $\mathbb{P}^{r-1}$ (A. Coble, I.Dolgachev, V.Shokurov).).
Second irred. component: related to $(1, r-1, r-1, 1)$ "generic" AG algebras (small tangent space argument J. Emsalem, A.I).

[2] [?] §6.2.

Deforming non-graded AG algebras.

II. Non-graded case: Hilbert function may change.

$r = 2$: May deform AG $\mathcal{A}$ to curvilinear $\mathcal{A}'$; $H(\mathcal{A}') = (1, 1, \ldots, 1)$[3].

$r = 3$: The Buchsbaum-Eisenbud structure theorem implies, AG algebras for $r = 3$ can be smoothed [**?**].

Question: Does the closure of $CI(n, 3)$ include $\mathrm{Gor}(n, 3)$? What is the dimension of $CI(H)$ in height 3?

$r \geq 4$: There have been some example deformation studies:

C. Bertone, F. Chioffi, M. Roggero $H = (1, 7, 7, 1)$ AG algebras are smoothable [**?**].

V. Kanev and I. [**?**, Chapter 6]. Using annihilator schemes.

G. Casnati and R. Notari [**?**, **?**]. AG $\mathcal{A}$, lengths 10,11.

G. Casnati, J. Jelisiejew, R. Notari [**?**] (ray families)

J. Jelisiejew [**?**]: $H = (1, 3, 3, \ldots, 3, 1)$.

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[3] [**?**].

Semicontinuity of $\dim \mathfrak{m}^i$ and $\dim 0 : \mathfrak{m}^a$.

We denote by $\ell(V)$ the *length*, or vector space dimension of V.

Lemma

Fix an integer n. Then the following invariants are semicontinuous (we give the open condition) on the family Artin(r, n) of Artinian quotients $\mathcal{A}$ of R having $\dim_{\mathsf{k}} \mathcal{A} = n$.

i. $\ell(0 : \mathfrak{m}^i) \leq a$.

ii. $\ell(\mathfrak{m}^i) \geq b$.

Upper semi-continuity of $\dim_k(0 : \mathfrak{m}_{\mathcal{A}}{}^b)$.

When $\mathcal{A}$ is Artinian Gorenstein, the Hilbert function $H = H(\mathcal{A})$ determines both $\dim_k \mathfrak{m}_{\mathcal{A}}{}^i$ and, by duality $\dim_k(0 : \mathfrak{m}_{\mathcal{A}}{}^b)$. We have

$\dim \mathfrak{m}_{\mathcal{A}}{}^i = \sum_{u=i}^{j} H(\mathcal{A})_u$ for any Artin; and for AG algebras we have
$\dim_k(0 : \mathfrak{m}_{\mathcal{A}}{}^b) = \dim_k \mathcal{A} - \dim_k \mathfrak{m}_{\mathcal{A}}{}^b$.

Also, for $V(t), W(t), t \in T$ two VS of fixed dimensions, we have

$\dim V(t) \cap W(t) \leq \dim_k V_{t_0} \cap W_{t_0}$
for t in a small neighborhood of t_0. (upper semicontinuity).

Lemma

Let $\mathcal{A}_t, t \in T$ (a parameter space) be a family of AG algebras having fixed Hilbert function H. Then for each pair (i, k) the dimension of $\mathfrak{m}_{\mathcal{A}(t)}{}^i \cap (0 : \mathfrak{m}_{\mathcal{A}(t)}{}^a)$ is upper semicontinuous on T.

The Hilbert function of $\mathfrak{m}^i \cap (0 : \mathfrak{m}^b)$

The HF of $\dim_k(\mathfrak{m}^i/(\mathfrak{m}^i \cap (0 : \mathfrak{m}^b))$ is
$(0, 0, \ldots, 0_{i-1}, t_{i,b}, t_{i+1,b}, \ldots)$ where $t_{i,b} = \sum_{a \geq j+1-b-i} H(a)_i$,

is the sum of the entries in the i-th column of $\mathcal{D}$ on or below the b-th from the right, rising diagonal of $\mathcal{D}$ through $H(0)_{j+1-b}$.

The HF of $\mathfrak{m}^i/(\mathfrak{m}^i \cap (0 : \mathfrak{m}^b))$ is

$(0, 0, \ldots, 0_{i-1}, n_{i,b}, n_{i+1,b}, \ldots)$ where $n_{i,b} = \sum_{a=0}^{j-b-i} H(a)_i$.

Let $N_{i,b} = \sum_{u \geq i} n_{u,b} = \dim_k(\mathfrak{m}^i/(\mathfrak{m}^i \cap (0 : \mathfrak{m}^b))$. $N_{i,b}$ is the content of a triangle of $\mathcal{D}$ bounded by $H(0)_i, \ldots, H(0)_{b-1}$ on top, by the rising diagonal from $H(j-b-i)_i$ to $H(0)_b$ below.

Lemma

$N_{i,b}$ is lower semicontinous on a family in $\mathrm{Gor}(H)$*:*
$N_{i,b}(t) \geq N_{i,b}(t_0)$ for t in a neighborhood of $t_0 \in T$.

Symmetric decomposition as an obstruction.

We can use the symmetric decomposition to identify irreducible components of Gor(H), when H is not symmetric.

