When do two nilpotent matrices commute?

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Abstract

The similarity class of an $n \times n$ nilpotent matrix B over a field k is given by its Jordan type, the partition P of n, specifying the sizes of the Jordan blocks. The variety $\mathcal{N}(B)$ parametrizing nilpotent matrices that commute with B is irreducible, so there is a partition $Q = \mathfrak{Q}(P)$ that is the generic Jordan type for matrices A in $\mathcal{N}(B)$. Here $\mathfrak{Q}(P)$ has parts that differ pairwise by at least two, and $\mathfrak{Q}(P)$ is stable: $\mathfrak{Q}(\mathfrak{Q}(P)) = \mathfrak{Q}(P)$.

We discuss what is known about the map P to $\mathfrak{Q}(P)$, in particular a recursive conjecture by P. Oblak (2008), very recently shown by R. Basili after partial results by P. Oblak, T. Kosir, L. Khatami, and others.

We then state a "Table theorem" when Q has two parts and discuss a "Box Conjecture" in general for the set of partitions P having a given partition Q as maximum commuting orbit: so $Q = \mathfrak{Q}(P)$. This is both quite beautiful and very mysterious: we keep on asking ourselves "Why?"

Section 1: The map $\mathfrak{Q} : P \to \mathfrak{Q}(P)$

Definition (Nilpotent commutator $\mathcal{N}_B$)

$V \cong \mathsf{k}^n$ vector space over an infinite field k.
$A, B \in \mathrm{Mat}_n(\mathsf{k}) = \mathsf{Hom}_{\mathsf{k}}(\mathsf{V}, \mathsf{V})$;

$P \vdash n$ partition of n;
J_P = Jordan block matrix of Jordan type P

$\mathcal{C}_B \subset \mathrm{Mat}_n(\mathsf{k})$ centralizer of B.

$\mathcal{N}_B \subset \mathcal{C}_B$: the variety of nilpotent elements of $\mathcal{C}_B$.

P_A = Jordan type of A.

Fact: $\mathcal{N}_B$ is an irreducible variety [Bas1, BI].

Def: $\mathfrak{Q}(P) = P_A$ for A generic in $\mathcal{N}_B, B = J_P$.

Problem 1. Given the partition P, determine $\mathfrak{Q}(P)$

Fact. $\mathfrak{Q}(P)$ is Rogers-Ramanujan (RR): the parts of $\mathfrak{Q}(P)$ differ by at least two.

Problem 2. Given the RR partition Q determine $\mathfrak{Q}^{-1}(Q)$.

Prob. 1: Recursive conjecture of P. Oblak (2008) for $\mathfrak{Q}(P)$: work of P. Oblak, P. Oblak-T.Košir, L. Khatami, I-Khatami, R. Basili.

Prob 2: Table conjecture of P. Oblak and R. Zhao (2012,2013) is shown for $Q = (u, u - r), r \geq 2$. Box conjecture for $\mathfrak{Q}^{-1}(Q)$ is open for Q RR with $k > 2$ parts..

Classical problem: but not studied classically. Connected with Hilbert scheme work of J. Briançon, M. Granger, R. Basili, V. Baranovsky, A. Premet. See Ngo-Sivic.

In 2006, three groups began to work on the $P \to \mathfrak{Q}(P)$ problem, independently

P. Oblak and T. Košir (Ljubljana)

D. Panyushev (Moscow)

R. Basili, I.-, and L.Khatami (Perugia, Boston).

Links to work of E. Friedlander, J. Pevtsova, A. Suslin, on representations of Abelian p-groups [FrPS,CFrP]s

Definition (Almost rectangular)

Let $B = J_{(n)}$, and denote by $[n]^k = P_{B^k}$..

For $n = kq$, $[n]^k = (q^k) = (q, q, \dots, q)$.
For $n = kq + r, 0 < r < k$, $[n]^k = ((\lceil n/k \rceil)^r, (\lfloor n/k \rfloor)^{k-r})$
Here $[n]^k$ has k parts that differ at most by 1.
We term $[n]^k$ *almost rectangular (AR)*.

Ex. $n = 5$,
$[5]^2 = (3, 2), [5]^3 = (2, 2, 1), [5]^4 = (2, 1, 1, 1), [5]^5 = (1, 1, 1, 1, 1)$.

Theorem ((R. Basili) $\mathfrak{Q}$ for $r_P = 1$)

For $P = [n]^k, \mathfrak{Q}(P) = [n]$ *and* $\mathfrak{Q}^{-1}([n]) = \{[n]^k, 1 \leq k \leq n\}$

Example ($P = (3, 1)$ does not commute with $[4]$.)

$$A = \begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad A^2 = \begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad J_P = \begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Figure : $A = J_{[5]}, A^2$, and J_P where $P = [5]^2 = (3, 2)$. Here A^2 is conjugate to J_P.

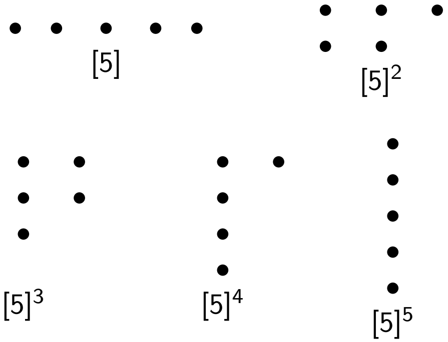


Figure : The AR partitions of 5.

Theorem (R. Basili [Bas1])

$\mathfrak{Q}(P)$ has r_P parts, where r_P= minimum number of AR partitions P_i such that $P = \bigcup P_i$.

Theorem (R. Basili and I.- [BI])

$\mathfrak{Q}(P) = P \Leftrightarrow P$ is RR: the parts of P differ pairwise by at least 2.

Def. We call a $P \mid \mathfrak{Q}(P) = P$ " stable'

also "super-distinct" or "Rogers-Ramanujan" [AlBe, An].

