BRIDGELAND STABILITY CONDITIONS ON THREEFOLDS II: AN APPLICATION TO FUJITA’S CONJECTURE

AREND BAYER, AARON BERTRAM, EMANUELE MACRÌ, AND YUKINOBU TODA

ABSTRACT. We apply a conjectured inequality on third chern classes of stable two-term complexes on threefolds to Fujita’s conjecture. More precisely, the inequality is shown to imply a Reider-type theorem in dimension three which in turn implies that $K_X + 6L$ is very ample when $L$ is ample, and that $5L$ is very ample when $K_X$ is trivial.

CONTENTS

1. Introduction 1
2. Setup 4
3. Reduction to curves 7
4. A Reider-type theorem 13
References 16

1. INTRODUCTION

A Bogomolov-Gieseker-type inequality on Chern classes of “tilt-stable” objects in the derived category of a threefold was conjectured in [BMT11] in the context of constructing Bridgeland stability conditions. In this paper, we show how the same inequality would allow one to extend Reider’s stable-vector bundle technique ([Rei88]) from surfaces to threefolds, and in particular to obtain Fujita’s conjecture in the threefold case. This follows a line of reasoning that was suggested in [AB11].

While we use the setup of tilt-stability from [BMT11], this paper is intended to be self-contained, and to be readable by birational geometers with a passing familiarity with derived categories.

Tilt-stability depends on two numerical parameters: an ample class $\omega \in \text{NS}_\mathbb{Q}(X)$ and an arbitrary class $B \in \text{NS}_\mathbb{Q}(X)$. It is a notion of stability on a particular abelian category, $\mathcal{B}_{\omega,B}$, of two-term complexes in $\text{D}^b(X)$, and codimension three Chern classes of stable

Date: September 28, 2012.

2000 Mathematics Subject Classification. 14F05 (Primary); 14C20, 14J30, 14J32, 18E30 (Secondary).

Key words and phrases. Bogomolov-Gieseker inequality, Bridgeland stability conditions, Derived category, adjoint line bundles, Fujita conjecture.
objects $E$ in this category (and not stable vector bundles) are conjectured to satisfy a Bogomolov-Gieseker inequality in Conjecture 2.3. Assuming this conjecture, we prove the following Reider-type theorem for threefolds:

**Theorem 4.1.** Let $X$ be a smooth projective threefold over $\mathbb{C}$, and let $L$ be an ample line bundle on $X$ such that Conjecture 2.3 holds when $B$ and $\omega$ are scalar multiples of $L$. Fix a positive integer $\alpha$, and assume that $L$ satisfies the following conditions:

(A) $L^3 > 49\alpha$;
(B) $L^2.D \geq 7\alpha$, for all integral divisor classes $D$ with $L^2.D > 0$ and $L.D^2 < \alpha$;
(C) $L.C \geq 3\alpha$, for all curves $C$.

Then $H^1(X, K_X \otimes L \otimes I_Z) = 0$ for any zero-dimensional subscheme $Z \subset X$ of length $\alpha$.

Theorem 4.1 would give an effective numerical criterion for an adjoint line bundle to be globally generated ($\alpha = 1$) or very ample ($\alpha = 2$):

**Corollary 1.1 (Fujita’s Conjecture).** Let $L$ be an ample line bundle on a smooth projective threefold $X$. Assume Conjecture 2.3 holds for $\omega$ and $B$ as above. Then:

(a) $K_X \otimes L^\otimes m$ is globally generated for $m \geq 4$. Moreover, if $L^3 \geq 2$, then $K_X \otimes L^\otimes 3$ is also globally generated.
(b) $K_X \otimes L^\otimes m$ is very ample for $m \geq 6$.

In Proposition 4.2, we also show (assuming the conjecture) that $K_X \otimes L^5$ is very ample as long as its restriction to special degree one curves is very ample. As a consequence, $K_X \otimes L^5$ is very ample when $K_X$ is trivial, or, more generally, when $K_X.C$ is even for all curves $C \subset X$.

Ein and Lazarsfeld proved that $K_X \otimes L^\otimes 4$ is globally generated [EL93]. In the case $L^3 \geq 2$, Fujita, Kawamata, and Helmke proved that $K_X \otimes L^\otimes 3$ is globally generated as well [Fuj93, Kaw97, HEL97]. In fact, in Proposition 4.4, we show that these results conversely give some evidence for Conjecture 2.3. Case (b) in Corollary 1.1 instead is not known in general; but also note that the strongest form of Fujita’s conjecture predicts that $K_X \otimes L^\otimes 5$ is already very ample. For further references, we refer to [Laz04, Section 10.4]. Notice that the bounds in Theorem 4.1 are very similar to those in [Fuj93] when $\alpha = 1$ (see also [Kaw97, HEL97]) and, when $\alpha = 2$ and $Z$ consists of two distinct points, to those in [Fuj94].

**Approach.** We explain our approach, which was outlined in [AB11, Section 5], but can now be made precise using the strong Bogomolov-Gieseker conjecture of [BMT11]. It is closer to Reider’s original approach [Rei88] for surfaces via stability of sheaves (generalized to threefolds by extending it to derived categories), than to the Ein-Lazarsfeld-Kawamata approach mentioned above, via vanishing theorems.

Let us give first a brief recall on Reider’s method for proving Fujita’s Conjecture in the case of $X$ being a surface. By Serre duality, an adjoint linear system $K_X \otimes L$ is very ample
if and only if $\text{Ext}^1(L \otimes I_Z, \mathcal{O}_X) = H^1(X, K_X \otimes L \otimes I_Z)^\vee = 0$, for all zero-dimensional subscheme $Z \subset X$ of length one or two. If this group was non-zero, we would get a rank 2 torsion-free sheaf $E$ as the non-trivial extension $\mathcal{O}_X \hookrightarrow E \twoheadrightarrow L \otimes I_Z$. Reider’s idea is to consider the slope-stability of $E$. If $E$ is stable, then the classical Bogomolov-Gieseker inequality gives a bound on the degree $L^2$ of $L$ in terms of the length of $Z$. If $E$ is not stable, then the destabilizing subsheaf gives a curve of bounded degree with respect to $L$. Hence, if we assume that $L$ satisfies inequalities similar to (A) and (C), we would get a contradiction.

We generalize this approach to threefolds as follows. We suppose the conclusion of Theorem 4.1 is false. Then by Serre duality,

$$0 \neq \text{Ext}^2(L \otimes I_Z, \mathcal{O}_X) = \text{Ext}^1(L \otimes I_Z, \mathcal{O}_X[1]).$$

For appropriate choices of $\omega$ and $B$, both $L \otimes I_Z$ and $\mathcal{O}_X[1]$ are objects in the abelian category $\mathcal{B}_{\omega,B}$, and thus this extension class corresponds to another object $E$ of $\mathcal{B}_{\omega,B}$. In Section 3.1, we will show that for $\omega \to 0$, the complex $E$ violates the inequality of Conjecture 2.3, thus it must become unstable. We show in Section 3.2 that the Chern classes of a destabilizing subobject give a contradiction to Assumptions (A) and (B) of the Theorem unless it is of the form $L \otimes I_C$, where $I_C$ is the ideal sheaf of a curve containing $Z$. In Section 4, we apply our conjecture and Assumption (C) to this remaining case and deduce Theorem 4.1.

