1. Part (a) was worth 6 points and part (b) was worth 4 points.
   
   (a) We just check the parts of the subspace criterion. Note that the vectors in \( S \) have the form \( \langle x_1, x_2, x_3, x_1 + x_2 \rangle \).
   
   - [S1]: The zero vector satisfies both conditions.
   - [S2]: If \( v = \langle x_1, x_2, x_3, x_1 + x_2 \rangle \) and \( w = \langle y_1, y_2, y_3, y_1 + y_2 \rangle \) then
     \[
     v + w = \langle x_1 + y_1, x_1 + y_1, x_3 + y_3, (x_1 + y_1) + (x_2 + y_2) \rangle
     \]
     which is of the desired form.
   - [S3]: If \( v = \langle x_1, x_2, x_3, x_1 + x_2 \rangle \) then \( cv = \langle cx_1, cx_2, cx_3, cx_1 + cx_2 \rangle \) which is of the desired form.

   (b) As noted above, the vectors in \( S \) are those of the form \( \langle x_1, x_2, x_3, x_1 + x_2 \rangle \). Since \( \langle x_1, x_2, x_3, x_1 + x_2 \rangle = x_1 \langle 1, 0, 0, 0, 1 \rangle + x_2 \langle 0, 1, 0, 0, 1 \rangle + x_3 \langle 0, 0, 1, 1, 0 \rangle \), we see that \( \langle 1, 0, 0, 0, 1 \rangle, \langle 0, 1, 0, 0, 1 \rangle, \langle 0, 0, 1, 1, 0 \rangle \) span \( S \). Furthermore, since they are clearly linearly independent, they are a basis for \( S \). So we get the basis \( \{\langle 1, 0, 0, 0, 1 \rangle, \langle 0, 1, 0, 0, 1 \rangle, \langle 0, 0, 1, 1, 0 \rangle\} \) and get \( \dim(S) = 3 \).

2. Part (a) was worth 5 points each, part (b) was worth 6 points, and part (c) was worth 1 point.

   (a) Suppose that we have a dependence \( b_1(v_1 - v_2) + b_2(v_2 - v_3) + \cdots + b_{n-1}(v_{n-1} - v_n) + b_n v_n = 0 \). Expanding and rearranging yields \( b_1 v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + \cdots + (b_n - b_{n-1})v_n = 0 \). But now since \( S \) is linearly independent, each coefficient must be zero: this gives \( b_1 = b_2 - b_1 = b_3 - b_2 = \cdots = b_n - b_{n-1} = 0 \), so clearly each of \( b_1, b_2, \ldots, b_n \) must be zero.

   (b) Suppose \( w \) is in \( V \). If \( T \) spans \( V \), then there exist scalars \( a_1, a_2, \ldots, a_n \) such that \( w = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \). In order to show that \( T \) spans \( V \), we need to show that there exist scalars \( b_1, b_2, \ldots, b_n \) such that \( w = b_1(v_1 - v_2) + b_2(v_2 - v_3) + \cdots + b_{n-1}(v_{n-1} - v_n) + b_n v_n \). Expanding and collecting terms yields \( w = b_1 v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + \cdots + (b_n - b_{n-1})v_n \). Comparing this to the linear combination we had for \( w \) above, we should try \( b_1 = a_1, b_2 - b_1 = a_2, b_3 - b_2 = a_3, \ldots, b_n - b_{n-1} = a_n \). This yields \( b_1 = a_1, b_2 = a_1 + a_2, b_3 = a_1 + a_2 + a_3, \ldots, b_n = a_1 + a_2 + \cdots + a_n \). So, by the calculation above, we can write \( w = a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + \cdots + (a_1 + \cdots + a_n)v_n \), meaning that \( w \) is in \( \text{span}(T) \).

   (c) If \( S \) is a basis for \( V \), then since \( S \) spans \( V \), part (a) implies \( T \) spans \( V \). Also, since \( S \) is linearly independent, part (b) implies \( T \) is linearly independent. Then \( T \) spans \( V \) and is linearly independent, so it is a basis.

3. Each part was worth 4 points.

   (a) We check the two requirements:
   
   - [T1]: We have \( S(A + B) = (A + B) + (A + B)^T = (A + AT^T) + (B + BT^T) = S(A) + S(B) \).
   - [T2]: We have \( S(cA) = (cA)^T + (cA)^T = c(A + AT^T) = cS(A) \).