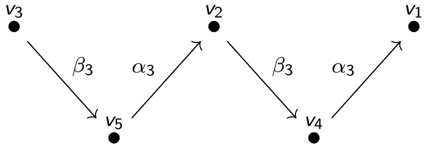
Example

$$P = (\underbrace{3}, \underbrace{1}), \quad \mathfrak{Q}(P) = (3, 1).$$

$$P = (\underbrace{5, 4}, \underbrace{3, 3, 2}, \underbrace{1}), \quad \mathfrak{Q}(P) = (12, 5, 1).$$

Poset $\mathcal{D}_P$

Rows of vertices: Span the maximal irreducible B - invariant subspaces of V: each row corresponds to a part of P.
Arrows: non-zero elements in $A \in \mathcal{U}_B$ (max subalgebra of $\mathcal{N}_B$).



$$A = \left(\begin{array}{ccc|cc} 0 & x_{\alpha_3\beta_3} & x_{(\alpha_3\beta_3)^2} & x_{\alpha_3} & x_{\alpha_3\beta_3\alpha_3} \\ 0 & 0 & x_{\alpha_3\beta_3} & 0 & x_{\alpha_3} \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & x_{\beta_3} & x_{\beta_3\alpha_3\beta_3} & 0 & x_{\beta_3\alpha_3} \\ 0 & 0 & x_{\beta_3} & 0 & 0 \end{array}\right), v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix}$$

Figure : Generic element A of $\mathcal{U}_B$, $B = J_P$ where $P = (3, 2)$.

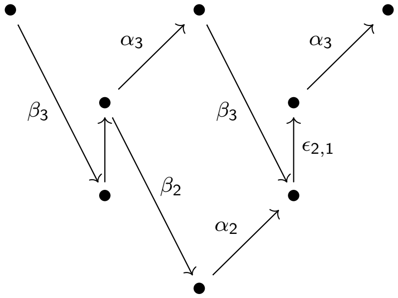


Figure : $\mathrm{Diag}(\mathcal{D}_P)$ for P=(3,2,2,1).

$$\left(\begin{array}{ccc|cc|cc|c}
0 & x_{c_3} & x_{(c_3)^2} & x_{\alpha_3} & x_{\alpha_3 c_2} & x_{\alpha_3 e_{21}} & x_{\alpha_3 c_2'} & x_{\alpha_3 e_{21} \alpha_2} \\
0 & 0 & x_{c_3} & 0 & x_{\alpha_3} & 0 & x_{\alpha_3 e_{21}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & x_{e_{21}\beta_3} & x_{43} & 0 & x_{c_2} & x_{e_{21}} & x_{c_2'} & x_{\alpha_3 e_{21}} \\
0 & 0 & x_{e_{21}\beta_3} & 0 & 0 & 0 & x_{e_{21}} & 0 \\
\hline
0 & 0 & x_{63} & 0 & x_{\alpha_2 \beta_2} & 0 & x_{\alpha_2 \beta_2 e_{21}} & x_{\alpha_2 \beta_2} \\
0 & 0 & x_{\beta_3} & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & x_{\beta_2 e_{21} \beta_3} & 0 & x_{\beta_2} & 0 & x_{\beta_2 e_{21}} & 0
\end{array}\right)$$

$$x_{63} = x_{\alpha_2 \beta_2 e_{21} \alpha_3}$$

Figure : Generic element A of $\mathcal{U}_B$ for $P = (3, 2, 2, 1)$.

Relation with Artin algebras

Let $PCN_n = \{\text{pairs } A, B \text{ of } n \times n \text{ nilp. matrices}, [A, B] = 0\}$.

V. Baranovsky (2001) showed that PCN_n is irreducible.
When $\mathrm{char}\ \mathsf{k} = 0$ he used a result of J. Briançon (1978) and a proof of M. Granger (1983) that the Hilbert scheme $\mathrm{Hilb}^n \mathsf{k}\{x, y\}$ parametrizing length-n Artin algebras is irreducible.

R. Basili (2003, $\mathrm{char}\ k \geq n/2$) and A. Premet (2003, all infinite k) showed the irreducibility of PCN_n directly.
This implies the irreducibility of $\mathrm{Hilb}^n \mathsf{k}\{x, y\}$ for all infinite k.

Pencil Lemma (I.-R. Basili)

Let $\mathcal{A} = \mathsf{k}[A, B]$ be an Artin algebra with FHS $H = H(\mathcal{A})$ and $\mathrm{char}\ \mathsf{k} \geq n = \dim_{\mathsf{k}} \mathcal{A}$. Then $P_C = H^\vee$, the conjugate of H, for $C = A + \lambda B, \lambda \in \mathsf{k}$ generic.

Theorem (P. Oblak and T.Košir [KO])

For $A \in \mathcal{N}_B$ generic, the Artin algebra $\mathrm{k}[A, B]$ *is Gorenstein, so a complete intersection (CI).*

Proof. Uses an involution of the poset $\mathcal{D}_P$ of $\mathcal{N}_B$. See also [BIK, Thm. 2.20].

Corollary (ibid. vith F.H.S. Macaulay [Mac])

$\mathfrak{Q}(P)$ is stable! ($\mathfrak{Q}(P)$ is RR: Parts differ pairwise by at least two)

Proof. After Macaulay, if $\mathcal{A}$ is CI, the jumps $e_i = H_i - H_{i+1}$ of $H = H(\mathcal{A})$ are each less or equal 1, which implies $H^\vee$ is RR.

Example

For $H = (1, 2, 3, 4, 3, 2, 2, 1)$, $H^\vee = (8, 6, 3, 1)$, which is RR.

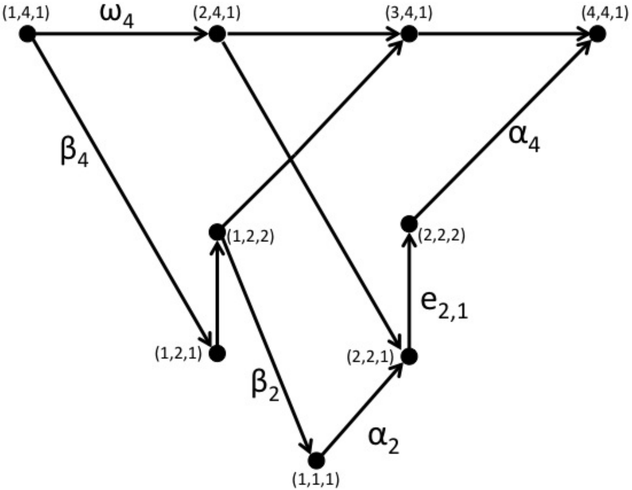


Diagram of the poset $\mathcal{D}_P$ and maps, $P = (4, 2, 2, 1)$.