Example

Let $H = (1, 3, 3, 2, 1, 1)$

$$\mathcal{D}_1 = \begin{matrix} 1 & 1 & \color{red}{1} & \color{red}{1} & 1 & 1 \\ 0 & 1 & \color{red}{1} & 1 & & \\ 0 & 1 & 1 & & & \end{matrix} \qquad \mathcal{D}_2 = \begin{matrix} 1 & 1 & \color{red}{1} & \color{red}{1} & 1 & 1 \\ 0 & 1 & \color{red}{2} & 1 & & \\ 0 & 0 & & & & \\ 0 & 1 & & & & \end{matrix}$$

Here $N_{2,2}(\mathcal{D}_1) = 3, \mathcal{N}_{2,2}(\mathcal{D}_2) = 4$. So there is no specialization from $\mathcal{D}_1$ (dim 18) to $\mathcal{D}_2$ (dim 16). But generically an $I \in \mathcal{D}_1$ is a CI, and those in $\mathcal{D}_2$ are not CI. So there is no specialization from any family in $\text{Gor}_{\mathcal{D}_2}(H)$ to a general element of $\text{Gor}_{\mathcal{D}_1}(H)$.

Open questions about AG algebras

Answers are known for $r = 2$. Begin with $r = 3$:

Question

What are the possible Hilbert functions $H(\mathcal{A})$, say given the socle degree j?
Given H what are the possible symmetric decompositions?

Question

Fix a Gorenstein sequence H: describe the irreducible component structure of $\mathrm{Gor}(H)$.

Question

Determine the "geography" of the family of length-n AG algebras: what are the irreducible components and their intersections?
Find "generic" local AG algebras"? Are any (except k) rigid?

"Linear algebra" questions

Question: For a fixed, arbitrary $f \in \mathfrak{D}_{2a}$ and a general enough linear form L, is

$$L^{[a]} \notin (AnnL)_a \circ f?$$

We show the answer is "Yes" in odd degree, $f \in \mathfrak{D}_{2a+1}$.

Conjecture (typical)

Given f_j, the vector space $V = R_a \circ f_j$ and a basis B for $R_a/(\mathrm{Ann}\, f_j)_a$ so $\langle B \rangle \circ f_j = V$, let $h = h_b, b = j - u - 1$ be generic. For $k = b - (j - a)$ the vector space $W = (R_k \circ h) \cap V$ has an expected dimension by compressed algebra theory. Now write $W = C \circ f_j$, so C is some matrix in terms of B that depends on the choice of h. Then we claim conjecturally that $\langle C \circ h \rangle$ satisfies

$$\dim(C \circ h) = \min\{\dim_{\mathsf{k}} C, \dim_{\mathsf{k}} R_{s-a}\}$$

and $C \circ h$ is maximally disjoint from $(\mathrm{Ann}\, f_j)_a \circ h$.

Generalize to ideals $J \subset \mathfrak{m}_{\mathcal{A}}$ in place of $\mathfrak{m}_{\mathcal{A}}$.

T. Harima and J. Watanabe studied "central simple (CS) modules" of a pair $(\ell, \mathcal{A})$ where $\mathcal{A}$ is an Artinian algebra and ℓ an element of $\mathcal{A}$ (see [**?**]). A CS module is equivalent to taking all strings of a given length in the action of ℓ on $\mathcal{A}$, so the CS numerical data about L is equivalent to the *Jordan type* of ℓ.[4]

We can generalize both this notion of central simple modules, and the $C(a)$ filtration to the associated graded algebra $\mathrm{Gr}_J(\mathcal{A})$ of an Artinian algebra $\mathcal{A}$ with respect to powers of any ideal $J \subset \mathfrak{m}_{\mathcal{A}}$. When $\mathcal{A}$ is also AG, then we may construct as we did for $J = \mathfrak{m}$ ideals $C_J(a)$ of $\mathrm{Gr}_J(\mathcal{A})$ and subquotients $Q_J(a)$ that are reflexive modules over $\mathrm{Gr}_J(\mathcal{A})$. Here $\mathcal{A}$ need not be homogeneous, but not all results of [**?**] extend to the non-homogeneous setting (see [**?**]).

[4] See the talk "Jordan type of multiplication maps."

Applications: Mapping germs, rank of forms

I. A mapping germ $f = (f_1, \ldots, f_m) : (\mathbb{C}^r, 0) \to (\mathbb{C}^m.0)$ is *finite* if the local algebra $\mathcal{A} = \mathsf{k}\{x_1, \ldots, x_r\}/(f_1, \ldots, f_m)$ is Artinian. Then $\mathcal{A}$ up to isomorphism is the right-left class of f. The symmetric HF decomposition of $H(\mathcal{A})$ for a complete intersection $(m = r)$ or Gorenstein mapping germ is an invariant, and may obstruct deforming the germ. In general the mapping germ specialists have not arrived at number of variables and lengths to use the theory.

II. Rank of forms: recently A. Bernardi, J. Jelisiejew, P. Marques, K. Ranestad [**?**, **?**] and others have studied the rank of cubic forms f in $\mathbb{P}^r$ by dehomogenizing, then considering (Gorenstein) non-homogeneous schemes $\mathfrak{Z}$ whose ideal $I_{\mathfrak{Z}} \subset \mathrm{Ann}(f_0)$. Bounding the dimension of these families by analyzing the possible HF decompositions has been a part of their study.

Comments, conclusion.

The symmetric decomposition of AG algebras has come into more use in the last several years, motivated by the applications and a desire to parametrize them. This talk is based on different sources, including Pedro Marques and my preprint [5].

Note that the "linear algebra" conjectures referred to in a previous slide, are not just linear. The first question come up when trying to start with a given AG graded algebra, and trying to arrange a new algebra with Hilbert function $(H(A), 1^s)$ with a tail of ones; the conjecture has to do with showing that, given f, we may choose h general enough such that $f + Zh$ has an expected HF.

Thank you!

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[5] We plan to post this summer 2016.

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