1. If $V$ is an inner product space and $\langle v, x \rangle = \langle w, x \rangle$ for all $x \in V$, prove that $v = w$.
   - Subtracting yields $(v - w, x) = 0$ for all $x \in V$, which by one of our lemmas (basic idea: set $x = v - w$) tells us that $v - w = 0$ whence $v = w$.
2. Prove the Pythagorean theorem: If $a$ and $b$ are orthogonal vectors in an inner product and $c = a + b$, then $||c||^2 = ||a||^2 + ||b||^2$.
   - $||c||^2 = \langle c, c \rangle = \langle a + b, a + b \rangle = \langle a, a \rangle + \langle b, b \rangle + \langle a, b \rangle + \langle b, a \rangle = ||a||^2 + ||b||^2$ since $a, b$ are orthogonal.
3. If $T : V \to V$ has an adjoin $T^*$ and $T^*T$ is the zero transformation, show that $T$ is the zero transformation.
   - If $T^*T$ is zero then for any $v \in V$ we have $\langle T(v), T(v) \rangle = \langle T^*T(v), v \rangle = 0$, so $T(v) = 0$ for all $v \in V$.
4. If $V$ is finite-dimensional, show that $T : V \to V$ is invertible if and only if $0$ is not an eigenvalue of $T$.
   - Since $V$ is finite-dimensional, $T$ is invertible $\iff$ $T$ is one-to-one $\iff$ $Tv = 0$ implies $v = 0$ $\iff$ there is no nonzero 0-eigenvector $v$ $\iff$ $0$ is not an eigenvalue of $T$.
5. If $T : V \to V$ is invertible, show that $\lambda$ is an eigenvalue of $T$ if and only if $\lambda^{-1}$ is an eigenvalue of $T^{-1}$.
   - If $Tv = \lambda v$ then multiplying by $T^{-1}$ gives $v = \lambda(T^{-1}v)$ so $T^{-1}v = \lambda^{-1}v$. The other direction is equivalent to this one upon swapping $T$ and $T^{-1}$.
6. If $V$ is a complex vector space and $T : V \to V$ is diagonalizable and only has one eigenvalue $\lambda$, prove that $T$ is the multiplication-by-$\lambda$ map $\lambda I$.
   - If $T$ is diagonalizable then the diagonal entries are all eigenvalues, so this says $[T]_B^B$ must be $\lambda$ times the identity matrix. But then $[\lambda I]_B^B = \lambda I_n = [T]_B^B$ so $T = \lambda I$.
7. For any field $F$, prove that similarity is an equivalence relation on $M_{n \times n}(F)$.
   - Clearly $A = I_n^{-1}AI_n$ so $A \sim A$. Also, if $A \sim B$ then $B = Q^{-1}AQ$ in which case $A = QBQ^{-1} = P^{-1}BP$ with $P = Q^{-1}$, so $B \sim A$. Finally, if $A \sim B$ and $B \sim C$ then $B = Q^{-1}AQ$ and $C = R^{-1}BR$ in which case $C = (QR)^{-1}A(QR)$ so $A \sim C$.
8. For any $n \times n$ matrix $A$, prove that $A$ and $A^T$ have the same characteristic polynomial and hence the same eigenvalues.
   - Observe $\det(tI - A^T) = \det(tI^T - A^T) = \det((tI - A)^T) = \det(tI - A)$ since $\det(B^T) = \det(B)$ for any $B$. So the characteristic polynomials of $A^T$ and $A$ are the same, hence so are their roots the eigenvalues.
9. Suppose $A \in M_{n \times n}(F)$ has two distinct eigenvalues $\lambda$ and $\mu$. If the $\lambda$-eigenspace has dimension $n - 1$, prove that $A$ is diagonalizable.
   - The dimension of the $\mu$-eigenspace is at least 1 (since it is an eigenspace), so since the sum of the dimensions of the two eigenspaces cannot exceed $n$, it must equal $n$. But this is precisely our criterion for diagonalizability, so $A$ is diagonalizable.
10. Prove that $T : V \to V$ is diagonalizable if and only if $V$ is the direct sum of the eigenspaces of $T$.
    - The intersections of the eigenspaces are all zero, as we proved, so the sum of the eigenspaces is the same as the direct sum.
    - The diagonalizability criterion says that $T$ is diagonalizable if and only if the sum of the (dimensions of) the eigenspaces is equal to the (dimension of) the whole space $V$. This yields the result.
11. If $A$ is an $n \times n$ matrix, define the subspace $W \subseteq M_{n \times n}(F)$ as $W = \text{span}(I_n, A, A^2, A^3, \ldots)$. Prove that $\dim_F W \leq n$. This problem will be used in the next section.
12. If $\beta = \{v_0, v_1, \ldots, v_k\}$ is a chain of generalized $\lambda$-eigenvectors (i.e., with $(T - \lambda I)v_i = v_{i+1}$ for each $0 \leq i \leq k - 1$ and $(T - \lambda I)v_k = 0$) show that $\beta$ is linearly independent.