Definition (Poset $\mathcal{D}_P$ [Obl1, KO, BIK, Kh1])

Let $P \vdash n, P = (\dots i^{n_i} \dots), S_P = \{i \mid n_i > 0\}$.The poset $\mathcal{D}_P$ has rows of the Ferrers graph of P, each row centered on the y-axis.
There are n_i rows of length i:

$$(u, i, k), 1 \leq u \leq i, 1 \leq k \leq n_i.$$

Let i^-, i^+ be the next smaller, next larger elements of S_P. The edges of $\mathcal{D}_P$ correspond to *elementary maps:*

Maps and edges of the diagram $\mathcal{D}_P$

(i) $\beta_i = \beta_{i,i^-} : (u, i, n_i) \to (u, i^-, 1)$ for $u \leq i^-$.

(ii) $\alpha_i = \alpha_{i^-,i} : (u, i^-, n_{i^-}) \to (u + i - i^-, i, 1)$.

(iii) $e_{i,k} : (u, i, k) \to (u, k, k + 1), 1 \leq u_i \leq i, 1 \leq k < n_i$.

(iv) When i is isolated: $i - 1 \notin S_P, i + 1 \notin S_P$,
$\omega_i : (u, i, n_i) \to (u + 1, i, n_i)$ for $1 \leq u < i$.

(Each map is 0 on the points of $\mathcal{D}_P$ not listed)

The *diagram* of a poset has the covering edges only.
The $\mathcal{D}_P$ is related to a a maximum nilpotent subalgebra $\mathcal{U}_B \subset N_B, B = J_P$: $v < v'$ if $\exists A \in \mathcal{U}_B \mid A_{v,v'} \neq 0$.

Def: U-chain in $\mathcal{D}_P$ determined by an AR $P' \subset P$: a chain that includes all vertices of $\mathcal{D}_P$ from an AR subpartition P', $+$ two tails.

The first tail descends from the source of $\mathcal{D}_P$ to the AR chain of P', and the second tail ascends from the AR chain to the sink of $\mathcal{D}_P$.

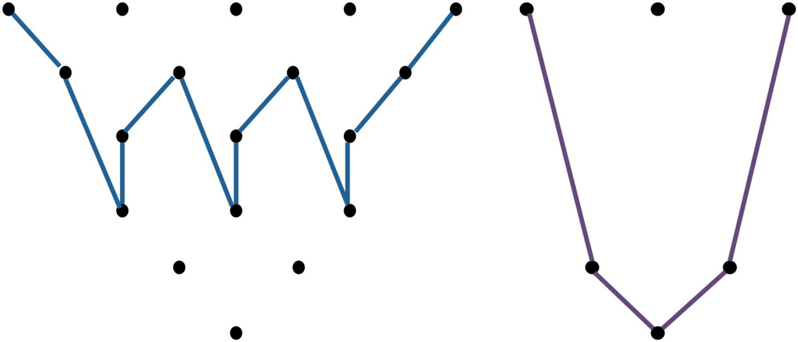


Figure : U-chain C_4: $P = (5,4,3,3,2,1)$ and new U-chain of $P' = (3,2,1)$.[Source: LK NU GASC talk 2013]

Oblak Recursive Conjecture

One obtains $\mathfrak{Q}(P)$ from $\mathcal{D}_P$:

(i) Let C be a longest U-chain of $\mathcal{D}_P$. Then $|C| = q_1$, the biggest part of $\mathfrak{Q}(P)$.

(ii) Remove the vertices of C from $\mathcal{D}_P$, giving a partition $P' = P - C$. If $P' \neq \emptyset$ then $\mathfrak{Q}(P) = (q_1, \mathfrak{Q}(P'))$ (Go to (i).).

Warning! The poset $\mathcal{D}_{P'}$ is not a subposet of $\mathcal{D}_P$.

Theorem (P. Oblak[Obl1] – Index of $\mathfrak{Q}(P)$)

The index of $\mathfrak{Q}(P) =$ is the length of the longest U-chain C of $\mathcal{D}_P$.

Theorem (L. Khatami [Kh1] – $Ob(P) = \lambda_U(\mathcal{D}_P)$)

The partition $Ob(P)$ obtained by Oblak recursion is independent of choices of AR subpartitions, and $Ob(P) = \lambda_U(\mathcal{D}_P)$, obtained in the same way as $\lambda(\mathcal{D}_P)$ but using U-chains. [1]

Work ofl I-L. Khatami (1/2 Oblak Rec Conj), L. Khatami (smallest part of $\mathfrak{Q}(P)$), and R. Basili (Oblak Rec Conj for $\mathrm{char}\ k = 0$, 2014) shows the Recursive Conjecture.

[1] A theory of E.R. Gansner, D. Kleitman, C. Greene, S. Poljak, T. Britz and S. Fomin assigns a partition $\lambda(\mathcal{P})$, using the lengths of multichains of a poset $\mathcal{P}$

Section 2: Table conjecture for $\mathfrak{Q}^{-1}(Q)$.

The set $\mathfrak{Q}^{-1}(Q)$ is mysterious, even for $Q = (u, u-r), r \geq 2$ where $P \to \mathfrak{Q}(P)$ is explicit. P. Oblak (2012) [Obl2] and R. Zhao (2013) made a very beautiful conjecture.

Table conjecture for $\mathfrak{Q}^{-1}(Q)$ (P. Oblak, R. Zhao)

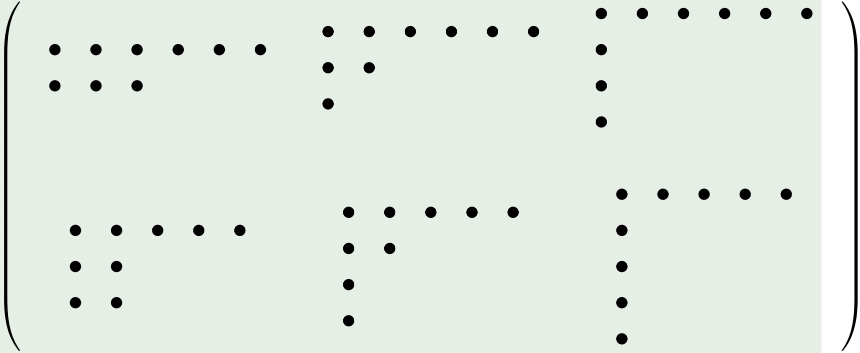
The elements of $\mathfrak{Q}^{-1}(Q), Q = (u, u-r), r \geq 2$ form a $(r-1) \times (u-r)$ table $\mathcal{T}(Q)$ such that $\mathcal{T}(Q)_{i,j}$ has $i+j$ parts.

