1. If $V$ is an inner product space and $\langle \mathbf{v}, \mathbf{x} \rangle = \langle \mathbf{w}, \mathbf{x} \rangle$ for all $\mathbf{x} \in V$, prove that $\mathbf{v} = \mathbf{w}$.

2. Prove the Pythagorean theorem: If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal vectors in an inner product and $\mathbf{c} = \mathbf{a} + \mathbf{b}$, then $||\mathbf{a}||^2 + ||\mathbf{b}||^2 = ||\mathbf{c}||^2$.

3. If $T : V \to V$ has an adjoint $T^*$ and $T^*T$ is the zero transformation, show that $T$ is the zero transformation.

4. If $V$ is finite-dimensional, show that $T : V \to V$ is invertible if and only if $0$ is not an eigenvalue of $T$.

5. If $T : V \to V$ is invertible, show that $\lambda$ is an eigenvalue of $T$ if and only if $\lambda^{-1}$ is an eigenvalue of $T^{-1}$.

6. If $V$ is a complex vector space and $T : V \to V$ is diagonalizable and only has one eigenvalue $\lambda$, prove that $T$ is the multiplication-by-$\lambda$ map $\lambda I$.

7. For any field $F$, prove that similarity is an equivalence relation on $M_{n\times n}(F)$.

8. For any $n \times n$ matrix $A$, prove that $A$ and $A^T$ have the same characteristic polynomial and hence the same eigenvalues.

9. Suppose $A \in M_{n\times n}(F)$ has two distinct eigenvalues $\lambda$ and $\mu$. If the $\lambda$-eigenspace has dimension $n - 1$, prove that $A$ is diagonalizable.

10. Prove that $T : V \to V$ is diagonalizable if and only if $V$ is the direct sum of the eigenspaces of $T$.

11. If $A$ is an $n \times n$ matrix, define the subspace $W \subseteq M_{n\times n}(F)$ as $W = \text{span}(I_n, A, A^2, A^3, \ldots)$. Prove that $\dim_F W \leq n$.

12. If $\beta = \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a chain of generalized $\lambda$-eigenvectors (i.e., with $(T - \lambda I)\mathbf{v}_i = \mathbf{v}_{i+1}$ for each $0 \leq i \leq k - 1$ and $(T - \lambda I)\mathbf{v}_k = \mathbf{0}$) show that $\beta$ is linearly independent.

13. Prove that, up to similarity, there are exactly 5 different matrices in $M_{6\times 6}(\mathbb{C})$ with characteristic polynomial $p(t) = t^6 - t^4$.

14. Suppose $A \in M_{n\times n}(\mathbb{C})$ such that $A^2 - 2A = 0$. Prove that the Jordan canonical form of $A$ must be diagonal.

15. Suppose $T : V \to V$ is diagonalizable. Prove that for any $\lambda$ in the scalar field of $V$, it is true that $\text{rank}(T - \lambda I) = \text{rank}(T - \lambda I^2)$.

16. Suppose $T : V \to V$ has the property that $\text{rank}(T - \lambda I) = \text{rank}(T - \lambda I)^2$ for every $\lambda$ in the scalar field of $V$. Prove that $T$ is diagonalizable. [Hint: Consider the $\lambda$-blocks in the Jordan canonical form.]

17. If $V$ is a complex inner product space and $T : V \to V$ is Hermitian, prove that $||T(\mathbf{v}) + i\mathbf{v}||^2 = ||T(\mathbf{v})||^2 + ||\mathbf{v}||^2$, and deduce that $T + iI$ is invertible. [Hint: Show $T + iI$ is one-to-one.]

18. If $V$ is a finite-dimensional inner product space and $T : V \to V$ has an adjoint $T^*$, prove that all eigenvalues of $T^*T$ are nonnegative real numbers, and deduce that $I + T^*T$ is invertible.