Acknowledgements. A. Ba. is partially supported by NSF grant DMS-0801356/DMS-1001056. A. Be. is partially supported by NSF grant DMS-0901128. E. M. is partially supported by NSF grant DMS-1001482/DMS-1160466, Hausdorff Center for Mathematics, Bonn, and by SFB/TR 45. Y. T. is supported by World Premier International Research Center Initiative (WPI initiative), MEXT, Japan, and Grant-in-Aid for Scientific Research grant (22684002), partly (S-19104002), from the Ministry of Education, Culture, Sports, Science and Technology, Japan. The authors would like to thank the Isaac Newton Institute and its program on “Moduli Spaces”, during which this paper was finished.

Notation and Convention. Throughout the paper, $X$ will be a smooth projective threefold defined over $\mathbb{C}$ and $\text{D}^b(X)$ its bounded derived category of coherent sheaves. Given a line bundle $L$ on $X$, we will denote by $\mathbb{D}_L: \text{D}^b(X) \to \text{D}^b(X)$ the following local dualizing functor on its derived category:

$$\mathbb{D}_L(\_):= (\_)^\vee[1] \otimes L = R\text{Hom}(\_, L[1]).$$

We identify a line bundle $L$ with its first Chern class $c_1(L)$, and write $K_X$ for the canonical line bundle. While $L^{\otimes m}$ denotes the tensor powers of the line bundle, $L^k$ denotes the intersection product of its first Chern class.
2. Setup

In this section, we briefly recall the notion of “tilt-stability” defined in [BMT11, Section 3] and its most important properties.

Let $X$ be a smooth projective threefold, and let $\omega, B \in \text{NS}_Q(X)$ be rational numerical divisor classes such that $\omega$ is ample. We use $\omega, B$ to define a slope function $\mu_{\omega,B}$ for coherent sheaves on $X$ as follows: For torsion sheaves $E$, we set $\mu_{\omega,B}(E) = +\infty$, otherwise

$$\mu_{\omega,B}(E) = \frac{\omega^2 \text{ch}^B_1(E)}{\omega^3 \text{ch}^B_0(E)} = \frac{\omega^2 \text{ch}_1(E)}{\omega^3 \text{ch}^B_0(E)} - \frac{\omega^2 B}{\omega^3}$$

where $\text{ch}^B(E) = e^{-B} \text{ch}(E)$ denotes the Chern character twisted by $B$ (explicitly, $\text{ch}^B_0 = \text{rk}$, $\text{ch}^B_1 = c_1 - B \text{rk}$, etc.).

A coherent sheaf $E$ is slope-(semi)stable (or $\mu_{\omega,B}$-(semi)stable) if, for all subsheaves $F \hookrightarrow E$, we have

$$\mu_{\omega,B}(F) < (\leq) \mu_{\omega,B}(E/F).$$

Due to the existence of Harder-Narasimhan filtrations (HN-filtrations, for short) with respect to slope-stability, there exists a “torsion pair” $(\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B})$ defined as follows:

$$\mathcal{T}_{\omega,B} = \{ E \in \text{Coh } X : \text{any quotient } E \to G \text{ satisfies } \mu_{\omega,B}(G) > 0 \}$$

$$\mathcal{F}_{\omega,B} = \{ E \in \text{Coh } X : \text{any subsheaf } F \hookrightarrow E \text{ satisfies } \mu_{\omega,B}(F) \leq 0 \}$$

Equivalently, $\mathcal{T}_{\omega,B}$ and $\mathcal{F}_{\omega,B}$ are the extension-closed subcategories of $\text{Coh } X$ generated by slope-stable sheaves of positive or non-positive slope, respectively.

**Definition 2.1.** We let $\mathcal{B}_{\omega,B} \subset D^b(X)$ be the extension-closure

$$\mathcal{B}_{\omega,B} = \langle \mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B} [1] \rangle.$$

More explicitly, $\mathcal{B}_{\omega,B}$ is the subcategory of two-term complexes $E : E^{-1} \xrightarrow{d} E^0$ with $H^{-1}(E) = \ker d \in \mathcal{F}_{\omega,B}$ and $H^0(E) = \text{cok} d \in \mathcal{T}_{\omega,B}$. We can characterize isomorphism classes of objects in $\mathcal{B}_{\omega,B}$ by extension classes: to give an object $E \in \mathcal{B}_{\omega,B}$ is equivalent to giving $T \in \mathcal{T}_{\omega,B}$, $F \in \mathcal{F}_{\omega,B}$, and a class $\xi \in \text{Ext}_X^2(T, F)$.

By the general theory of torsion pairs and tilting [HRO96], $\mathcal{B}_{\omega,B}$ is the heart of a bounded t-structure on $D^b(X)$. For the most part, we only need that $\mathcal{B}_{\omega,B}$ is an abelian category: Exact sequences in $\mathcal{B}_{\omega,B}$ are given by exact triangles in $D^b(X)$. For any such exact sequence

$$0 \to E \to F \to G \to 0$$

in $\mathcal{B}_{\omega,B}$, we have a long exact sequence in $\text{Coh } X$:

$$0 \to H^{-1}(E) \to H^{-1}(F) \to H^{-1}(G) \to H^0(E) \to H^0(F) \to H^0(G) \to 0.$$
Using the classical Bogomolov-Gieseker inequality and Hodge Index theorem, we defined the following slope function on $\mathcal{B}_{\omega,B}$: We set

$$\nu_{\omega,B}(E) = \begin{cases} +\infty & \text{when } \omega^2 \text{ch}_1^B(E) = 0, \\ \omega \text{ch}_2^B(E) - \frac{1}{9} \omega^3 \text{ch}_0^B(E) & \text{otherwise.} \end{cases}$$

We showed that this is a slope function, in the sense that it satisfies the weak see-saw property for short exact sequences in $\mathcal{B}_{\omega,B}$: for any subobject $F \hookrightarrow E$, we have $\nu_{\omega,B}(F) \leq \nu_{\omega,B}(E) \leq \nu_{\omega,B}(E/F)$ or $\nu_{\omega,B}(F) \geq \nu_{\omega,B}(E) \geq \nu_{\omega,B}(E/F)$.

**Definition 2.2.** An object $E \in \mathcal{B}_{\omega,B}$ is “tilt-(semi)stable” if, for all non-trivial subobjects $F \hookrightarrow E$, we have $\nu_{\omega,B}(F) < (\leq) \nu_{\omega,B}(E/F)$.