[P. Oblak: $\# \mathfrak{Q}^{-1}(Q) = (r-1)(u-r)$; R. Zhao: table $\mathcal{T}(Q)$].

Example (Table $\mathcal{T}(Q)$ for $Q = (6,3)$)

Let $Q = (6,3)$.

$$\mathcal{T}(Q) = \begin{pmatrix} (6,3) & (6,[3]^2) & (6,[3]^3) \\ (5,[4]^2) & (5,[4]^3) & (5,[4[^4) \end{pmatrix} = \begin{pmatrix} A & A & A \\ B & B & B \end{pmatrix}$$



Definition (Type A,B,C partitions in $\mathfrak{Q}^{-1}(Q)$)

Let $Q = (u, u - r), r \geq 2, \mathfrak{Q}(P) = Q$ et $S_P = (a, a-1, b, b-1)$, $a > b + 2$, or $S_P = (a, a-1, a-2)$. The largest part u of Q comes from a U-row C_a (type A), or C_b (type B) or C_{a-1} (type C).

Example

Type A: $P = (\underbrace{5, 4}, 2, 1)$. Type B: $P = (5, 4, \underbrace{2, 2, 2})$. $|C_2| = 10$

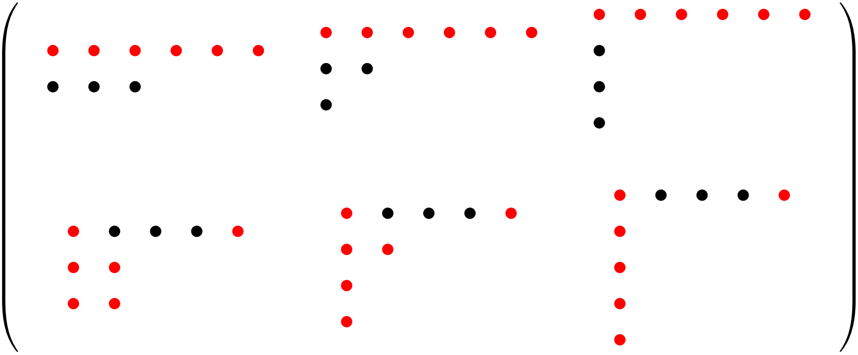
Type C: $P = (5, \underbrace{4, 4, 4, 3, 3}, 2)$, $|C_4| = 20$

Theorem ([Obl2, Z] Special $\mathfrak{Q}^{-1}(u, u - r)$)

The table conjecture $\mathfrak{Q}^{-1}$ is shown for $2 \leq r \leq 4$ (P. Oblak); and also for $u >> r$ – the "normal pattern" case when each A row is followed immediately by a B hook (R.Zhao).

Example (Normal pattern)

The table $\mathcal{T}(Q)$ for $Q = (6,3)$ has "normal pattern": the first row $(6,3), (6,[3]^2), (6,[3]^3)$ is type A, the second $(5,[4]^2), (5,[4]^3), (5,[4]^4)$ is a hook of type B.



Theorem ([IKvSZ] Table $\mathcal{T}^{-1}(Q)$)

Let $Q = (u, u - r)$. We can fill the $(r - 1) \times (u - r)$ table $\mathcal{T}(Q)$ with partitions that are $\mathfrak{Q}^{-1}(Q)$, arranged in rows of type A and hooks with partitions of type B or C,B.

$\mathcal{T}(Q)$ contains all the set $\mathfrak{Q}^{-1}(Q)$.

Idea of proof:

(i) *Specify the elements, showing they are in $\mathfrak{Q}^{-1}(Q)$.* ✓

(ii) *Use GF to show $\#\{P \mid r_P = 2, P \vdash n\}$ is the same as $\sum |\mathcal{T}(Q)| = \sum (r - 1)(u - r)$, the sum over all RR partitions $Q = (u, u - r), r \geq 2$ de n, (?)* *OR*

(ii') *Determine all the partitionsl P of type C having $\mathfrak{Q}(P) = Q$ and show that are in $\mathcal{T}(Q)$.* ✓

Example ($\mathcal{T}(Q)$ for $Q = (8,3)$, normal pattern)

$$\mathfrak{Q}^{-1}(8,3) = \begin{pmatrix} (8,3) & (8,[3]^2) & (8,[3]^3) \\ (\mathbf{5},[\mathbf{6}]^{\mathbf{2}}) & (\mathbf{5},[\mathbf{6}]^{\mathbf{3}}) & (\mathbf{5},[\mathbf{6}]^{\mathbf{4}}) \\ ([8]^2,[3]^2) & ([8]^2,[3]^3) & (\mathbf{5},[\mathbf{6}]^{\mathbf{5}}) \\ ([7]^2,[4]^3) & ([7]^2,[4]^4) & (\mathbf{5},[\mathbf{6}]^{\mathbf{6}}) \end{pmatrix}$$

$$= \begin{pmatrix} A & A & A \\ B & B & B \\ A & A & B \\ B' & B' & B \end{pmatrix}.$$

Note two B hooks.

Example ($\mathcal{T}(Q)$ for $Q = (12, 3)$, First $C \backslash A \cup B$ case [Z].)