   (b) We have \( S\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad S\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \) and \( S\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \). Therefore we see \( [S]_B^B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \).

   (c) Row-reducing \( S \) yields the reduced row-echelon form \( E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \).
4. Equivalently, we must show that $T$ is an isomorphism.

- Approach 1: The condition $T^3 = I$ is equivalent to $T \circ T^2 = I = T^2 \circ T$; thus $T$ has a two-sided inverse given by $T^2$, so $T$ is an isomorphism.
- Approach 2: First suppose $v \in \ker(T)$ so that $T(v) = 0$. Then applying $T^2$ yields $v = T^3(v) = T^2(0) = 0$ so $\ker(T) = \{0\}$ and so $T$ is one-to-one. Now let $w \in V$ be arbitrary: then $T[T^2(w)] = T^3(w) = w$, and so $w \in \im(T)$ and so $T$ is onto.

5. By the nullity-rank theorem, we know that $\dim(\ker(T)) + \dim(\im(T)) = \dim(V) = 300$. Also, since $\im(T)$ is a subspace of $W$, we have $\dim(\im(T)) \leq \dim(W) = 200$. Hence $\dim(\ker(T)) = 300 - \dim(\im(T)) \geq 300 - 200 = 100$, as required.

6. Parts (b) and (c) were worth 4 points, and parts (a) and (d) were worth 3 points.
   
   (a) Suppose $w$ is in $\im(T)$. Then there exists $v$ with $w = T(v)$. Then $T(w) = T(T(v)) = 0$, meaning that $w$ is in $\ker(T)$. Thus, $\im(T)$ is contained in $\ker(T)$.
   
   (b) By part (a), $\dim(\im(T)) \leq \dim(\ker(T))$, and by nullity-rank, $\dim(\im(T)) + \dim(\ker(T)) = 2$. Thus, $\dim(\im(T))$ is either 0 or 1. But the dimension of the image cannot be zero, since this would imply that $T$ is the zero transformation. Thus, $\dim(\im(T)) = 1$.

   (c) Since $\{v, w\}$ has size 2 = $\dim(\mathbb{R}^2)$ it is enough to show that $v$ and $w$ are linearly independent. But if $0 = av + bw$ then $0 = T(0) = T(av + bw) = aT(v) + bT(w) = bv$, so since $v$ is nonzero, $b = 0$. Then $av = 0$ so $a = 0$. So $\{v, w\}$ is linearly independent, hence a basis.

   (d) Since $T(v) = 0 = 0v + 0w$ and $T(w) = v = 1v + 0w$, the matrix is $[T]_\beta^\alpha = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ as claimed.

7. Each item was worth 1.25 points (with the total rounded up to the nearest integer).

   (a) False  since $P_3(\mathbb{R})$ has dimension 4, any spanning set must contain at least 4 vectors; the given set has only 3.

   (b) False  the zero space has dimension 0. (The dimension of any other space is positive.)

   (c) True  any basis must have exactly 8 elements.

   (d) True  if $\dim(V) = n$ then as we showed, a set of $n$ vectors spans $V$ if and only if it is linearly independent.

   (e) False  the set spans $\im(T)$ but is not necessarily linearly independent.

   (f) False  although the two dimensions are consistent with the nullity-rank theorem, the map cannot have rank 3 because the target space only has dimension 2.

   (g) True  the map $T(A) = 2A^T$ has an inverse map $T^{-1}(A) = \frac{1}{2}A^T$ so it is an isomorphism.

   (h) False  because $V$ could be infinite-dimensional, neither condition implies the other.

   (i) True  both spaces have the same dimension $(\dim V) \cdot (\dim W)$ so they are isomorphic.

   (j) False  the matrix $[I]_\alpha^\beta$ will only be the identity matrix when $\alpha = \beta$.

   (k) True  the statement $[S T]_\alpha^\beta = [S]_\alpha^\gamma [T]_\gamma^\beta$ is a correct application of the composition formula.

   (l) True  we can take $Q = [I]_\beta^\alpha$ to be the change-of-basis matrix from $\beta$-coordinates to $\alpha$-coordinates.