Motivated by the case of torsion sheaves ([BMT11, Proposition 7.1.1]), by projectively flat vector bundles ([BMT11, Proposition 7.4.2]), and the case of $X = \mathbb{P}^3$ ([BMT11, Theorem 8.2.1] and [Mac12]), we stated the following conjecture:

**Conjecture 2.3 ([BMT11, Conjecture 1.3.1]).** For any $\nu_{\omega,B}$-semistable object $E \in \mathcal{B}_{\omega,B}$ satisfying $\nu_{\omega,B}(E) = 0$, we have the following inequality

$$\text{ch}_3^B(E) \leq \frac{\omega^2}{18} \text{ch}_1^B(E).$$

Conjecture 2.3 is analogous to the classical Bogomolov-Gieseker inequality, which can be formulated as follows: For any $\mu_{\omega,B}$-semistable sheaf $E$ satisfying $\mu_{\omega,B}(E) = 0$, we have $\omega \text{ch}_2^B(E) \leq 0$.

The original motivation for Conjecture 2.3 is to construct examples of Bridgeland stability conditions on $\mathbb{D}^b(X)$. While any linear inequality of the form (2) would be sufficient to this end, the constant $\frac{1}{18}$ in equation (2) is chosen so that, if $\omega$ and $B$ are proportional to the first Chern class of an ample line bundle $L$, the inequality is an equality for tensor power $L^\otimes n$ of $L$. More generally, it is an equality when $E$ is a slope-stable vector bundles $E$ whose discriminant $\Delta = (\text{ch}_1^B)^2 - 2 \text{ch}_0^B \text{ch}_2^B$ satisfies $\omega \Delta(E) = 0$, and for which $\text{ch}_1^B(E)$ is proportional to $L$. Such vector bundles have a projectively flat connection, and are examples of tilt-stable objects:

**Proposition 2.4 ([BMT11, Proposition 7.4.1]).** Let $L$ be an ample line bundle, and assume that both $\omega$ and $B$ are proportional to $L$. Then any slope-stable vector bundle $E$, with $\omega \Delta(E) = 0$ and for which $\text{ch}_1^B(E)$ is proportional to $L$, is also tilt-stable with respect to $\nu_{\omega,B}$.

The proof is essentially the same as for line bundles $L^\otimes n$ in [AB11, Proposition 3.6].

By assuming Conjecture 2.3, we can also show conversely: if an object in $\mathcal{B}_{\omega,B}$ is tilt-stable and the inequality in Conjecture 2.3 is an equality, then it must have trivial
Since we also showed the existence of a continuous family of stability conditions depending on real classes \(\omega, B\):

**Proposition 2.5** ([BMT11, Corollary 3.3.3]). Let \(U \subset \text{NS}_\mathbb{R}(X) \times \text{NS}_\mathbb{R}(X)\) be the subset of pairs of real classes \((\omega, B)\) for which \(\omega\) is ample. There exists a notion of “tilt-stability” for every \((\omega, B) \in U\). For every object \(E\), the set of \((\omega, B)\) for which \(E\) is \(\nu_{\omega,B}\)-stable defines an open subset of \(U\).

By using Proposition 2.5, we can then prove the following.

**Proposition 2.6.** Let \(L\) be an ample line bundle, and assume that both \(\omega\) and \(B\) are proportional to \(L\). Assume also that Conjecture 2.3 holds for such \(B\) and \(\omega\). Let \(E \in \mathcal{B}_{\omega,B}\) be a \(\nu_{\omega,B}\)-stable object, with \(\text{ch}_0(E) \neq 0\) and \(\text{ch}_1(E)\) proportional to \(L\), and satisfying:

\[
\frac{\omega^3}{6} \text{ch}_0(E) = \omega \text{ch}_2^B(E) \quad \text{and} \quad \frac{\omega^2}{18} \text{ch}_3^B(E).
\]

Then \(\omega \Delta(E) = 0\).

**Proof.** Write \(d = L^3\), \(B = b_0L\), \(\omega = T_0L\) and \(\text{ch}_0(E) = r\). The idea for the proof is that, since stability is an open property, we can deform \(b = b_0\) and \(T = T_0\), as a function \(T = T(b)\) of \(b\), slightly such that \(E\) is still \(\nu_{T(b)L,bL}\)-stable with \(\nu_{T(b)L,bL}(E) = 0\); then we apply Conjecture 2.3 for the pairs \(\omega = T(b)L, B = bL\) depending on \(b\).

Evidently, \(\nu_{T(b)L,bL}(E) = 0\) is equivalent to

\[
T^2 = \frac{6}{rd} L \cdot \text{ch}_2^{bL}(E)
\]

Since \(T_0 > 0\), and since the equation is satisfied for \(T = T_0\) and \(b = b_0\), the equation defines a function \(T = T(b)\) for \(b\) nearby \(b_0\).

It is immediate to check from the definition that the chain rule

\[
\frac{\partial}{\partial b} \text{ch}^{bL}_i(E) = -L \cdot \text{ch}^{bL}_{i-1}(E)
\]

holds for \(i = 1, \ldots, 3\).

Consider

\[
f(b) = \text{ch}_3^{bL}(E) - \frac{(T(b)L)^2}{18} \cdot \text{ch}_1^{bL}(E) = \text{ch}_3^{bL}(E) - \frac{1}{3rd} L \cdot \text{ch}_2^{bL}(E) \cdot L^2 \cdot \text{ch}_1^{bL}(E)
\]

as a function of \(b\) in some neighborhood of \(b_0 \in \mathbb{R}\). By Proposition 2.5 and Conjecture 2.3, we have \(f(b) \leq 0\) for \(b\) close to \(b_0\), and by assumption \(f(b_0) = 0\); therefore \(f'(b_0) = 0\). Using equation (3), we obtain

\[
f'(b) = -L \cdot \text{ch}_2^{bL}(E) + \frac{1}{3rd} ((L^2, \text{ch}_1^{bL})^2 + L \cdot \text{ch}_2^{bL}(E) \cdot rd)
\]

\[
= \frac{1}{3r} (L \cdot (\text{ch}_1^{bL}(E))^2 - 2L \cdot \text{ch}_2^{bL}(E)r) = \frac{1}{3r} L \cdot \Delta(E).
\]
(Note that we used \((L^2, \text{ch}_1^{BL})^2 = L^3 \cdot L.(\text{ch}_1^{BL})^2\), which holds because \(\text{ch}_1^{BL}(E)\) is proportional to \(L\).) This proves the claim.

Finally, based on an alternate construction of tilt-stability, we also showed that it behaves well with respect to the dualizing functor \(\mathbb{D}_L(\_\,) = \mathbb{R}\text{Hom}(\_\,, L[1])\) for every line bundle \(L\). For this purpose, we fix \(B = \frac{L}{2}\):

**Proposition 2.7.** Let \(F \in \mathcal{B}_{\omega, \frac{L}{2}}\) be an object with \(\nu_{\omega, \frac{L}{2}}(A) < +\infty\) for every subobject \(A \subset F\). Then there is an exact triangle \(\tilde{F} \to \mathbb{D}_L(F) \to T_0[-1]\) where \(T_0\) is a zero-dimensional torsion sheaf and \(\tilde{F}\) an object of \(\mathcal{B}_{\omega, \frac{L}{2}}\) with \(\nu_{\omega, \frac{L}{2}}(\tilde{F}) = -\nu_{\omega, \frac{L}{2}}(F)\). The object \(\tilde{F}\) is \(\nu_{\omega, \frac{L}{2}}\)-semistable if and only if \(F\) is \(\nu_{\omega, \frac{L}{2}}\)-semistable.