$\mathcal{T}(Q)$	3	$[3]^2$	$[3]^3$
8	$(12, 3)$	$(12, [3]^2)$	$(12, [3]^3)$
$[8]^2$	$([12]^2, 3)$	$[12]^2, [3]^2)$	$([12]^2, [3]^3)$
$[8]^3$	$(5, [10]^3)$	$(5, [10]^4)$	$(5, [10]^5)$
$[8]^4$	$([12]^3, [3]^2)$	$([12]^3, [3]^3)$	$(5, [10]^6)$
$[8]^5$	$(4, [10]^4, 1)^C$	$([7]^2, [8]^5)$	$(5, [10]^7)$
$[8]^6$	$([12]^4, [3]^3)$	$([7]^2, [8]^6)$	$(5, [10]^8)$
$[8]^7$	$(\mathbf{[9]^3}, \mathbf{[6]^5})$	$([7]^2, [8]^7)$	$(5, [10]^9)$
$[8]^8$	$(\mathbf{[9]^3}, \mathbf{[6]^6})$	$([7]^2, [8]^8)$	$(5, [10]^{10})$

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Combinatorial Relation between $\mathcal{T}(Q)$ and Durfee squares

Definition ($\mathcal{DH}(Q)$, Q stable)

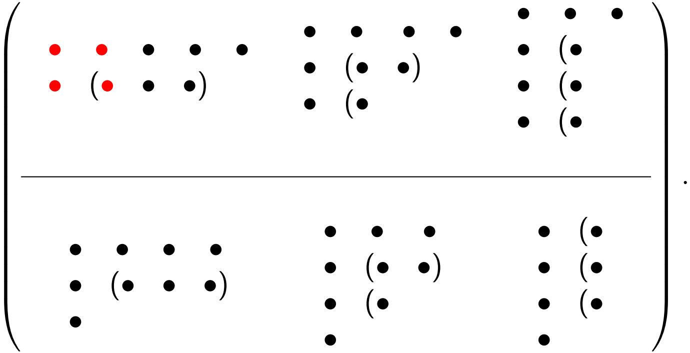
Let Q be a stable partition. Denote by $\mathcal{DH}(Q)$ the set of all partitions having diagonal hook lengths Q.

Example ($\mathcal{DH}(Q)$ for $Q = (6,3)$)

The inside diagonal hook h_{22} has length 3 so can be

$$P' = (3) \; \bullet \;\; \bullet \;\; \bullet \;, (2,1) \; \begin{array}{cc} \bullet & \bullet \\ \bullet & \end{array} \;, \text{ or } (1,1,1) \; \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \;.$$

Then a diagonal hook h_{11} of length 6 is folded around P'; in each case there are two positions: adding one, or two parts to P'. So $\mathcal{DH}(Q)$, $Q = (6,3)$ is



$$= \begin{pmatrix} (5,4) & (4,3,2) & (3,2,2,2) \\ (4,4,1) & (3,3,2,1) & (2,2,2,1) \end{pmatrix}.$$

Corollary (Bijection $\mathcal{T}(Q)$ et $\mathcal{DH}(Q)$.)

Let $Q = (u, u - r)$. There is a bijection $\theta : \mathcal{T}(Q) \to \mathcal{DH}(Q)$ that preserved the number of parts of P.

Proof. It is simple to write the tables $\mathcal{DH}(Q)$ by adding longer diagonal hooks; so it is easy to count $|\mathcal{DH}(Q|)$. It's the same number for $\mathcal{T}(Q)$ after the Theorem. We take $\theta(\mathcal{T}_{ij}(Q)) = \mathcal{DH}_{i,j}(Q)$.

Question

Can we define θ^{-1} combinatorially? A "jeu de taquin"?

If we can extend the definition of θ to Q with $k > 2$ parts, this can help construct the tables $\mathcal{T}(Q)$, as $\mathcal{DH}(Q)$ is easy to write down.

Example (θ for $Q = (6, 3)$)

The map $\theta(\mathcal{T}(Q)_{ij}) = \mathcal{DH}(Q)_{ij}$. Here

$$\mathcal{T}(Q) = \begin{pmatrix} (6,3) & (6,[3]^2) & (6,[3]^3) \\ (5,[4]^2) & (5,[4]^3) & (5,[4[^4) \end{pmatrix}.$$

$$\mathcal{DH}(Q) = \begin{pmatrix} (5,4) & (4,3,2) & (3,2,2,2) \\ (4,4,1) & (3,3,2,1) & (2,2,2,1) \end{pmatrix}.$$

Example (Case $r_P = 1$, $\mathrm{dh}(P)$ has 1×1 Durfee square.)

Soit $n = 5, Q = (5)$.
$\mathcal{T}(Q) = ([5], [5]^2, [5]^3, [5]^4, [5]^5)$
$DH(Q) = ((5), (4,1), (3,1^2), (2,1^4), (1^5))$ (single diagonal hook).

Lemma

GF for $\#P \vdash n \mid durf(Q) = 2$ is shifted http://oeis.org/A006918.

$$durf_2(n) = \begin{cases} \frac{(n+1)(n-1)(n-3)}{24} & \text{if } i \geq 5 \text{ is odd} \\ \frac{1}{4}\binom{n}{3} & \text{if } i \geq 4 \text{ is even,} \end{cases} \quad (1)$$

Definition (Key S_Q of a stable Q)

Let $Q = (q_1, q_2, \ldots, q_k), q_i \geq q_{i+1} + 2, 1 \leq i < k$ be stable. The *key* $S_Q = (q_1 - q_2 - 1, q_2 - q_3 - 1, \ldots, q_{k-1} - q_k - 1, q_k)$.

Example

For $Q = (u, u - r)$ the key is $S_Q = (r - 1, u - r)$.
For $Q = (12, 6, 2)$ the key is $S_Q = (5, 3, 2)$.

Corollary of the box conjecture

For Q RR, there is an isomorphism $\theta : \mathfrak{Q}^{-1}(Q) \to \mathcal{DH}(Q)$, that preserves numbers of parts.

Problems Find θ explicitly.
Give the table $\mathcal{T}(Q)$. (Find "hooks" for $k \geq 3$.)

Box conjecture for $\mathfrak{Q}^{-1}(Q)$

Let $Q = (q_1, \ldots, q_k)$ be stable of key S_Q. Then

(i) The partitions $\mathfrak{Q}^{-1}(Q)$ form a k-box $\mathcal{T}(Q)$ such that $\mathcal{T}(Q)_I, I = (i_1, \ldots, i_k)$ has $|I|$ parts.

(ii) The codimension of the similarity orbit of $\mathcal{T}(Q)_I$ in $\mathcal{N}_Q$ is $|I| - k$.