- Induction on $k$. Base case $k = 1$ is trivial.
- For the inductive step, suppose $a_0v_0 + \cdots + a_kv_k = 0$. Applying $(T - \lambda I)^k$ to both sides yields $a_0v_k = 0$, so since $v_k \neq 0$ we have $a_0 = 0$.
- Therefore we actually have $a_1v_1 + \cdots + a_kv_k = 0$. By the inductive hypothesis, all of the coefficients must then be zero, so we are done.

13. Prove that, up to similarity, there are exactly 5 different matrices in $M_{6 \times 6}(\mathbb{C})$ with characteristic polynomial $p(t) = t^6 - t^4$.

- The Jordan canonical form is unique up to similarity and equivalence so we just need to count the number of possible Jordan forms. From the characteristic polynomial, the eigenvalues must be $0, 0, 0, 0, 1, -1$.
- Then there must be a single $1 \times 1$ Jordan block with eigenvalue 1 and a single $1 \times 1$ block with eigenvalue $-1$, while the remaining blocks have eigenvalue 0 and the sum of their sizes is 0. There are 5 possibilities: 4, 3-1, 2-2, 2-1-1, and 1-1-1-1.

14. Suppose $A \in M_{n \times n}(\mathbb{C})$ such that $A^2 - 2A = 0$. Prove that the Jordan canonical form of $A$ must be diagonal.

- First note that if $Av = \lambda v$ then $(A^2 - 2A)v = (\lambda^2 - 2\lambda)v$ so $\lambda = 0, 2$.
- Also note that if $A^2 - 2A = 0$ then the same is true for the Jordan form $J$ of $A$.
- Now we just observe (e.g., by a direct calculation) that computing $J^2 - 2J$ on any block of size 2 or larger with eigenvalue 0 or 2 does not yield the zero matrix, so there cannot be any blocks of size larger than 1 in the Jordan form.

15. Suppose $T : V \to V$ is diagonalizable. Prove that for any $\lambda$ in the scalar field of $V$, it is true that rank$(T - \lambda I) = \text{rank}(T - \lambda I)^2$.

- Direct calculation: rank is unchanged under conjugation so it suffices to observe that this fact holds for the diagonalization $D$. But this is obvious: the rank is then the number of nonzero diagonal entries.
- Since all of the diagonal entries are either zero or not zero, and squaring them does not change the number of nonzero entries, the rank of $D - \lambda I_n$ equals the rank of $(D - \lambda I_n)^2$.

16. Suppose $T : V \to V$ has the property that rank$(T - \lambda I) = \text{rank}(T - \lambda I)^2$ for every $\lambda$ in the scalar field of $V$. Prove that $T$ is diagonalizable. [Hint: Consider the $\lambda$-blocks in the Jordan canonical form.]

- If $J$ is an $n \times n$ Jordan block with eigenvalue $\lambda$ with $n > 1$, by a trivial row-reduction we can see that the rank of $J - \lambda I$ is $n - 1$ while the rank of $(J - \lambda I)^2$ is $n - 2$.
- Thus, there are Jordan $\lambda$-blocks of size greater than 1 if and only if the ranks of $T - \lambda I$ and $(T - \lambda I)^2$ are unequal. Thus, the contrapositive says that if rank$(T - \lambda I) = \text{rank}(T - \lambda I)^2$ for every $\lambda$ then there are no Jordan blocks of size greater than 1, meaning that the Jordan canonical form is diagonal (and thus, $T$ is diagonalizable).

17. If $V$ is a finite-dimensional complex inner product space and $T : V \to V$ is Hermitian, prove that $\|T(v) + iv\|^2 = \|T(v)\|^2 + |v|^2$, and deduce that $T + iI$ is invertible. [Hint: Show $T$ is one-to-one.]

- Since $T^* = T$, we have $\|T(v) + iv\|^2 = \langle T(v) + iv, T(v) + iv \rangle = \langle T(v), T(v) \rangle + \langle iv, T(v) \rangle + \langle T(v), iv \rangle + \langle iv, iv \rangle = \|T(v)\|^2 - i\langle T(v), v \rangle + i\langle T^*(v), v \rangle + |v|^2 = \|T(v)\|^2 + |v|^2$ as claimed.
- The second statement follows by observing that if $(T + iI)v = 0$ then by the calculation we just made, we have $\|T(v)\|^2 + |v|^2 = 0$ from which $\|v\| = 0$ so that $v = 0$. Thus $T + iI$ is one-to-one hence invertible.
- Alternatively: all eigenvalues of $T$ are real (as we proved), so $-i$ cannot be one of them.

18. If $V$ is a finite-dimensional inner product space and $T : V \to V$ has an adjoint $T^*$, prove that all eigenvalues of $T^*T$ are nonnegative real numbers, and deduce that $I + T^*T$ is invertible.

- Suppose that $T^*T(v) = \lambda v$. Then $\lambda \langle v, v \rangle = \langle T^*T(v), v \rangle = \langle T(v), T(v) \rangle$, and so $\lambda = \langle T(v), T(v) \rangle / \langle v, v \rangle$ is a nonnegative real number because both inner products are nonnegative (and the denominator is not zero).
- Then the eigenvalues of $I + T^*T$ are just 1 plus the eigenvalues of $T^*T$, so they are all positive (in particular, nonzero). Then by problem 4 above, $I + T^*T$ is invertible.