**Proof.** Since \(\mathbb{D}_L(\_\,)\) can be written as the composition \(\_\, \otimes L \circ \mathbb{D}(\_\,)\), this follows from [BMT11, Proposition 5.1.3] and the fact that tensoring with \(L\) corresponds to replacing \(B\) with \(B - L\). 

\[\square\]

### 3. Reduction to Curves

In this section, we use Assumptions (A) and (B) of Theorem 4.1 to show that the non-vanishing of \(H^1(X, K_X \otimes L \otimes I_Z)\) implies the existence of special low-degree curves on \(X\). The approach, explained in the introduction, involves studying the tilt-stability of a certain object \(E\) in the category \(\mathcal{B}\) constructed in the previous section.


We will use Conjecture 2.3 in the case where \(L\) is an ample line bundle on \(X\), \(\omega = TL\) for some \(T > 0\), and \(B = \frac{L}{2}\). The abelian category \(\mathcal{B} := \mathcal{B}_{T,L,\frac{L}{2}}\) is independent of \(T\).

To simplify notation, we will rescale the slope function: set \(t = \frac{T^2}{6}\) and write \(\nu_t\) for

\[
\nu_t(\_\,) = T \cdot \nu_{T,L,\frac{L}{2}}(\_\,) = \frac{L \cdot \text{ch}_2^{L/2}(\_\,) - t d \cdot \text{ch}_0^{L/2}(\_\,)}{L^2 \cdot \text{ch}_1^{L/2}(\_\,)},
\]

where \(d := L^3\). Then the inequality of Conjecture 2.3 states that, for every \(\nu_t\)-stable object \(E\), we have

\[
\text{ch}_3^{L/2}(E) \leq \frac{t}{3} L^2 \cdot \text{ch}_1^{L/2}(E) \quad \text{if} \quad L \cdot \text{ch}_2^{L/2}(E) = dt \cdot \text{ch}_0^{L/2}(E).
\]

Let \(Z \subset X\) be a zero-dimensional subscheme of length \(\alpha\). Following [AB11], observe that if \(H^1(X, K_X \otimes L \otimes I_Z) \neq 0\), then by Serre duality, we also have \(\text{Ext}^2(L \otimes I_Z, \mathcal{O}_X) \neq 0\). Any non-zero element \(\xi \in \text{Ext}^2(L \otimes I_Z, \mathcal{O}_X)\) gives a non-trivial exact triangle in \(D^b(X)\)

\[
\mathcal{O}_X[1] \xrightarrow{E} E_\xi \xrightarrow{L \otimes I_Z \xi} \mathcal{O}_X[2].
\]

We will show that \(E\) is \(\nu_t\)-semistable for \(t = \frac{1}{8}\); its Chern classes invalidate the inequality of Conjecture 2.3 for \(t \ll 1\), and thus it must become unstable for \(t < t_0\) and some
$t_0 \in (0, \frac{1}{8}]$; finally, we will show that the Chern classes of its destabilizing factor would give special curves or divisors on $X$.

**Proposition 3.1.** Assume that $H^1(X, K_X \otimes L \otimes I_Z) \neq 0$, and let $E$ be an extension as given by equation (6).

(a) $E \in \mathcal{B}$ and

$$\text{ch}^{L/2}(E) = \left(0, L, 0, \frac{d}{24} - \alpha\right).$$

(b) If $t > \frac{1}{8}$, then (6) destabilizes $E$ with respect to $\nu_t$.

(c) If $t = \frac{1}{8}$, then $E$ is $\nu_t$-semistable.

(d) Assume Conjecture 2.3 and Assumption (A) of Theorem 4.1. Then $E$ is not $\nu_t$-semistable for $0 < t \ll 1$.

**Proof.** First of all, we have

$$\text{ch}^{L/2}(O_X) = \left(1, -\frac{L}{2}, \frac{L^2}{8}, -\frac{L^3}{48}\right),$$

$$\text{ch}^{L/2}(L \otimes I_Z) = \left(1, \frac{L}{2}, \frac{L^2}{8}, \frac{L^3}{48} - \alpha\right).$$

As $O_X$ and $L \otimes I_Z$ are slope-stable, with $\mu_{\omega,L/2}(O_X) < 0$ and $\mu_{\omega,L/2}(L \otimes I_Z) > 0$, we have $O_X \in \mathcal{F}$ and $L \otimes I_Z \in \mathcal{T}$. By the definition of $\mathcal{B}$, it follows that $O_X[1], L \otimes I_Z$ and $E$ are all objects of $\mathcal{B}$; in particular, we have proved (a).

Moreover, we have

$$\nu_t(O_X[1]) = 2 \left(t - \frac{1}{8}\right), \quad \nu_t(E) = 0$$

which immediately implies (b), since (6) is an exact sequence in $\mathcal{B}$.

To prove (c), simply observe that, by Proposition 2.4, both $O_X[1]$ and $L$ are $\nu_t$-stable for all $t > 0$. Moreover, since $\nu_t(L \otimes I_Z) = \nu_t(L)$, any destabilizing subobject $A \hookrightarrow L \otimes I_Z$ would also destabilize $L$ via the composition $A \hookrightarrow L \otimes I_Z \hookrightarrow L$ (which is an inclusion in $\mathcal{B}$); thus $L \otimes I_Z$ is also $\nu_t$-stable. For $t = \frac{1}{8}$, we have $\nu_t(O_X[1]) = \nu_t(L \otimes I_Z) = 0$, and thus the extension (6) shows that $E$ is $\nu_t$-semistable at $t = \frac{1}{8}$.

Finally, if $E$ was $\nu_t$-semistable for all $t \in (0, \frac{1}{8}]$, then by our conjectural inequality (5) we would get

$$\frac{d}{24} - \alpha \leq \frac{t}{3d}$$

for all such $t$. Hence $d \leq 24\alpha$, in contradiction to Assumption (A).

Notice that the previous proposition would answer Question 4 in [AB11]. Also observe that in part (d), instead of Assumption (A), already assuming $d > 24\alpha$ would have been
enough. Similarly, instead of Conjecture 2.3, any linear inequality between \( \text{ch}_3 \) and \( \text{ch}_4 \) would have been sufficient.

In the following proposition, we will show that our situation is self-dual with respect to the local dualizing functor \( \mathbb{D}_L(\_\_\_) = \mathbb{R} \text{Hom}(\_\_, L[1]) \). As a preliminary, let us first note that we may make the following assumption:

\[ (*) \quad H^1(X, K_X \otimes L \otimes I_{Z'}) = 0 \text{ for all subschemes } Z' \subset Z, \text{ and } H^1(X, K_X \otimes L \otimes I_{Z}) \cong \mathbb{C}. \]

Indeed, in order to show \( H^1(X, L \otimes I_Z \otimes K_X) = 0 \), we can proceed by induction on the length of \( Z \) (the case \( \alpha = 0 \) is, of course, given by Kodaira vanishing).