Example ($S_Q = (2,2,2)$)

Take $Q = (8,5,2)$ so $S_Q = (2,2,2)$.
The two floors of $\mathcal{T}(Q)$ are

$$\begin{pmatrix} (8,5,2) & (8,5,1^2) \\ (8,4,2,1) & (8,4,1^3) \end{pmatrix}, \begin{pmatrix} (\mathbf{7},\mathbf{4},\mathbf{2^2}) & (7,4,2,1^2) \\ (7,3^2,1^2) & (7,4,1^4) \end{pmatrix}.$$

The corresponding floors of $\mathcal{DH}(Q) = \theta(\mathcal{T}(Q))$ are

$$\begin{pmatrix} (6,5,4) & (5,4,3^2) \\ (5,4^2,2) & (4,3^3,2) \end{pmatrix}, \begin{pmatrix} (\mathbf{5^2},\mathbf{4},\mathbf{1}) & (4^2,3^2,1) \\ (4^3,2,1) & (3^4,2,1) \end{pmatrix}.$$

Question: Can we explain these results? *Not yet!*

Lie algebra perspective:

The columns of $\mathcal{D}(P)$ are weight spaces for the sl_2 triple of B. But the S_n irreps for $P \in \mathcal{T}(Q)$ and $\theta(P) \in \mathcal{DH}(Q)$ have different VS dimensions.

Map to the Hilbert scheme:

Let $B = J_Q$. The map

$$\pi : \mathcal{N}_B \to \mathrm{Hilb}^n \mathsf{k}[x, y] \colon \ A \to \mathsf{k}[A, B]$$

defines an image, whose fixed points under a $\mathbb{C}^*$ action correspond to the monomial ideals of $\mathcal{T}(Q)$, so to homology classes on $\pi(\mathcal{N}_B)$. Will this explain the codimensions in $\mathcal{T}(Q)$?

Combinatorial questions arising from $P \to \mathfrak{Q}(P)$.

(a) Poset $\mathcal{D}(P)$: Is $\lambda(\mathcal{D}_P) = \lambda_U(\mathcal{D}_P)$?

(b) Explain the map $\theta^{-1} : \mathcal{DH}(Q) \to \mathcal{T}(Q)$ combinatorially.

(c) Verify $\#\{P \vdash n$ with p parts and $r_P = k\}$ is the expected sum.

(d) An a-cluster is a partition $P = (p_1 \geq \ldots \geq p_t)$ with
$p_1 - p_t \leq a$.
$r_{a,P} = \min\{\ \#\ a$-clusters needed to cover $P\}$.
$V_{a,k}(n) = \{P \vdash n \mid r_{a,P} = k\}$.
Determine $|V_{a,k}(n)|$.
(e) Consider other posets $\mathcal{P}$ with multiplicities, and a linear action $B \to$ on vertices($\mathcal{P}$). Consider $A \in \mathfrak{I}(\mathcal{P})$ commuting with B.
Is $\lambda(\mathcal{P}) = \lambda^B(\mathcal{P})$?

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Thank you for your attention and questions!

Appendix: $\mathfrak{Q}(P)$ and its smallest part (L.Khatami)

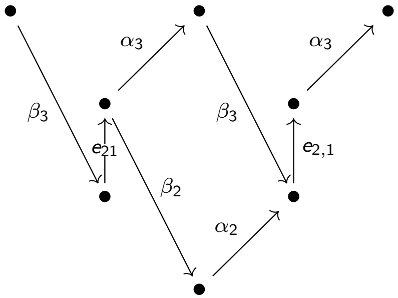

Figure : Diagram of the poset $\mathcal{D}_P$: $P = (3, 2, 2, 1)$.

Def. (U-chain)

A U-chain C_i in $\mathcal{D}_P$ is the saturated (maximal) chain through the union of three subsets of vertices:

(i) All rows of length $i, i-1$, corresponding to an AR subpartition of P.

(ii) A descending chain from the source – the top left vertex of $\mathcal{D}_P$ – to the vertex at the start of the lowest length-i row.

(iii) An ascending chain from the vertex at the end of the highest length-i row to the sink - the top right vertex of $\mathcal{D}_P$.

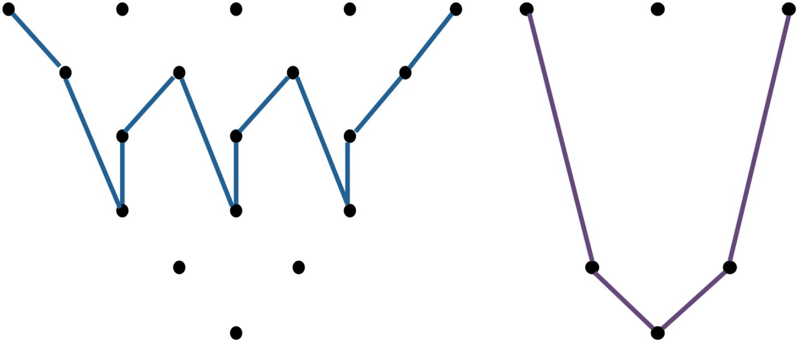

Figure : U-chain C_4 for $P = (5, 4, 3, 3, 2, 1)$ and new U-chain for $P' = (3, 2, 1)$.[Source: LK NU GASC talk 2013]

Oblak Recursive Conjecture

We obtain $\mathfrak{Q}(P)$ as follows from $\mathcal{D}_P$:

(i) Choose a maximum length U-chain in $\mathcal{D}_P$. Its length is q_1, the largest part of $\mathfrak{Q}(P)$.

(ii) Remove the vertices in the chain from $\mathcal{D}_P$, obtaining a smaller partition P'. If $P' \neq \emptyset$ then $\mathfrak{Q}(P) = (q_1, \mathfrak{Q}(P'))$ (go to (i).).

Warning. The poset $\mathcal{D}_{P'}$ in the Oblak recursion is *not* in general a subposet of $\mathcal{D}_P$.

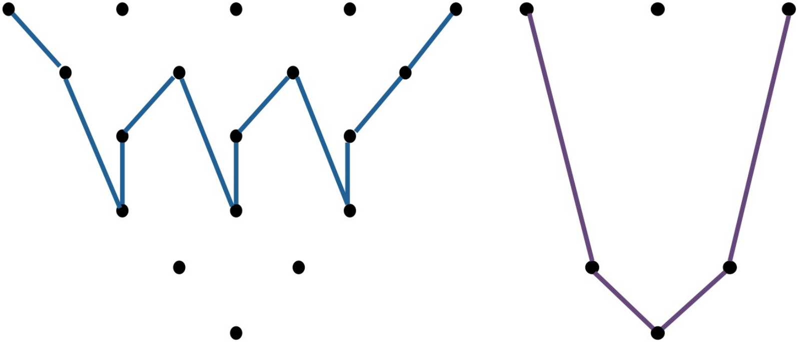

Figure : U-chain for $P = (5,4,3,3,2,1)$ and new U-chain for $P' = (3,2,1)$. So $\mathfrak{Q}(P) = (12,5,1)$.