**Proposition 3.2.** If Assumption \( (*) \) holds, and \( E \) is given by the unique non-trivial extension of the form (6), then \( E \cong \mathbb{D}_L(E) \).

**Proof.** Due to Assumption \( (*) \), it is sufficient to show that \( \mathbb{D}_L(E) \) is again a non-trivial extension of the form (6). Applying the octahedral axiom to the composition \( O_Z[-1] \rightarrow L \otimes I_Z \rightarrow O_X[2] \), and using the two exact triangles (6) and \( O_Z[-1] \rightarrow L \otimes I_Z \rightarrow L \), we obtain an exact triangle \( F \rightarrow E \rightarrow L \), where \( F \) itself fits into an exact triangle

\[ (9) \quad O_X[1] \rightarrow F \rightarrow O_Z[-1]. \]

We claim that \( \text{Hom}(k(x)[-1], F) = 0 \) for all skyscraper sheaves of points \( x \in X \). Using the long exact sequence for \( \text{Hom}(k(x), \_\_\_) \) applied to (9), we see that this is equivalent to the non-vanishing of the composition

\[ (10) \quad k(x)[-1] \rightarrow O_Z[-1] \rightarrow L \otimes I_Z \xrightarrow{\xi} O_X[2] \]

for every inclusion \( k(x) \hookrightarrow O_Z \). Given such an inclusion, let \( Z' \subset Z \) be the subscheme given by \( O_{Z'} \cong O_Z/k(x) \). If the composition (10) vanishes, then \( \xi \) factors via \( L \otimes I_Z \hookrightarrow L \otimes I_{Z'} \). This contradicts our assumption \( \text{Ext}^2(L \otimes I_{Z'}, O_X) = H^1(X, L \otimes I_{Z'} \otimes K_X)^\vee = 0 \).

Now we apply \( \mathbb{D}_L \) to the exact triangle \( O_X[1] \rightarrow F \rightarrow O_Z[-1] \). As \( \mathbb{D}_L(O_X[1]) = L \) and \( \mathbb{D}_L(O_Z[-1]) = O_Z[-1] \), dualizing (9) gives an exact triangle \( O_Z[-1] \rightarrow \mathbb{D}_L(F) \rightarrow L \rightarrow O_Z \). Since \( \text{Hom}(\mathbb{D}_L(F), k(x)[-1]) = \text{Hom}(k(x)[-1], F) = 0 \) for all \( x \in X \), the map \( L \rightarrow O_Z \) must be surjective, and hence \( \mathbb{D}_L(F) \cong L \otimes I_Z \). Consequently, applying \( \mathbb{D}_L \) to the exact triangle \( F \rightarrow E \rightarrow L \) shows that \( \mathbb{D}_L(E) \) is indeed a non-trivial extension of the form (6). \( \square \)

### 3.2. Chern classes of destabilizing subobjects

By Proposition 3.1 and Proposition 2.5, Conjecture 2.3 implies the existence of \( t_0 \in (0, \frac{1}{8}] \) with the following properties:

- \( E \) is \( \nu_{t_0} \)-semistable.
- There exists an exact sequence in \( \mathcal{B} \)

\[ (11) \quad 0 \rightarrow A \rightarrow E \rightarrow F \rightarrow 0, \]

with \( \nu_t(A) > 0 \) if \( t < t_0 \), and \( \nu_{t_0}(A) = 0 \).

In the remainder of this section, we will prove the following statement:
\textbf{Proposition 3.3.} Assume that \( X, L, \alpha \) satisfy Assumptions (A) and (B) of Theorem 4.1 and Assumption (*) of the previous section. Then in any destabilizing sequence \( (11) \), the object \( A \) is of the form \( L \otimes I_C \), for some purely one-dimensional subscheme \( C \subset X \) containing \( Z \).

We will first prove this for subobjects satisfying \( L^2.\text{ch}_1^{L/2}(A) \leq L^2.\text{ch}_1^{L/2}(F) \), or, equivalently,

\[
(12) \quad L^2.\text{ch}_1^{L/2}(A) \leq \frac{1}{2}L^2.\text{ch}_1^{L/2}(E) = \frac{d}{2}.
\]

(We will later use the derived duality \( \mathbb{D}_L(\_ \_] \) to reduce to this case.)

\textbf{Lemma 3.4.} Any subobject \( A \) satisfying \( (12) \) is a sheaf with \( \text{rk}(A) = \text{rk}(H^0(A)) > 0 \).

\textit{Proof.} Consider the long exact cohomology sequence for \( A \hookrightarrow E \twoheadrightarrow F \). If \( H^{-1}(A) \neq 0 \), then \( H^{-1}(A) \hookrightarrow \mathcal{O}_X \) is isomorphic to an ideal sheaf of some subscheme \( Y \) of \( X \). Since \( \mathcal{O}_Y \hookrightarrow H^{-1}(F) \) and \( H^{-1}(F) \) is torsion-free, we must have \( H^{-1}(A) \cong \mathcal{O}_X \). Then \( H^0(A) \) is also torsion-free, and \( (12) \) implies

\[
L^2.\text{ch}_1^{L/2}(H^0(A)) = L^2.\text{ch}_1^{L/2}(A) - L^2.\text{ch}_1^{L/2}(\mathcal{O}_X[1]) \leq \frac{d}{2} - \frac{d}{2} = 0.
\]

On the other hand, by construction of \( \mathcal{B} \), every HN-filtration factor \( U \) of \( H^0(A) \) satisfies \( L^2.\text{ch}_1^{L/2}(U) > 0 \); thus \( H^0(A) = 0 \) and \( A = \mathcal{O}_X[1] \). This contradiction proves \( H^{-1}(A) = 0 \).

Finally, note that if \( A = H^0(A) \) is a torsion-sheaf, then \( \nu_t(A) \) is independent of \( t \), again a contradiction. \( \Box \)

\textbf{Lemma 3.5.} Either \( A \) is torsion-free, or its torsion-part \( A_t \) satisfies

\[
L^2.\text{ch}_1(A_t) - 2L.\text{ch}_2(A_t) \geq 0 \quad \text{and} \quad L^2.\text{ch}_1(A_t) > 0.
\]

\textit{Proof.} The sheaf \( A_t \) is a subobject of \( E \) in \( \mathcal{B} \) with \( \text{rk} = 0 \). Hence \( L.\text{ch}_2^{L/2}(A_t) \leq 0 \), otherwise it would destabilize \( E \) at \( t = \frac{1}{k} \). Expanding \( \text{ch}_2^{L/2} \) gives the first inequality. To show the second inequality, we just observe that there are no non-trivial morphisms from sheaves supported in dimension \( \leq 1 \) to \( E \). \( \Box \)

\textbf{Lemma 3.6.} In the HN-filtration of \( A \) with respect to slope-stability, there exists a factor \( U \) of rank \( r \) such that \( \Gamma := L - \frac{\text{ch}_1(U)}{r} \) satisfies the following inequalities:

(I) \[
L^2.\Gamma \leq L.\Gamma^2 + 6\alpha
\]

(II) \[
\frac{d}{2} \left( 1 - \frac{1}{r} \right) \leq L^2.\Gamma < \frac{d}{2}.
\]

The case \( r = 1 \) and \( L^2.\Gamma = 0 \) only occurs when \( A \) is a torsion-free sheaf of rank one and \( H^{-1}(F) = \mathcal{O}_X \).
If $A$ was a line bundle, the above definition of $\Gamma$ would be just as Reider’s original argument for surfaces: in this case, $\Gamma$ is the support of the cokernel of $A \hookrightarrow H^0(E) \cong L \otimes I_Z$.