Theorem (P. Oblak[Obl1] – Index of $\mathfrak{Q}(P)$)

The index (largest part) of $\mathfrak{Q}(P)$ is the length of the longest U-chain in $\mathcal{D}_P$.

Theorem (L. Khatami [Kh1] – $Ob(P) = \lambda_U(\mathcal{D}_P)$)

The partition $Ob(P)$ obtained by the Oblak recursive process is independent of the choices of AR subpartitions; and $Ob(P) = \lambda_U(\mathcal{D}_P)$, obtained as $\lambda(\mathcal{D}_P)$ below by restricting to sets of U-chains.

Definition

$P \geq P'$ in the orbit closure (Bruhat) order if
$\forall i \ \sum_{u=1}^{i} p_u \geq \sum_{u=1}^{i} p'_u$.

Theorem (I,L.Khatami [IKh])

$\mathfrak{Q}(P) \geq Ob(P)$.

Proof idea. For each maximal-length set of s U-chains, we specify an $A \in \mathcal{N}_B$ such that $\dim_k k[A] \circ \{v_1, \ldots, v_s\}$ where the v_i are initial elements, agrees with the sum of the first s parts of $Ob(P)$. This involves an analysis of the sets of chains from the v_i to all the vertices covered by the s U-chains.

Def. (C. Greene et al, see[BrFo])

Let $\mathcal{D}$ be a poset without loops. Define $c_i = \max\#$ vertices covered by i chains. Set

$$\lambda(\mathcal{D}) = (c_1, c_2 - c_1, c_3 - c_2, \ldots).$$

Theorem (C. Greene, S. Poljak, E.R. Gansner, see [BrFo])

Let $\mathcal{D}$ be any finite poset without loops, and let A be a generic nilpotent matrix in the incidence algebra $\mathfrak{I}(\mathcal{D}_P)$. Then the Jordan type $P_A = \lambda(\mathcal{D})$.

Definition ([Kh1])

$\lambda_U(\mathcal{D}_P)$ is obtained by replacing arbitrary chains c_i in the definition of $\lambda(\mathcal{D}_P)$ by U-chains.

Question: Combinatorial Oblak conjecture

Is $\mathfrak{Q}(P) = \lambda(\mathcal{D}_P)$?

Since $Ob(P) = \lambda_U(\mathcal{D}_P) \geq \mathfrak{Q}(P)$ this is equivalent to
Is $\lambda_U(\mathcal{D}_P) = \lambda(\mathcal{D}_P)$?

The key issue is that $A \in \mathcal{N}_B$ commutes with B, that acts by moving vertices of $\mathcal{D}_P$ to the right: this greatly restricts $A \in \mathfrak{I}(\mathcal{D}_P)$. Does it matter for the Jordan type P_A?

R. Basili answers "No" in her talk and ArXiv post that describe an apparent proof of the Oblak Recursive Conjecture.[2]

[2] R. Basili posted a preprint in June 2012 on her proof of the Oblak conjecture. It appears to implicitly assume $\mathrm{char}\ k = 0$; The $\mathrm{char}\ k = 0$ case by [IKh] implies the Oblak conjecture over any infinite field k.

Theorem (L. Khatami [Kh2] – Minimum part)

The minimum part of $\mathfrak{Q}(P)$ is a specified combinatorial invariant $\mu(P)$. Also

$$Ob(P)_{\min} = \mathfrak{Q}(P)_{\min} = \lambda(\mathcal{D}_P)_{\min} = \mu(P) \qquad *$$

Proof idea. First show $\mu(P)$ is the minimum part of $\lambda_U(\mathcal{D}_P) = Ob(P)$. Then an intensive study of the antichains of $\mathcal{D}_P$ shows $\lambda(\mathcal{D}_P)_{\min} = \mu(P)$. By [IKh], $Ob(P) \leq \mathfrak{Q}(P) \leq \lambda(\mathcal{D}_P)$, showing $(*)$.

Corollary ([Obl1, KO, Kh2])

$\mathfrak{Q}(P)$ is explicitly known for $r_P \leq 3$, over any infinite field k.

The invariant $\mu(P)$ for a spread.

Let $P = ((p+s-1)^{n_s}, \dots, p^{n_1})$ be an s-spread: $n_i > 0$ for $1 \le i \le s$. Set

$$\mu(P) = \min\{pn_{2i-1} + (p+1)n_{2j} | 1 \le i \le j \le r_P\}$$

Note: For s odd $r_P = (s+1)/2$ so $n_{2r_P} = 0$ and $\mu(P) = p \cdot \min\{n_{2i-1} \mid 1 \le i \le r_P\}$.

Theorem (L.Khatami [Kh2])

For P a spread, $\mu(P)$ is the $\#$ of disjoint length-r_P antichains in $\mathcal{D}_P$.

Fact: [Gre, BrFo] $\lambda(\mathcal{D}_P)_{\min} = \#$ length r_P anti-chains in $\mathcal{D}_P$.

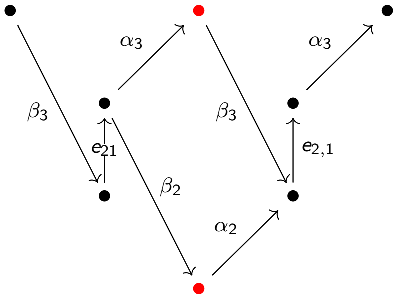

Figure : $\mu(P) = 1$ for $P = (3, 2, 2, 1)$, $Q(P) = (7, 1)$

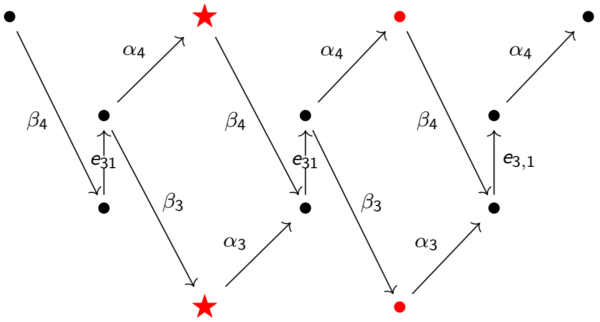

Figure : $\mu(P) = 2$ for $P = (4, 3, 3, 2)$, $\mathfrak{Q}(P) = (10, 2)$

- K. Alladi, A. Berkovish: *New weighted Rogers-Ramanjan partition theorems and their implications*, Trans. Amer. Math. Soc. 354 (2002) no. 7, 2557–2577.
- G. Andrews: *The Theory of Partitions*, Cambridge University Press, 1984, paper 1988 ISBN 0-521-63766-X.
- V. Baranovsky: *The variety of pairs of commuting nilpotent matrices is irreducible*, Transform. Groups 6 (2001), no. 1, 3–8.
- R. Basili: *On the irreducibility of commuting varieties of nilpotent matrices*. J. Algebra 268 (2003), no. 1, 58–80.
- R. Basili: *On the maximum nilpotent orbit intersecting a centralizer in $M(n, K)$*, preprint, Feb. 2014, arXiv:1202.3369 v.5.

- R. Basili and A. Iarrobino: *Pairs of commuting nilpotent matrices, and Hilbert function*. J. Algebra **320** # 3 (2008), 1235–1254.
- R. Basili, A. Iarrobino and L. Khatami, *Commuting nilpotent matrices and Artinian Algebras, J. Commutative Algebra (2) #3 (2010) 295–325.*
- R. Basili, T. Košir, P. Oblak: *Some ideas from Ljubljana*, (2008), preprint.
- J. Briançon: *Description de* $\mathrm{Hilb}^n C\{x, y\}$, Invent. Math. 41 (1977), no. 1, 45–89.
- J.R. Britnell and M. Wildon: *On types and classes of commuting matrices over finite fields*, J. Lond. Math. Soc. (2) 83 (2011), no. 2, 470–492.

- T. Britz and S. Fomin: *Finite posets and Ferrers shapes*, Advances Math. 158 #1 (2001), 86–127.
- J. Brown and J. Brundan: *Elementary invariants for centralizers of nilpotent matrices*, J. Aust. Math. Soc. 86 (2009), no. 1, 1–15. ArXiv math/0611.5024.
- J. Carlson, E. Friedlander, J. Pevtsova: *Representations of elementary abelian p-groups and bundles on Grassmannians*, Adv. Math. 229 (2012), # 5, 2985–3051.
- D. Collingwood, W. McGovern: *Nilpotent Orbits in Semisimple Lie algebras*, Van Norstrand Reinhold (New York), (1993).
- E. Friedlander, J. Pevtsova, A. Suslin: *Generic and maximal Jordan types*, Invent. Math. 168 (2007), no. 3, 485–522.

- E.R. Gansner: *Acyclic digraphs, Young tableaux and nilpotent matrices*, SIAM Journal of Algebraic Discrete Methods, 2(4) (1981) 429–440.
- M. Gerstenhaber: *On dominance and varieties of commuting matrices*, Ann. of Math. (2) 73 1961 324–348.
- C. Greene: *Some partitions associated with a partially ordered set*, J. Combinatorial Theory Ser A **20** (1976), 69–79.
- R. Guralnick and B.A. Sethuraman: *Commuting pairs and triples of matrices and related varieties*, Linear Algebra Appl. 310 (2000), 139–148.
- C. Gutschwager: *On prinicipal hook length partition and Durfee sizes in skew characters*, Ann. Comb. 15 (2011) 81–94.

T. Harima and J. Watanabe: *The commutator algebra of a nilpotent matrix and an application to the theory of commutative Artinian algebras*, J. Algebra 319 (2008), no. 6, 2545–2570.

A. Iarrobino and L. Khatami: *Bound on the Jordan type of a generic nilpotent matrix commuting with a given matrix*, J. Alg. Combinatorics, 38, #4 (2013), 947–972. On line DOI: 10.1007/s10801–013–0433–1. ArXiv 1204.4635.

A. Iarrobino, L. Khatami, B. Van Steirteghem, and R. Zhao *Combinatorics of two commuting nilpotent matrices*, preprint in progress, 2013 (52 p.).

L. Khatami: *The poset of the nilpotent commutator of a nilpotent matrix*, Linear Algebra and its Applications 439 (2013) 3763-3776.

- L. Khatami: *The smallest part of the generic partition of the nilpotent commutator of a nilpotent matrix*, arXiv:1302.5741, to appear, J. Pure and Applied Algebra.
- T. Košir and P. Oblak: *On pairs of commuting nilpotent matrices*, Transform. Groups 14 (2009), no. 1, 175–182.
- T. Košir and B. Sethuranam: *Determinantal varieties over truncated polynomial rings*, J. Pure Appl. Algebra 195 (2005), no. 1, 75–95.
- F.H.S. Macaulay: *On a method for dealing with the intersection of two plane curves*, Trans. Amer. Math. Soc. 5 (1904), 385–410.
- G. McNinch: *On the centralizer of the sum of commuting nilpotent elements*, J. Pure and Applied Alg. 206 (2006) # 1-2, 123–140.

- P. Oblak: *The upper bound for the index of nilpotency for a matrix commuting with a given nilpotent matrix*, Linear and Multilinear Algebra 56 (2008) no. 6, 701–711. Slightly revised in ArXiv: math.AC/0701561.
- P. Oblak: *On the nilpotent commutator of a nilpotent matrix*, Linear Multilinear Algebra 60 (2012), no. 5, 599–612.
- D. I. Panyushev: *Two results on centralisers of nilpotent elements*, J. Pure and Applied Algebra, 212 no. 4 (2008), 774–779.
- S. Poljak: *Maximum Rank of Powers of a Matrix of Given Pattern*, Proc. A.M.S., 106 #4 (1989), 1137–1144.
- A. Premet: *Nilpotent commuting varieties of reductive Lie algebras*, Invent. Math. 154 (2003), no. 3, 653–683.

- G. de B. Robinson and R.M.Thrall: *The content of a Young diagramme,* Michigan Math.Journal 2 No.2 (1953), 157–167.
- H.W. Turnbull and A.C. Aitken: *An Introduction to the Theory of Canonical Matrices*, Dover, New York, 1961.
- R. Zhao: *Commuting nilpotent matrices and Oblak's proposed formula*, preprint in preparation, 2013.

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