Proof. From $\nu_t (A) = 0$ we obtain

$$t_0 = \frac{L \cdot \text{ch}_2^{L/2}(A)}{\text{rk}(A)d}.$$  \hspace{1cm} (13)

Applying the conjectured inequality (5) to $E$, and plugging in $t_0$ gives

$$\frac{d}{24} - \alpha = \text{ch}_3^{L/2}(E) \leq \frac{L^2 \cdot \text{ch}_1^{L/2}(E) t_0}{3} = \frac{d L \cdot \text{ch}_2^{L/2}(A)}{3 \cdot \text{rk}(A)d} = \frac{1}{3} \cdot \frac{L \cdot \text{ch}_2^{L/2}(A)}{\text{rk}(A)}.$$  

We want to bound $L \cdot \text{ch}_2^{L/2}(A)$. First we expand $\text{ch}_2^{L/2}(A)$:

$$\text{ch}_2^{L/2}(A) = \text{ch}_2(A) - \frac{L \cdot \text{ch}_1(A)}{2} + \text{rk}(A) \frac{L^2}{8}.$$  

Substituting, we deduce

$$\frac{L^2 \cdot \text{ch}_1(A)}{\text{rk}(A)} - 2 \frac{L \cdot \text{ch}_2(A)}{\text{rk}(A)} \leq 6\alpha.$$  \hspace{1cm} (14)

Let $A_{tf}$ denote the torsion-free part of $A$, and consider its HN-filtration. Among the HN factors, we choose a torsion-free sheaf $U$ for which the function

$$\eta(U) := \frac{L^2 \cdot \text{ch}_1(U) - 2L \cdot \text{ch}_2(U)}{\text{rk}(U)}$$

is minimal. Notice that $\eta$ satisfies the see-saw property: for an exact sequence of torsion-free sheaves

$$0 \to M \to N \to P \to 0,$$

we have $\eta(N) \geq \min\{\eta(M), \eta(P)\}$. Hence we get a chain of inequalities leading to

$$\eta(U) \leq \eta(A_{tf}) \leq \eta(A) \leq 6\alpha$$  \hspace{1cm} (15)

where we used Lemma 3.5 for the second inequality.

To abbreviate, we now write $D := \text{ch}_1(U)$ and $r := \text{rk}(U)$. Since $U$ is $\mu_{\omega,L/2}$-semistable, we can combine the classical Bogomolov-Gieseker inequality with (15) to obtain

$$L^2 \cdot \frac{D}{r} = \frac{2L \cdot \text{ch}_2(U)}{r} + \eta(U) \leq L \cdot \frac{D^2}{r^2} + 6\alpha.$$  

Substituting $D = rL - r\Gamma$ yields the inequality (I).
To prove the chain of inequalities (II), we observe on the one hand that \( L^2, \text{ch}^{L/2}(U) > 0 \) by the definition of \( T_{\omega, B} = B \cap \text{Coh} X \). On the other hand, \( U \) is a subquotient of \( A \) in \( T_{\omega, B} \); combined with inequality (12) we obtain
\[
0 < L^2, \text{ch}^{L/2}(U) \leq L^2, \text{ch}^{L/2}(A) \leq \frac{d}{2}.
\]
Plugging in \( \text{ch}^{L/2}(U) = -\frac{c}{2}I + D = \frac{c}{2}L - r \Gamma \) shows the inequality (II).

Finally, note that in the case \( r = 1 \) and \( L^2, \Gamma = 0 \) the chain of inequalities leading to the first part of (II) must be equalities; in particular \( L^2, \text{ch}^{L/2}(U) = L^2, \text{ch}^{L/2}(A) \). This shows that \( A_{tf} \) cannot have any other HN-filtration factors besides \( U \), i.e., \( U = A_{tf} \). Additionally it implies that \( \text{ch}^{L/2}(A_t) = 0 \), in contradiction to Lemma 3.5; hence \( A_t = 0 \) and \( A = U \) is a torsion-free rank one sheaf.

As \( L \otimes I_Z \) is torsion-free, if the image of \( H^{-1}(F) \to A \) is non-trivial, then the map is surjective, and the inclusion \( A \hookrightarrow E \) factors via \( A \hookrightarrow O_X[1] \hookrightarrow E \), in contradiction to the stability of \( O_X[1] \) for all \( t \) and \( \nu_0 - \varepsilon(A) > 0 > \nu_0 - \varepsilon(O_X[1]) \). Thus \( H^{-1}(F) = O_X \). \( \square \)

Proof. (Proposition 3.3) We combine (I) and (II) with the Hodge Index Theorem (just as in [AB11, Corollary 3.9]) to obtain
\[
(L, \Gamma^2) \leq (L^2, \Gamma)^2 \leq \frac{d}{2} (L, \Gamma^2 + 6 \alpha),
\]
and so \( L, \Gamma^2 \leq 6 \alpha \).

In the case \( r > 1 \), we use (I) and (II) again to get
\[
\frac{d}{4} \leq L^2, \Gamma \leq L, \Gamma^2 + 6 \alpha \leq 12 \alpha,
\]
and so \( d \leq 48 \alpha \) in contradiction to Assumption (A).

Reider’s original argument in [Rei88] deals with the case \( r = 1 \); in case \( L^2, \Gamma \neq 0 \), then \( L^2, \Gamma \geq 1 \). Let \( \kappa := L, \Gamma^2 \leq 6 \alpha \). Again combining the Hodge Index Theorem with (I), we obtain
\[
(L, \Gamma^2) \leq (L, \Gamma^2 + 6 \alpha)^2,
\]
and so
\[
d \leq 12 \alpha + \frac{\kappa^2 + 36 \alpha^2}{\kappa}.
\]
The RHS is strictly decreasing function for \( \kappa \in (0, 6 \alpha] \) and equals \( 49 \alpha \) for \( \kappa = \alpha \); thus Assumption (A) implies \( \kappa < \alpha \). On the other hand, \( \Gamma \) is integral, and hence Assumption (B) implies \( L^2, \Gamma \geq 7 \alpha \), in contradiction to (I).

Finally, if \( L^2, \Gamma = 0 \); then, according to Lemma 3.6, we have \( H^{-1}(F) \cong O_X \). Hence \( A \) is a subsheaf of \( L \otimes I_W \) with \( \text{ch}_1(A) = \text{ch}_1(L) \); this is only possible if \( A \cong L \otimes I_W \), for some closed subscheme \( W \subset X \) with \( \dim(W) \leq 1 \). If \( W \) is zero-dimensional, then \( \text{ch}_2^{L/2}(A) = \frac{1}{2}L^2 \) and equation (13) gives \( t_0 = \frac{1}{2} \), in contradiction to \( t_0 \in (0, \frac{1}{8}] \). Hence \( W \)
is one-dimensional, and we have shown that any subobject $A$ with $\text{ch}_1^{L/2}(A) \leq \frac{d}{2}$ is of the form $A \cong L \otimes I_W$. In particular $\text{ch}_1^{L/2}(A) = \frac{d}{2}$ in this case, so there are no subobject with $\text{ch}_1^{L/2}(A) < \frac{d}{2}$.

Now assume $\text{ch}_1^{L/2}(A) > \frac{d}{2}$. We can apply Proposition 3.2 and Proposition 2.7 to the short exact sequence (11) obtain a short exact sequence in $\mathcal{B}$

$$0 \to \tilde{F} \to E \to E/\tilde{F} \to 0$$

which is again destabilizing. Indeed, since $\mathcal{B}$ is the heart of a bounded t-structure, there exists a cohomology functor $H^*_B(\_ \_ \_)$.

As $D_L$ preserves $L^2$, $\text{ch}_1^{L/2}(\_ \_ \_)$, we have that $\tilde{F}$ is a destabilizing subobject with $\text{ch}_1^{L/2}(F) = \text{ch}_1^{L/2}(E) - \text{ch}_1^{L/2}(A) < \frac{d}{2}$, which does not exist.

Finally, note that the long exact sequence (16) also implies that $D_L(A) = \tilde{A} \in \mathcal{B}$. This gives the vanishing of $\text{Hom}(D_L(A), k(x)[-1]) = \text{Hom}(k(x)[-1], A)$. This is equivalent to the claim that $W$ is a purely one-dimensional scheme, as any subsheaf $k(x) \hookrightarrow O_W$ gives an extension of $k(x)$ by $L \otimes I_W$. This finishes the proof of Proposition 3.3.

\[ \square \]

4. A Reider-type theorem

In this section we prove our main theorem:

**Theorem 4.1.** Let $L$ be an ample line bundle on a smooth projective threefold $X$, and assume Conjecture 2.3 holds for $\mathcal{B}$ and $\omega$ proportional to $L$. Fix a positive integer $\alpha$, and assume that $L$ satisfies the following conditions:

(A) $L^3 > 49\alpha$;

(B) $L^2.D \geq 7\alpha$, for all integral divisor classes $D$ with $L^2.D > 0$ and $L.D^2 < \alpha$;

(C) $L.C \geq 3\alpha$, for all curves $C$.

Then $H^1(X, K_X \otimes L \otimes I_Z) = 0$, for any zero-dimensional subscheme $Z \subset X$ of length $\alpha$.

**Proof.** As explained in Section 3.1, we may proceed by induction on the length of $Z$ and may use Assumption (*). Let $t_0 \in (0, \frac{1}{8}]$ be as in Section 3.2 and let $t = t_0 - \epsilon$. Truncating the Harder-Narasimhan filtration of $E$ with respect to $\nu_t$-stability gives a short exact sequence

$$0 \to A \to E \to F \to 0$$
with \( \nu_t(A) > 0 \), such that any subobject \( A' \hookrightarrow E \) with \( \nu_t(A') > 0 \) factors via \( A' \hookrightarrow A \).

By Proposition 3.3, \( A \) is of the form \( L \otimes I_C \) for some purely one-dimensional subscheme \( C' \subset X \); it also implies that \( A \) is stable, as any destabilizing subobject \( A' \) of \( A \) would again be of the form \( A' \cong L \otimes I_{C'} \), so that the quotient \( A/A' \) would be a torsion sheaf with \( \nu_t(A/A') = +\infty \).

Let \( \tilde{F} \) be the object obtained by dualizing \( F \) and applying Proposition 2.7. The map \( \mathbb{D}_L(F) \to \mathbb{D}_L(E) \cong E \) induces a map \( \tilde{F} \to E \) which is an injection in \( B \). Since

\begin{equation}
\text{ch}_i^{L/2}(\tilde{F}) = \text{ch}_i^{L/2}(\mathbb{D}_L(F))
\end{equation}

for \( i \leq 2 \), we have \( \nu_t(\tilde{F}) = -\nu_t(F) > 0 \); thus the map factorizes as \( \tilde{F} \hookrightarrow A \hookrightarrow E \).

By Proposition 3.3, the object \( \tilde{F} \) is of the form \( L \otimes I_{C'} \) for some purely one-dimensional subscheme \( C' \subset X \). Equation (17) also implies \( \text{ch}_i^{L/2}(\tilde{F}) = \text{ch}_i^{L/2}(A) \) for \( i \leq 2 \); thus the (non-trivial) map \( L \otimes I_{C'} \to L \otimes I_C \) has zero-dimensional cokernel. It follows that

\begin{equation}
\text{ch}_3^{L/2}(F) = \text{ch}_3^{L/2}(\mathbb{D}_L(F)) \leq \text{ch}_3^{L/2}(\tilde{F}) \leq \text{ch}_3^{L/2}(A).
\end{equation}

This implies that

\begin{equation}
2 \text{ch}_3^{L/2}(A) \geq \text{ch}_3^{L/2}(A) + \text{ch}_3^{L/2}(F) = \text{ch}_3^{L/2}(E) = \frac{d}{24} - \alpha,
\end{equation}

and the difference of the two sides is a non-negative integer.

On the other hand, as \( A \) is stable, by Conjecture 2.3, by (13) and (18), and by expanding \( \text{ch}^{L/2} \) we have

\begin{equation}
\frac{d}{48} - \frac{\alpha}{2} \leq \text{ch}_3^{L/2}(A) \leq \frac{t_0}{3} L^2. \text{ch}_1^{L/2}(A) = \frac{1}{6} L. \text{ch}_2^{L/2}(A) = \frac{d}{48} - \frac{L.C}{6}.
\end{equation}

We now use Assumption (C): \( L.C \geq 3\alpha \). This contradicts (19), unless \( L.C = 3\alpha \) and

\[ \frac{d}{48} - \frac{\alpha}{2} = \text{ch}_3^{L/2}(A) = \frac{t_0}{3} L^2. \text{ch}_1^{L/2}(A). \]

Since \( (TL) \Delta(A) = 3\alpha T \neq 0 \), this in turn contradicts Proposition 2.6. \( \square \)

We also obtain the following result characterizing the only possible counter-examples to Fujita’s very ampleness conjecture in case \( L = M^5 \):

**Proposition 4.2.** Assume that Conjecture 2.3 holds for \( X \), \( \omega = tL \) and \( B = \frac{L}{2} \) and \( L \cong M^5 \) for an ample line bundle \( M \). Then either \( K_X \otimes L \) is very ample, or there exists a curve \( C \) of degree \( M.C = 1 \) and arithmetic genus \( g_a(C) = \frac{5}{2} + \frac{1}{2} K_X.C \) such that \( K_X \otimes L|_C \) is a line bundle of degree \( 2g_a(C) \) on \( C \) which is not very ample.

**Proof.** Assume that \( K_X \otimes L \) is not very ample. We follow the logic and the notation of the proof of Theorem 4.1, with \( \alpha = 2 \). As before, let \( A = L \otimes I_C \) be the destabilizing subobject of \( E \) for \( t = t_0 - \varepsilon \); here \( C \) is a purely one-dimensional subscheme of \( X \). By the proof of Theorem 4.1, we have \( L.C < 6 \) and thus necessarily \( M.C = 1 \) and \( L.C = 5 \). In
particular, $C$ is reduced and irreducible. We claim that $\text{ch}^L_3(A) = \frac{d}{48} - 1$. Indeed, setting $\alpha = 2$ in (19) gives

$$
\frac{d}{48} - 1 \leq \text{ch}^L_3(A) \leq \frac{d}{48} - \frac{5}{6}.
$$

On the other hand, if $\text{ch}^L_3(A) \neq \frac{d}{48} - 1$, then, by (18), $\text{ch}^L_3(A) \geq \frac{d}{48} - \frac{1}{2}$, a contradiction to the inequality (20).

From the claim, we obtain

$$
\text{ch}_3(L \otimes O_C) = \text{ch}_3(L) - \text{ch}_3(A) = \frac{7}{2}
$$

and thus

$$
\text{ch}_3(O_C) = \text{ch}_3(L \otimes O_C) - L.C = \frac{3}{2}
$$

By Hirzebruch-Riemann-Roch, we get

$$
1 - g_a(C) = \text{ch}_3(O_C) - \frac{1}{2}K_X.C.
$$

Plugging in the previous equation and solving for $K_X.C$ shows that $K_X \otimes L|C$ is a line bundle of degree $2g_a(C)$ on $C$. The explicit expression for $g_a(C)$ follows immediately.

Finally, the cohomology sheaves of the quotient $F \cong E/A$ are $H^{-1}(F) \cong O_X$ and $H^0(F) \cong L \otimes O_C(-Z)$ (where $O_C(-Z)$ denotes the ideal sheaf of $Z \subset C$). If $F$ were decomposable, $\tilde{F}$ would be a decomposable destabilizing subobject of $E$, which cannot exist. Hence

$$
0 \neq \text{Ext}^2(L \otimes O_C(-Z), O_X) = H^1(C, K_X \otimes L|C(-Z))^\vee.
$$

On the other hand, $K_X \otimes L|C$ is a line bundle of degree $2g_a(C)$ on an irreducible Cohen-Macaulay curve, and thus $H^1(K_X \otimes L|C) = 0$. Hence $K_X \otimes L|C$ is not very ample. \hfill \square

**Remark 4.3.** Notice that Proposition 4.2 implies Fujita’s conjecture when $K_X$ is numerically trivial (or, more generally, when $K_X.C$ is even for all integral curve classes $C$).

In case the curve $C \subset X$ of Proposition 4.2 is l.c.i, one can be even more precise. Let $\omega_C$ be the dualizing sheaf (which agrees with the dualizing complex, as $O_C$ is pure and thus $C$ Cohen-Macaulay). The sheaf $K_X \otimes L(-Z)|C$ is torsion-free of rank one and degree $2g_a(C) - 2$ with $H^1(K_X \otimes L(-Z)|C) \neq 0$, and thus Serre duality implies $K_X \otimes L(-Z)|C \cong \omega_C$. If $N$ is the normal bundle, adjunction gives $\Lambda^2 N \cong L(-Z)$. In particular, the normal bundle has degree 3. Since $M.C = 1$, bend-and-break implies that such a curve cannot be rational.

In conclusion, we show how to reverse part of the argument in this section when $Z$ has length one. Indeed, in such a case we can use Ein-Lazarsfeld theorem (or better, its variant
by Kawamata and Helmke) to show that Conjecture 2.3 holds true for this particular case, coherently with our result:

**Proposition 4.4.** Let $L$ be an ample line bundle on a smooth projective threefold $X$. Assume that $L$ satisfies the following conditions:

(a) $L^3 \geq 28$;
(b) $L^2 \cdot D \geq 9$, for all integral effective divisor classes $D$.

Assume also that there exists $x \in X$ such that $H^1(X, K_X \otimes L \otimes I_x) \neq 0$. Then Conjecture 2.3 holds for all objects $E \in \mathcal{B}$ given as non-trivial extensions

$$\mathcal{O}_X[1] \to E \to L \otimes I_x \to \mathcal{O}_X[2].$$

**Proof.** The argument is very similar to [Kaw97], Proposition 2.7 and Theorem 3.1, Step 1. We freely use the notation from [Laz04, Sections 9 & 10]. By [Kaw97, Lemma 2.1], given a rational number $t$ satisfying $3/\sqrt{L^3} < t < 1$, there exists a $\mathbb{Q}$-divisor $D$ numerically equivalent to $tL$ such that $\text{ord}_x D = 3$. Let $c \leq 1$ the log-canonical threshold of $D$ at $x$.

By [Kaw97, Theorem 3.1] (also [Hel97]) and our assumptions, the LC-locus $\text{LC}(cD; x)$ (i.e., the zero-locus of the multiplier ideal $\mathcal{J}(c \cdot D)$ passing through $x$) must be a curve $C$ satisfying $1 \leq L.C \leq 2$. We can now apply Nadel's vanishing theorem to $cD$ to deduce that $H^1(X, K_X \otimes L \otimes I_C) = 0$, and so that the restriction map $H^0(X, K_X \otimes L) \to H^0(X, K_X \otimes L|_C)$ is surjective.

Consider the composition $u: L \otimes I_C \to L \otimes I_x \to \mathcal{O}_x[2]$. Then, $u \neq 0$ if and only if $x$ is a base point of $K_X \otimes L$ which is not a base point of $K_X \otimes L|_C$. The surjectivity of the restriction map implies that $u = 0$. Hence, we get an inclusion $L \otimes I_C \hookrightarrow E$ in $\mathcal{B}$ which destabilizes $E$, if (2) is not satisfied. \qed

**References**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT U-3009, 196 AUDITORIUM ROAD, STORRS, CT 06269-3009, USA

E-mail address: bayer@math.uconn.edu

URL: http://www.math.uconn.edu/~bayer/

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, 155 S 1400 E, SALT LAKE CITY, UT 84112, USA

E-mail address: bertram@math.utah.edu

URL: http://www.math.utah.edu/~bertram/

MATHEMATICAL INSTITUTE, UNIVERSITY OF BONN, ENDENICHER ALLEE 60, D-53115 BONN, GERMANY & DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, 155 S 1400 E, SALT LAKE CITY, UT 84112, USA

Current address: Department of Mathematics, The Ohio State University, 231 W 18th Avenue, Columbus, OH 43210, USA

E-mail address: macri.6@math.osu.edu

URL: http://www.math.osu.edu/~macri.6/

INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE, UNIVERSITY OF TOKYO, 5-1-5 KASHIWANOHA, KASHIWA, 277-8583, JAPAN

E-mail address: yukinobu.toda@ipmu.jp